UNIVERSITY OF LYON-1 / UNIVERSITY OF JOHANNESBURG

Extra-dimensional theory & phenomenology in the era of gravitational waves

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JRJC2022 | 28 October 2022

What is the significance of these extra dimensions?

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Historically: *A path to unification?*

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Today: *A path to physics BSM?*

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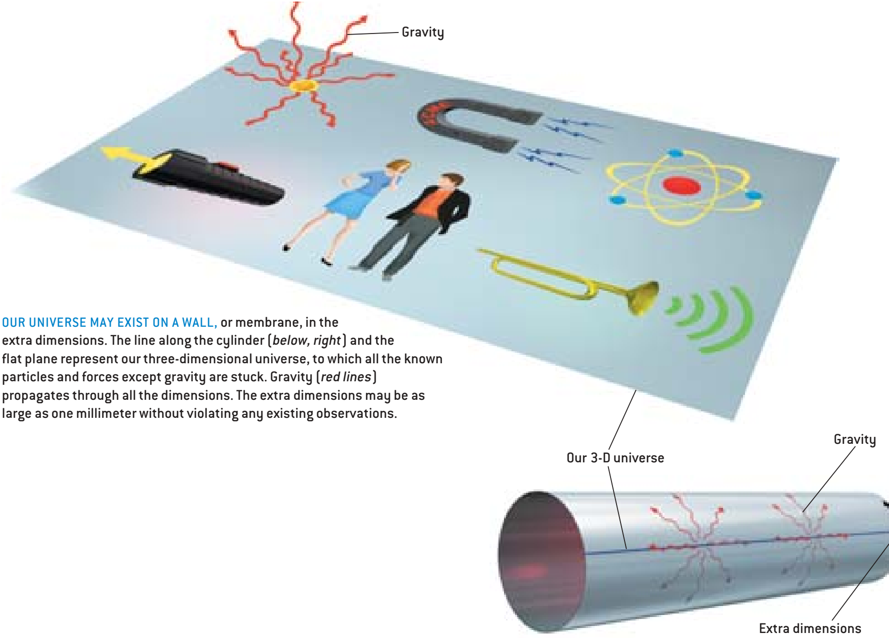

OUR UNIVERSE MAY EXIST ON A WALL, or membrane, in the extra dimensions. The line along the cylinder (*below, right*) and the flat plane represent our three-dimensional universe, to which all the known particles and forces except gravity are stuck. Gravity (*red lines*) propagates through all the dimensions. The extra dimensions may be as large as one millimeter without violating any existing observations.

Taken from ADD's 2002 Scientific American article, "The universe's unseen dimensions"

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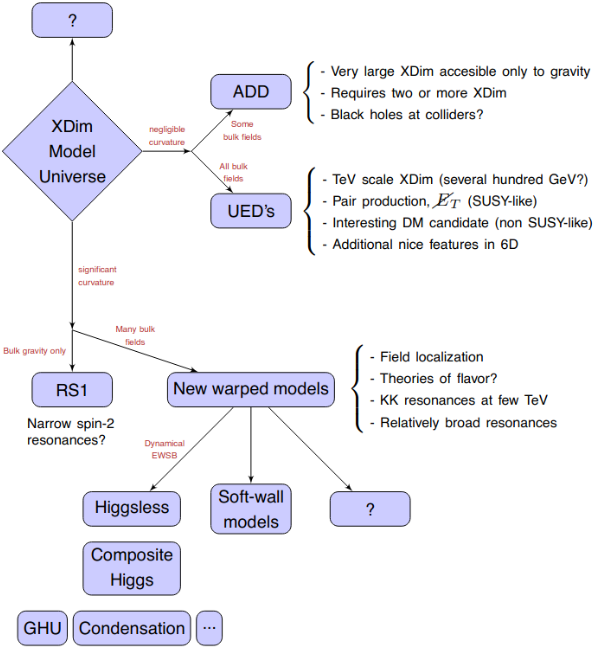
Today: *A path to physics BSM?*

- 1980s: “KK renaissance”, 1984 “superstring revolution”
- 1998: Arkani-Hamed, Dimopoulos, Dvali: large EDs
- 1999: Randall & Sundrum: warped EDs

...



Taken from E. Pontón's 2011 TASI lectures, "TeV scale EDs"

Why the negativity?

Negatively-curved EDs: a BSM landscape of untapped potential?

Phenomenological implications:

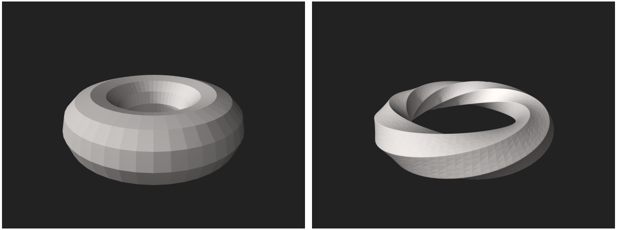
- natural resolution to the hierarchy problem

  $\rightarrow$ volume grows exponentially with ℓ_G/ℓ_c

  $\rightarrow$ RSI-like KK mass spectrum w/o light KK modes

- zero modes of Dirac operator emerges w/o gauge breaking
- enables homogeneity & flatness of observed universe

Any Lie group G of dimension d can be understood as a d-dimensional differentiable manifold. To compactify solvable G, we quotient by the lattice Γ. For nilpotent groups, the resultant twisted torus is a nilmanifold.

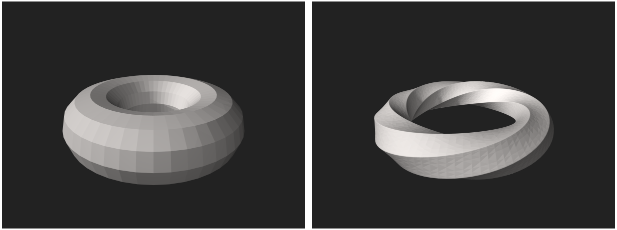


Wikimedia Commons, Torus & Twisted torus

Negatively-curved extra dimensions: the nilmanifold

Any Lie group G of dimension d can be understood as a d-dimensional differentiable manifold. To compactify solvable G, we quotient by the lattice Γ. For nilpotent groups, the resultant twisted torus is a nilmanifold.

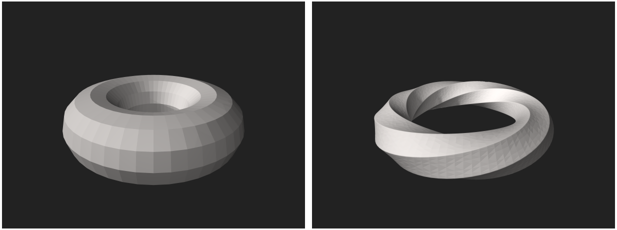
$$\mathcal{R} = -\frac{1}{4}\delta_{ad}\,\delta^{bc}\,\delta^{cg} f^{a}{}_{bc} f^{d}{}_{cg}$$



Wikimedia Commons, Torus & Twisted torus

Heisenberg algebra...

$$[Z_1, Z_2] = -\mathrm{f} Z_3 \ , \quad [Z_1, Z_3] = [Z_2, Z_3] = 0$$



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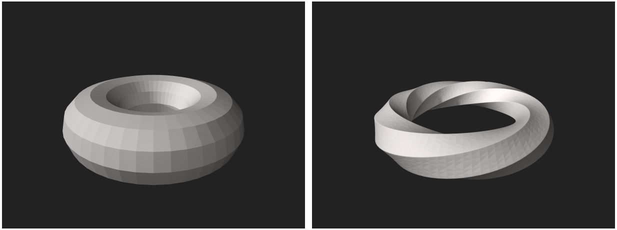
Heisenberg algebra...

$$[Z_1, Z_2] = -\mathrm{f} Z_3 \ , \quad [Z_1, Z_3] = [Z_2, Z_3] = 0$$

$$\mathrm{d}e^3 = \mathrm{f} e^1 \wedge e^2 \ , \quad \mathrm{d}e^1 = 0 \ , \quad \mathrm{d}e^2 = 0$$

$$e^1 = r^1 dy^1 \ , \quad e^2 = r^2 dy^2 \ , \quad e^3 = r^3 (dy^3 + N r^1 dy^2) \ , \quad N = \frac{r^1 r^2}{r^3} \mathrm{f}$$

...gives us the metric $ds^2_{\mathcal{H}} = g^{\mathcal{H}}_{ij} dx^i dx^j$



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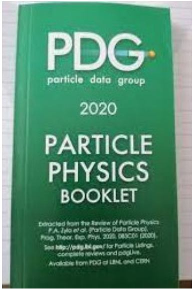
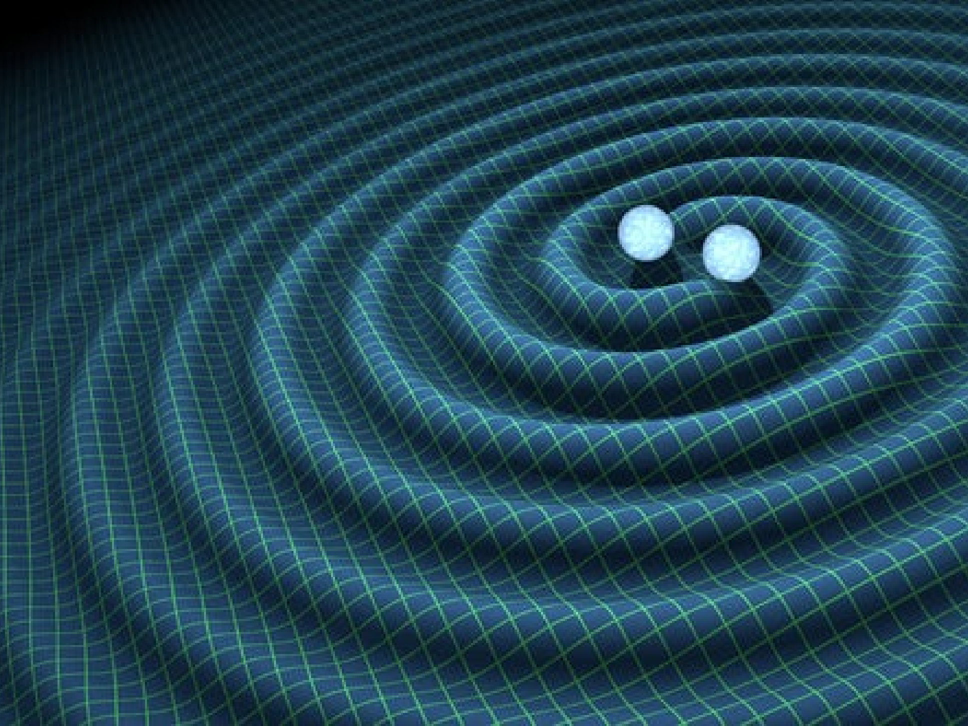
- Limits on R from Deviations in Gravitational Force Law
- Limits on R from On-Shell Production of Gravitons: $\delta = 2$
- Mass Limits on M_{TT}
- Limits on $1/R = M_c$
- Limits on Kaluza-Klein Gravitons in Warped Extra Dimensions
- Limits on Kaluza-Klein Gluons in Warped Extra Dimensions
- Black Hole Production Limits
  – Semiclassical Black Holes
  – Quantum Black Holes

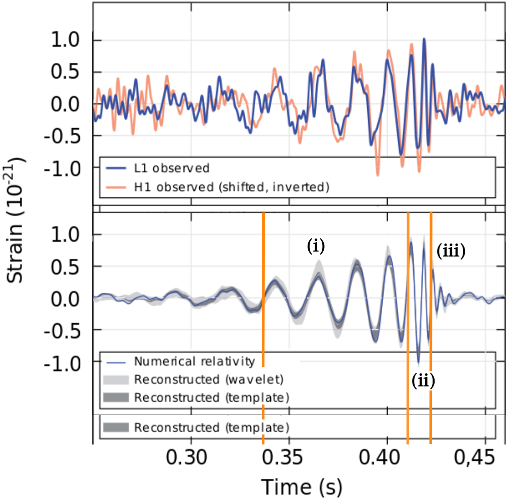
ATLAS, CMS, DELPHI, ALEPH, CDF, D0, OPAL, etc.





Objective

Using new techniques to probe underexplored BSM landscapes...



(i) inspiral
(ii) merger
(iii) ringdown

B. P. Abbott *et al.*, PRL **116**, 061102 (2016).

1957: Regge and Wheeler's grav. perturbations

The birth of black hole perturbation theory:

L REVIEW VOLUME 108, NUMBER 4 NOVEMBER

Stability of a Schwarzschild Singularity

TULLIO REGGE, *Istituto di Fisica della Università di Torino, Torino, Italy*

AND

JOHN A. WHEELER, *Palmer Physical Laboratory, Princeton University, Princeton, New Jersey*

(Received July 15, 1957)

It is shown that a Schwarzschild singularity, spherically symmetrical and endowed with mass, will undergo small vibrations about the spherical form and will therefore remain stable if subjected to a small nonspherical perturbation.

$$
\begin{aligned}
g_{\mu\nu}dx^{\mu}dx^{\nu} &= -f(r)dt^2 + f(r)^{-1}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \\
g_{\mu\nu} \to g'_{\mu\nu} &= g_{\mu\nu} + h_{\mu\nu} \qquad (g_{\mu\nu} \gg h_{\mu\nu})
\end{aligned}
$$

Quasinormal mode and frequency

$$\Psi(x^\mu) = \sum_{n=0}^{\infty} \sum_{\ell,m} \frac{\psi_{sn\ell}(r)}{r} e^{-i\omega t} Y_{\ell m}(\theta, \phi) \ , \qquad \omega_{sn\ell} = \omega_R - in\omega_I$$

- $\mathbb{R}e\{\omega\}$ = physical oscillation frequency
- $\mathbb{I}m\{\omega\}$ = damping $\rightarrow$ dissipative, "quasi"

Quasinormal mode and frequency

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- s: spin of perturbing field
- m: azimuthal number for spherical harmonic decomposition in θ_i
- ℓ: angular/multipolar number for spherical harmonic decomposition in θ, ϕ
- n: overtone number labels QNMs by a monotonically increasing $|\mathbb{I}m\{\omega\}|$

Black hole wave equation:

$$\frac{d^2}{dx^2}\varphi(x) + \left[\omega^2 - V(r)\right]\varphi(x) = 0\ , \qquad \frac{dr}{dx} = f(r)$$

$\rightarrow$ just a second-order ODE?

The QNM eigenvalue problem

Black hole wave equation:

$$\frac{d^2}{dx^2}\varphi(x) + \left[\omega^2 - V(r)\right]\varphi(x) = 0\,, \qquad \frac{dr}{dx} = f(r)$$

$\rightarrow$ actually: **QNM boundary conditions**

purely ingoing:	$\varphi(x) \sim e^{+i\omega x}$	$x \rightarrow -\infty \; (r \rightarrow r_H)$
purely outgoing:	$\varphi(x) \sim e^{-i\omega x}$	$x \rightarrow +\infty \; (r \rightarrow +\infty)$

Waves escape domain of study at the boundaries $\Rightarrow$ dissipative

The 7D metric

$$ds^2_{7D} = g^{BH}_{\mu\nu} dx^\mu dx^\nu + g^{\mathcal{H}}_{ij} dx^i dx^j$$

$$\Psi^s_{n\ell m}(t, r, \theta, \phi, y^1, y^2, y^3) = \sum_{n=0}^{\infty} \sum_{\ell,m} \frac{\psi_{sn\ell}(r)}{e^{i\omega t} r} Y^s_{m\ell}(\theta, \phi) Z(y^1, y^2, y^3)$$

$$ds^2_{BH} = -f(r)dt^2 + f(r)^{-1} dr^2 + r^2(\sin^2 d\theta^2 + d\phi^2)$$

$$f(r) = 1 - 2M/r$$

The 7D metric

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Laplacian of a product space is the sum of its parts

$$\left(\nabla^2_{BH} + \nabla^2_{\mathcal{H}}\right) \sum \Phi(x) Z_k(y) = 0 \ ,$$

$$\nabla^2 Z_k(y) = -\mu_k^2 Z_k(y)$$

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“mogette parameter"

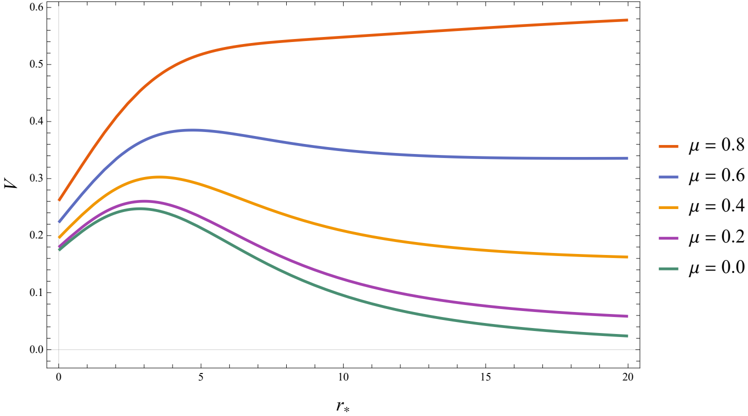
The wavelike equation

$$\frac{d^2\psi}{dr_*^2} + \left(\omega^2 - V(r)\right)\psi = 0$$

$$V(r) = \left(1 - \frac{2M}{r}\right)\left(\frac{\ell(\ell+1)}{r^2} + \frac{2M}{r^3} + \mu^2\right)$$

The QNM spectrum

The fundamental mode: $n = 0$, $\ell = 2$



The fundamental mode: $n = 0$, $\ell = 2$

μ	ω (WKB)	ω (PT)	ω (DO)
0.0	$0.4836 - 0.0968i$	$0.4874 - 0.0979i$	$0.4836 - 0.0968i$
0.1	$0.4868 - 0.0957i$	$0.4909 - 0.0968i$	$0.4868 - 0.0957i$
0.2	$0.4963 - 0.0924i$	$0.5015 - 0.0936i$	$0.4963 - 0.0924i$
0.3	$0.5123 - 0.0868i$	$0.5192 - 0.0881i$	$0.5124 - 0.0868i$
0.4	$0.5351 - 0.0787i$	$0.5443 - 0.0800i$	$0.5352 - 0.0787i$
0.5	$0.5649 - 0.0676i$	$0.5770 - 0.0690i$	$0.5653 - 0.0676i$
0.6	$0.6022 - 0.0528i$	$0.6181 - 0.0541i$	$0.6032 - 0.0532i$
0.7	$0.1396 + 0.2763i$	$0.6695 - 0.0312i$	$0.6500 - 0.0343i$

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$\Rightarrow$ An upper bound on our QNM probe (“sensitivity range cutoff")

Bounds from GWs?

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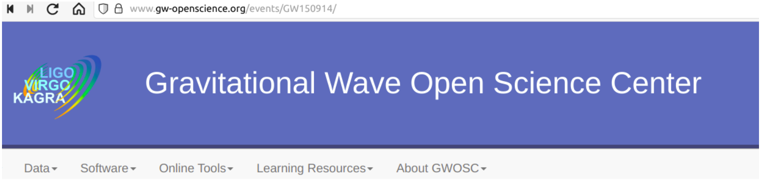
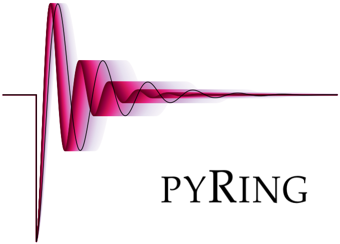


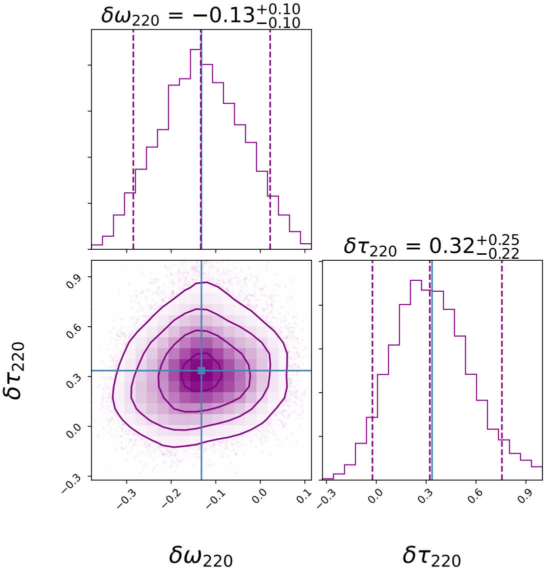


$$\delta\omega = \omega^{GR}(1+\delta\omega)$$

$$\delta\tau = \tau^{GR}(1+\delta\tau)$$

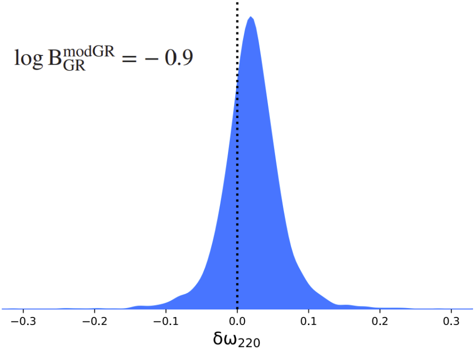
The fundamental mode: $n = 0, \ell = 2$

μ	$\delta\omega$	$\delta\tau$
0.0	0.0000	0.0000
0.1	0.0065	0.0113
0.2	0.0262	0.0473
0.3	0.0594	0.1149
0.4	0.1066	0.2302
0.5	0.1687	0.4306
0.6	0.2472	0.8206
0.7	0.3440	1.8181


$\delta\omega_{220} = -0.13^{+0.10}_{-0.10}$
$\delta\tau_{220} = 0.32^{+0.25}_{-0.22}$
$\delta\tau_{220}$
$\delta\omega_{220}$
$\delta\tau_{220}$
0.9
0.6
0.3
0.0
−0.3
−0.3
−0.2
−0.1
0.0
0.1
−0.3
0.0
0.3
0.6
0.9

Tests of GR with GWTC-3 [2112.06861]

$$\delta\omega_{O3} = 0.02^{+0.07}_{-0.07}$$
$$\delta\tau_{O3} = 0.13^{+0.21}_{-0.22}$$



$$0.1747 < \mu < 0.3681$$

Interpretation?

From the dimensionless parameter $M\mu$,

$$
\begin{aligned}
M\mu &= \frac{G m^{\mathrm{BH}} m}{\hbar c} \\
&\Rightarrow \quad m = \frac{1}{m^{\mathrm{BH}}} \frac{\hbar c}{G} M\mu \\
&\qquad m \sim 10^{-\chi} 10^{-46} \mathrm{kg} \sim 10^{-(\chi+10)} \mathrm{eV/c^2}
\end{aligned}
$$

$m \sim 10^{-13}$ eV/c^2 for $M_f \sim 62 M_\odot$ of GW150914

light scalar hypotheses

Conclusions

- Rich phenomenology awaits in the mathematicians' playground!
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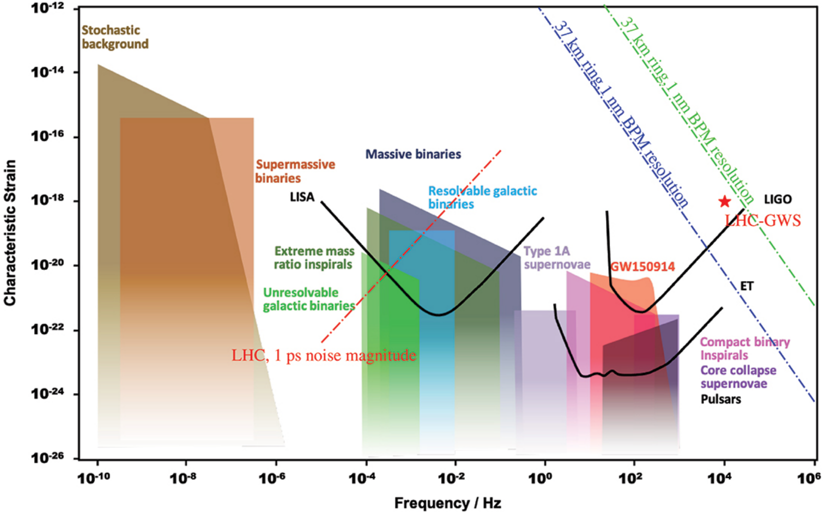
- Rich phenomenology awaits in the mathematicians' playground!
- Connecting theory and observation is non-trivial ("the gap")
- Detecting modifications to GR is considered to be beyond the sensitivity of modern detectors
- Using QNM theory, we have introduced a possible new observable + applied naive constraints

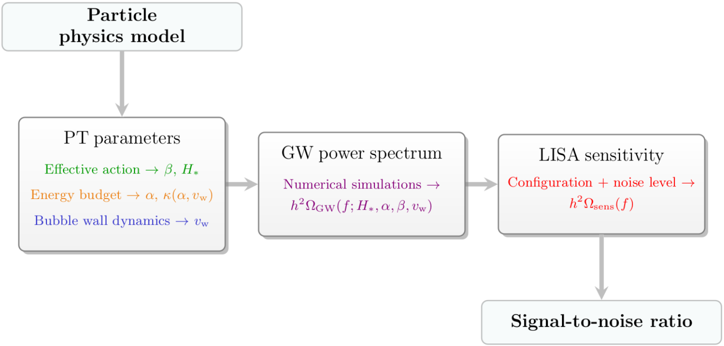
What's next?

Using established techniques to probe the GW BSM landscape...



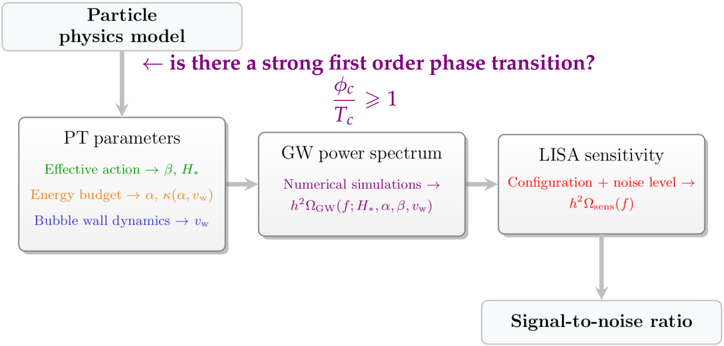
Credit to S. Rao et al. / K. Oide, G. Franchetti & F. Zimmermann / P. Chen [http://gwplotter.com]

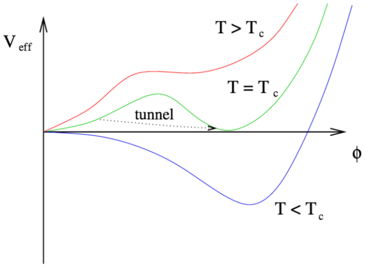
Algorithm for finding GWs from PTs

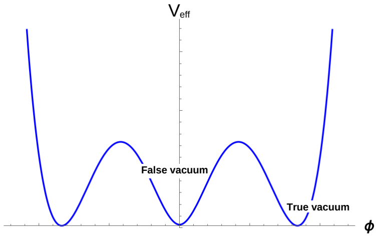


C. Caprini *et al.* JCAP 03 (2020) 024.

Algorithm for finding GWs from PTs

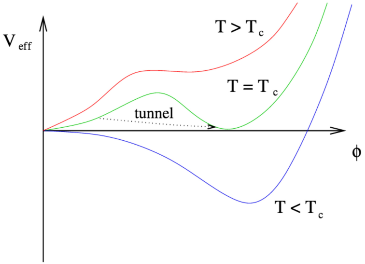


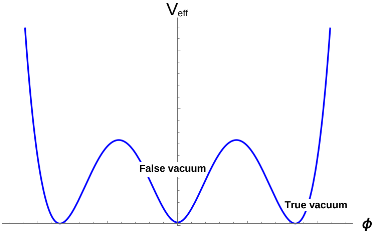




G. White, *A Pedagogical Intro to Baryogenesis*; W.C. Huang, F. Sannino, and Z.W. Wang, Phys. Rev. D 102 (2020) 095025

$$V_{eff} \approx V_{tree} + V_{1loop} + V_T$$





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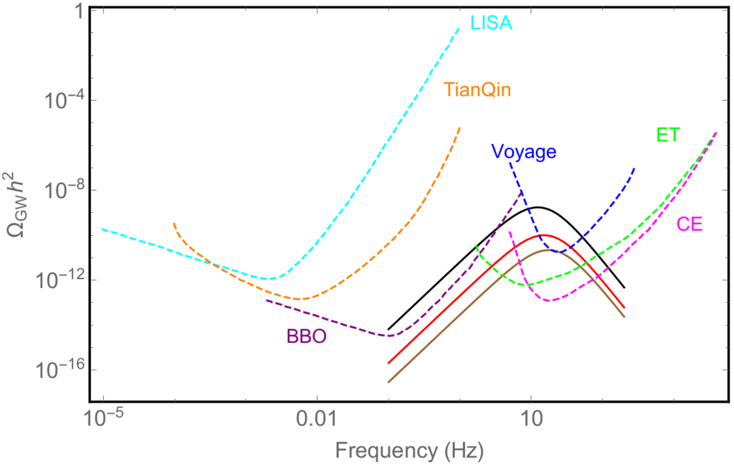
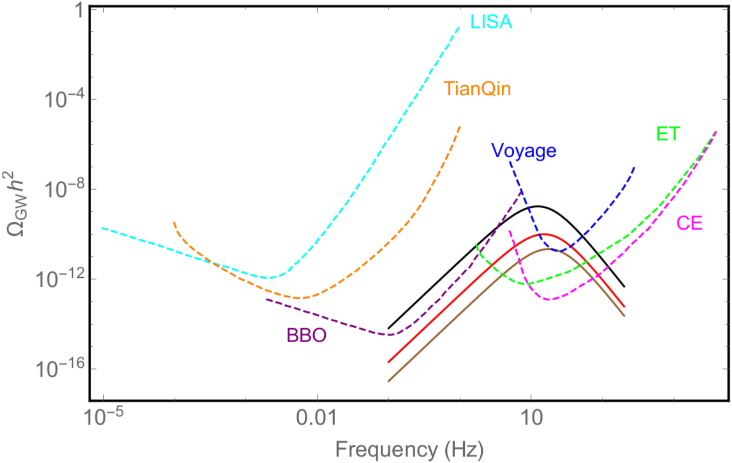


W.C. Huang, F. Sannino, and Z.W. Wang, Phys. Rev. D 102 (2020) 095025

Stay tuned!



W.C. Huang, F. Sannino, and Z.W. Wang, Phys. Rev. D 102 (2020) 095025

Thank you

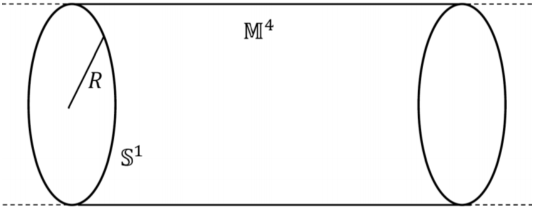
Thank you

And a warm thanks to

Compactification: infinite $(3+1)$ dims; finite x_5
periodic BCs: $x_5 \to x_5 + 2\pi R$

KK tower of states:

$$\Phi(x^\mu, x^5) = \sum_{n=0}^{\infty} \Phi^{(n)}(x^\mu)\, e^{inx^5/R}\,,\; m_n = \sqrt{m_0^2 + \left(\frac{n}{R}\right)^2}$$



In the Kaluza-Klein 5D universe

$$V_{eff} \approx V_{tree} + V_{1loop} + V_T$$

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For $\beta = 1/T,\ L_5 = 2\pi R$:

$$\begin{aligned} V_T &= -\frac{3}{4\pi^2}\zeta(5)\frac{1}{L_5^4} \\ &\quad -\frac{3}{4\pi^2}\zeta(5)\frac{L_5}{\beta^2} - \frac{\Gamma(5/2)}{\pi^{5/2}L_5^4}\, 2\sum_{m,n=1}^{\infty}\left[\left(\frac{\beta m}{L_5}\right)^2 + n^2\right]^{-5/2} \end{aligned}$$

$$V_T \sim -\frac{3}{4\pi^2}\zeta(5)\frac{1}{L_5^4} \qquad L_5 \ll \beta$$

$$V_T \sim -\frac{3}{4\pi^2}\zeta(5)\frac{1}{\beta^5} \qquad L_5 \gg \beta$$

Dolan & Ottewill (2009)

A new computation method for BH QNMs through a novel ansatz based on **null geodesics** + expansion of the QNF in inverse powers of $L = \ell + 1/2$

$$\Phi(r) = e^{i\omega z(x)} v(r)\ , \quad \omega = \sum_{k=-1}^{\infty} \omega_k L^{-k}$$

We explore the method for Schwarzschild, RN, and SdS in 4D:

- more efficient means of calculating detectable BH QNMs?
- explore interplay of θ, λ in large-ℓ limit

Components of the ansatz

$$v(r) = \exp\left\{ \sum_{k=0}^{\infty} S_k(r) L^{-k} \right\}, \quad z(x) = \int^x \rho(r) dx = \int^x b_c k_c(r) dx$$

$$k_c(r)^2 = \frac{1}{b^2} - \frac{f(r)}{r^2}$$

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$$r_c = \frac{2f(r)}{\partial_r f(r)}\bigg|_{r=r_c}, \quad b_c = \sqrt{\frac{r^2}{f(r)}}\,\bigg|_{r=r_c}, \quad k_c(r)^2 = \frac{1}{b^2} - \frac{f(r)}{r^2}$$

Components of the ansatz

$$v(r) = \exp\left\{\sum_{k=0}^{\infty} S_k(r) L^{-k}\right\}, \quad z(x) = \int^x \rho(r)dx = \int^x b_c k_c(r) dx$$

$$r_c = \left.\frac{2f(r)}{\partial_r f(r)}\right|_{r=r_c}, \quad b_c = \left.\sqrt{\frac{r^2}{f(r)}}\right|_{r=r_c}, \quad k_c(r)^2 = \frac{1}{b^2} - \frac{f(r)}{r^2}$$

We generalise the consequent ODE

$$f(r)\frac{d}{dr}\left(f(r)\frac{dv}{dr}\right) + 2i\omega\rho(r)\frac{dv}{dr} + \left[i\omega f(r)\frac{d\rho}{dr} + \left(1-\rho(r)^2\right)\omega^2 - V(r)\right]v(r) = 0$$

We solve iteratively for ω_k and $S'_k(r)$ and sub into ω_k

$$r_c = 3\ ,\ b_c = \sqrt{27} \quad \Rightarrow \quad \rho(r) = \left(1 - \frac{3}{r}\right)\sqrt{1 + \frac{6}{r}}$$

$$\begin{aligned}\omega(L,\mu) = &+\frac{1}{3}L - \frac{i}{6}L^0 + \left[\frac{3\mu^2}{2} + \frac{7}{648}\right]L^{-1} \\ &+ \left[\frac{5i\mu^2}{4} - \frac{137i}{23328}\right]L^{-2} + \left[\frac{9\mu^4}{8} - \frac{379\mu^2}{432} + \frac{2615}{3779136}\right]L^{-3} \\ &+ \left[\frac{27i\mu^4}{16} - \frac{2677i\mu^2}{5184} + \frac{590983i}{1088391168}\right]L^{-4} \\ &+ \left[\frac{63\mu^6}{16} - \frac{427\mu^4}{576} + \frac{362587\mu^2}{1259712} - \frac{42573661}{117546246144}\right]L^{-5} \\ &+ \left[\frac{333i\mu^6}{32} + \frac{6563i\mu^4}{6912} + \frac{100404965i\mu^2}{725594112} + \frac{11084613257i}{25389989167104}\right]L^{-6}\ .\end{aligned}$$

The fundamental mode: $n = 0$, $\ell = 2$

μ	ω (WKB)	ω (PT)	ω (DO)
0.0	$0.4836 - 0.0968i$	$0.4874 - 0.0979i$	$0.4836 - 0.0968i$
0.1	$0.4868 - 0.0957i$	$0.4909 - 0.0968i$	$0.4868 - 0.0957i$
0.2	$0.4963 - 0.0924i$	$0.5015 - 0.0936i$	$0.4963 - 0.0924i$
0.3	$0.5123 - 0.0868i$	$0.5192 - 0.0881i$	$0.5124 - 0.0868i$
0.4	$0.5351 - 0.0787i$	$0.5443 - 0.0800i$	$0.5352 - 0.0787i$
0.5	$0.5649 - 0.0676i$	$0.5770 - 0.0690i$	$0.5653 - 0.0676i$
0.6	$0.6022 - 0.0528i$	$0.6181 - 0.0541i$	$0.6032 - 0.0532i$
0.7	$0.1396 + 0.2763i$	$0.6695 - 0.0312i$	$0.6500 - 0.0343i$

In agreement with massive scalar QNFs of S. Dolan, Phys. Rev. D 76 (2007) 084001

Suppose we place a 4D Schwarzschild black hole within a 7D spacetime, perturbed by a 7D scalar test field of mass μ:

$$\text{KG:}\ \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\Psi\right)-\mu^2\Psi=0\ ,$$

$$g_{\mu\nu}dx^\mu dx^\nu=g_{ab}(x)dx^a dx^b+g_{ij}(y)dx^i dx^j\ ,$$

$$g_{\mu\nu}=\begin{bmatrix} -f(r) & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & f(r)^{-1} & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & +1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & +1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & r_1^2 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & r_2^2+r_3^2N^2y_1^2 & r_3^2Ny_1\\ 0 & 0 & 0 & 0 & 0 & r_3^2Ny_1 & r_3^2 \end{bmatrix},$$

where $f(r)=1-2M/r$

QNMs: Deriving the radial equation

Variable-separable QNM solution:

$$\Psi^{s}_{n\ell m \mu}(t,r,\theta,\phi,y_1,y_2,y_3) = \sum_{n=0}^{\infty}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} R^{s}_{n\ell\mu}(r) Y^{s}_{m\ell}(\theta,\phi) Z_{\mu}(y_1,y_2,y_3) e^{i\omega t} \,.$$

Laplacian of a product space is the sum of its parts

$$\left(\nabla^2_{BH} + \nabla^2_{nil}\right) \sum \Phi(x) Z_k(y) = 0 \,,$$

- $\nabla^2 Y^{s}_{m\ell}(\theta,\phi) = \frac{-\ell(\ell+1)}{r^2} Y^{s}_{m\ell}(\theta,\phi)$
- $\nabla^2 Z_k(y) = -\mu_k^2 Z_k(y)$

$$\mu^2_{k,j,m} = \frac{4\pi^2 k^2}{(r_3)^2}\left[1 + \frac{(2m+1) r_3}{2\pi |k|} |\mathtt{f}|\right]$$

Gravitational perturbations

Table I. Stabilities of generalised static black holes. In this table, "d" represents the spacetime dimension, $n+2$. The results for tensor perturbations apply only for maximally symmetric black holes, while those for vector and scalar perturbations are valid for black holes with generic Einstein horizons, except in the case with $K=1, Q=0, \lambda>0$ and $d=6$.

		Tensor		Vector		Scalar	
		$Q=0$	$Q\neq 0$	$Q=0$	$Q\neq 0$	$Q=0$	$Q\neq 0$
$K=1$	$\lambda=0$	OK	OK	OK	OK	OK	$d=4,5$ OK $d\geq 6$?
	$\lambda>0$	OK	OK	OK	OK	$d\leq 6$ OK $d\geq 7$?	$d=4,5$ OK $d\geq 6$?
	$\lambda<0$	OK	OK	OK	OK	$d=4$ OK $d\geq 5$?	$d=4$ OK $d\geq 5$?
$K=0$	$\lambda<0$	OK	OK	OK	OK	$d=4$ OK $d\geq 5$?	$d=4$ OK $d\geq 5$?
$K=-1$	$\lambda<0$	OK	OK	OK	OK	$d=4$ OK $d\geq 5$?	$d=4$ OK $d\geq 5$?

$$\mathcal{R}_{ED} = (d-3)K\gamma_{ij}$$