Journées de Recontres Jeunes Chercheurs 2022
Microscopic ab initio approach for the antiproton-nucleus annihilation

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Outline

1. Nuclear physics with antiprotons

2. Research project

3. The $p\bar{p}$ annihilation
   - Schrödinger equation
   - $N\bar{N}$ interaction
   - S-matrix fit

4. The $d\bar{p}$ annihilation
   - Faddeev equations
   - Optical model
   - Coupled-channel model

5. Conclusion and prospects

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Nuclear physics with antiprotons

- The first experiment with antiprotons to study neutron skins and halos was performed in Brookhaven National Laboratory in the 1970s.
- The interest for the physics of the interactions between matter and antimatter has been revived with the development of new facilities at CERN: LEAR (1982-1996), AD , ELENA (2016).
- Low-energy antiprotons are expected to provide a unique sensitivity to the nuclear density tail.

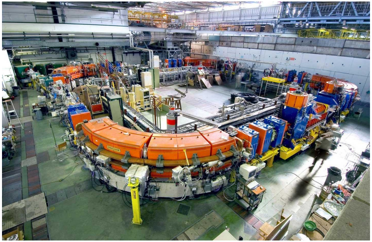


Figure: LEAR (Low Energy Antiproton Ring). Credits: CERN

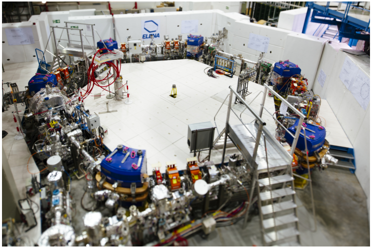
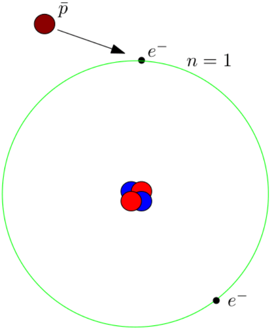


Figure: ELENA (Extra-Low Energy Antiproton Ring). Credits: CERN

Antiproton-nucleus interaction

- The evolution of the antiproton-nucleus system takes place in three main steps[1]
    1. Capture on a highly excited Coulomb orbital ($n \gg 1$) and formation of a quasi-bound system.
    2. Decay of the antiprotonic atom via X-ray and Auger emissions.
    3. Annihilation with a nucleon of the nucleus followed by the emission of mesons.



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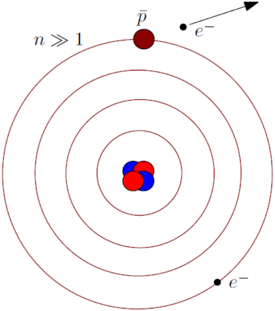

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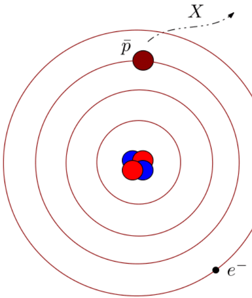

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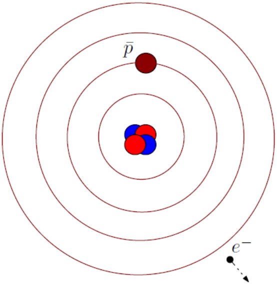

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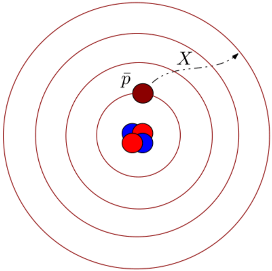

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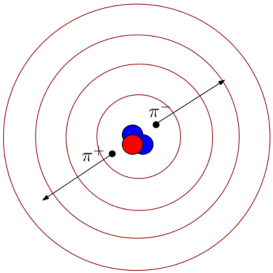

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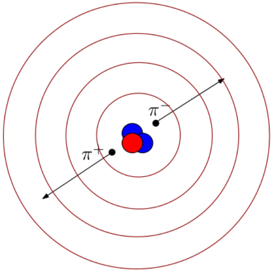

- The annihilation process involves a very complex dynamics and contains multitude of meson-producing channels, namely $N\bar{N} \to \pi\bar{\pi}, K\bar{K}, \rho\bar{\rho}, \pi\bar{\pi}\pi\bar{\pi}$.
- The measurement of annihilation products allows us to infer the neutron to proton annihilation ratio $\longrightarrow$ since the annihilation is expected to happen in the nuclear periphery, it could be an interesting tool to investigate the nature of the nuclear tail density.

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The PUMA experiment

- PUMA[1]: antiProton Unstable Matter Annihilation.
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- Aims to study nucleus skin densities of short lived nuclear isotopes, produced by ISOLDE, using low energy antiprotons transported from ELENA.

- The first objectives of the PUMA experiment are
  1. to study the density tail of radioactive nuclei, namely the n-to-p annihilation ratio.
  2. to characterise the density tail of known neutron halos and neutron skins.
  3. to evidence new proton and neutron halos.

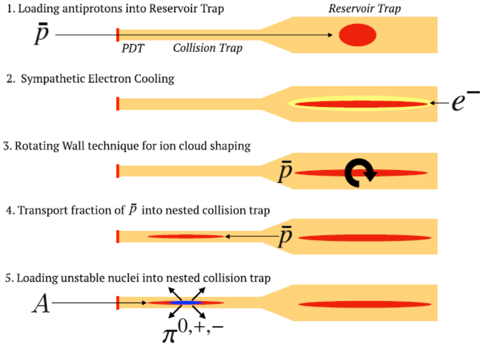

Figure: Schematic view of the PUMA experiment[1]

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- Key question: how can the measured antiproton-nucleon annihilations be related with the nuclear surface densities ?
- PhD goal: answering to this question by developing a **microscopic** *ab initio* approach to study the simplest cases of antiproton-nucleus ($\bar{p} - A$) system.

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- Progress of the work
  1. Study of the $p\bar{p}$ system.
  2. Study of the $d\bar{p}$ annihilation (in progress).

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- The $p\bar{p}$ annihilation involves many meson-producing channels, mainly pions:

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$$\sum_{c'} \left[\left(T_{l_c} + E_c - E\right) \delta_{cc'} + V_{cc'} \right] u_{c'}(r) = 0.$$

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Optical model

- The annihilation is usually accounted using an optical potential: $V_{N\bar{N}} = U_{N\bar{N}} + iW$.
- The **imaginary part** induces a loss of probability current in the initial channel which simulates the effect of **all annihilation channels**.
- Present work: Kohno-Weise potential[1] ($W \equiv$ Wood-Saxon well).

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[1] M. Kohno and W. Weise. *Nuclear physics A* **454** (1986) 429

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Coupled-channel model

- Alternative method explored: the annihilation is simulated by the addition of effective $m\bar{m}$ channels coupled to $p\bar{p}$ and $n\bar{n}$ channels by short-range Yukawa potentials[1].

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- Still phenomenological but includes very different dynamics.
- The parameters of the couped-channel potential are adjusted to fit the S-matrix obtained with the optical model.

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[1] E. Ydrefors and J. Carbonell. *The European physical journal A* **57** (2021) 303

S-matrix fit: ${}^1S_0$

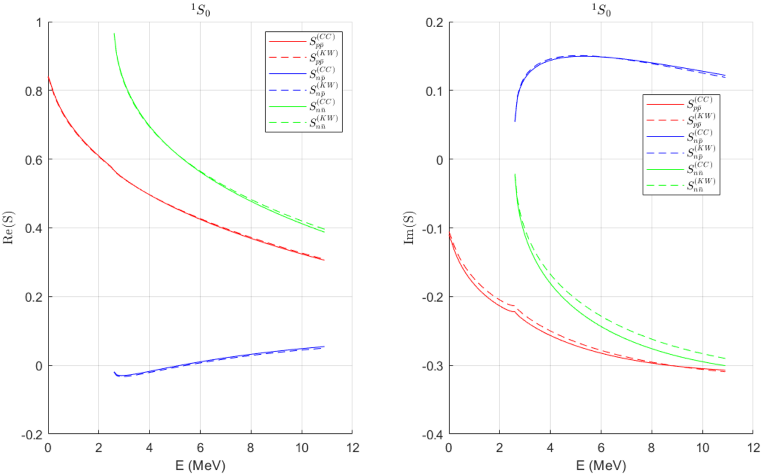

Figure: S-matrix for the ${}^1S_0$ wave computed with the optical model (dotted line) and the coupled-channel model (full line).

S-matrix fit: ${}^1P_1$

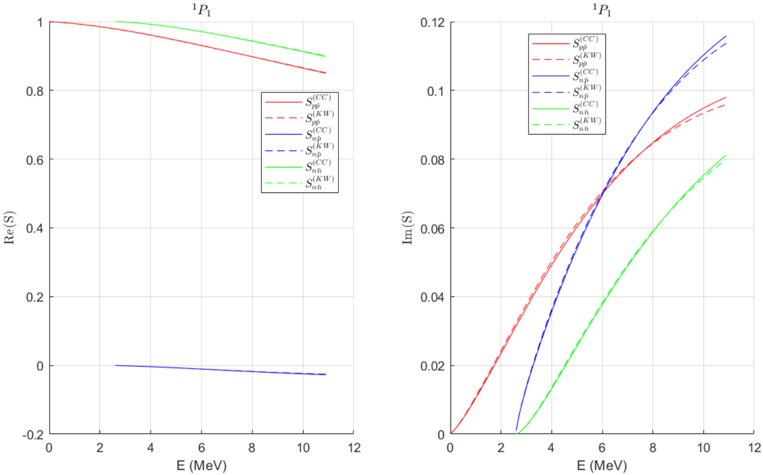

Figure: S-matrix for the ${}^1P_1$ wave computed with the optical model (dotted line) and the coupled-channel model (full line).

S-matrix fit: ${}^3P_1$

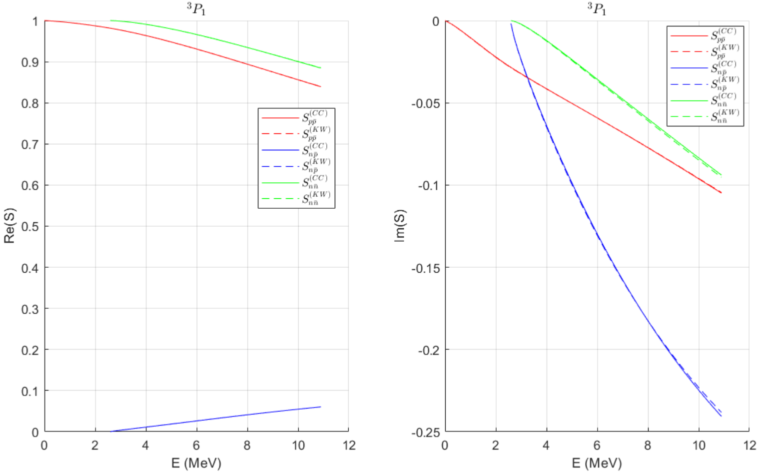

Figure: S-matrix for the ${}^3P_1$ wave computed with the optical model (dotted line) and the coupled-channel model (full line).

Outline

The Faddeev equations

- The 3-body Schrödinger equation reads $(E - H_0 - V_{23} - V_{13} - V_{12})\Psi = 0$.
- Faddeev decomposition: $\Psi = \Phi_1(\boldsymbol{x}_1, \boldsymbol{y}_1) + \Phi_2(\boldsymbol{x}_2, \boldsymbol{y}_2) + \Phi_3(\boldsymbol{x}_3, \boldsymbol{y}_3)$.
- The Faddeev components Φ_i are solutions of the Faddeev equations[1]:

$$(E - H_0 - V_{23})\,\Phi_1(\boldsymbol{x}_1, \boldsymbol{y}_1) = V_{23}\left[\Phi_2(\boldsymbol{x}_2, \boldsymbol{y}_2) + \Phi_3(\boldsymbol{x}_3, \boldsymbol{y}_3)\right]$$

$$(E - H_0 - V_{13})\,\Phi_2(\boldsymbol{x}_2, \boldsymbol{y}_2) = V_{13}\left[\Phi_3(\boldsymbol{x}_3, \boldsymbol{y}_3) + \Phi_1(\boldsymbol{x}_1, \boldsymbol{y}_1)\right]$$

$$(E - H_0 - V_{12})\,\Phi_3(\boldsymbol{x}_3, \boldsymbol{y}_3) = V_{12}\left[\Phi_1(\boldsymbol{x}_1, \boldsymbol{y}_1) + \Phi_2(\boldsymbol{x}_2, \boldsymbol{y}_2)\right]$$

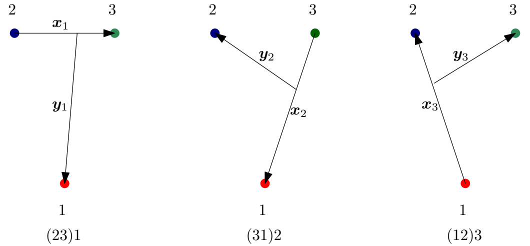

Figure: Jacobi coordinates for a three-body system

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[1] L. D. Faddeev. *Journal of Experimental and Theoretical Physics* **39** (1960) 1459

Optical model

- In the optical model, the Faddeev components include the $p\bar{p}$ and $n\bar{n}$ channels coupled together by the $N\bar{N}$ interaction.

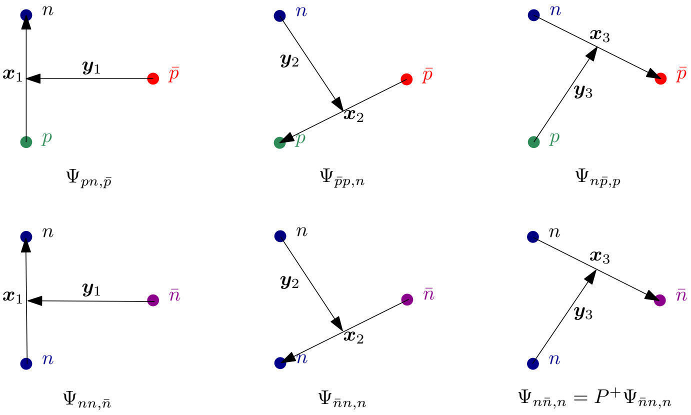

Figure: Faddeev components for the $d\bar{p}$ annihilation (optical model).

Optical model

- From the wavefunction, we can extract scattering phase shift and annihilation densities[1] ⟶ related to the annihilation probability
- The P-wave annihilation density scales nicely with the deuteron density ⟶ confirming the intuitive conjecture on which the PUMA project is based.
- Model dependence ?

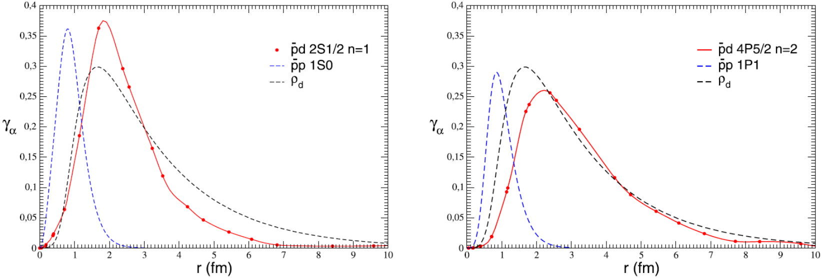

Figure: $d\bar{p}$ annihilation densities for the $^2S_{1/2}$ (left panel) and $^4P_{5/2}$ (right panel) states[1]

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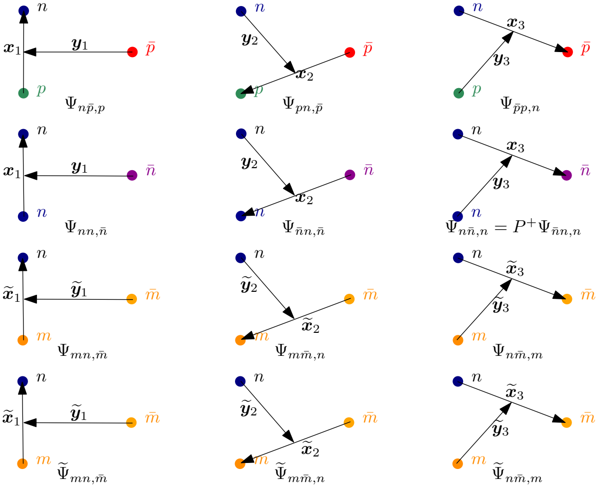
[1] R. Lazauskas and J. Carbonell. *Physics letters B* **820** (2021) 136573

Coupled-channel model

- In the coupled channel model, the Faddeev components include the $p\bar{p}$, $n\bar{n}$ and $m\bar{m}$ channels.



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  3. 3B: Resolution of the Faddeev equations for the $d\bar{p}$ collision with the coupled-channel model $\longrightarrow$ complex scaling method to handle the three-body breakup in meson channels.
- Long-term prospects: Resolution of the Faddeev-Yakubovsky equations for the systems ${}^{3}\mathrm{H} + \bar{p}$ and ${}^{3}\mathrm{He} + \bar{p}$.

Backup - Partial wave expansion

- Resolution with a partial wave expansion:

$$\Phi_i(\boldsymbol{x}_i, \boldsymbol{y}_i) = \sum_{n=l_x,l_y,L,s_x,S,t_x,T} \frac{\phi_n^{(i)}(x_i, y_i)}{x_i y_i} \left\{ [l_x l_y]_L \left[(s_j s_k)_{s_x} s_i\right]_S \right\}_J |(t_j t_k)_{t_x} t_i\rangle_T \, .$$

- The radial functions are expressed as a linear combination of Lagrange functions:

$$\phi_n^{(i)}(x_i, y_i) = \sum_{\alpha,\beta} c_{n\alpha\beta}^{(i)} \hat{f}_\alpha \left(\frac{x_i}{h_x^{(i)}}\right) \hat{f}_\beta \left(\frac{y_i}{h_y^{(i)}}\right)$$

- Lagrange function: polynomial multiplied by an exponential function, behaving as r^{l+1} close to the origin.
- Resolution of a generalised eigenvalue problem for bound states.
- Resolution of linear systems for scattering states.
- Matrices of large dimension $\longrightarrow$ iterative methods (Power method, Lanczos, GMRES, BICGSTAB,...).

Backup - Faddeev equations: practically

- Faddeev equation for component i:

$$\underbrace{(E - H_0 - V_i)\Phi_i(\boldsymbol{x}_i, \boldsymbol{y}_i)}_{\text{Same component} \rightarrow \text{symmetric matrix, x and y decoupled}} = \underbrace{V_i[\Phi_j(\boldsymbol{x}_j, \boldsymbol{y}_j) + \Phi_k(\boldsymbol{x}_k, \boldsymbol{y}_k)]}_{\text{Different components} \rightarrow \text{non-symmetric matrix}} \quad .$$

- Once projected over angular and radial functions, the right-hand side includes 6-dimensional integrals of type

$$I = \int \frac{x_i y_i}{x_j y_j} \, \hat{f}_\beta(x_i, y_i) \, \hat{f}_\alpha(x_j, y_j) \, \left[Y_{l_{x\,i}}(\hat{x}_i) \otimes Y_{l_{y\,i}}(\hat{y}_i)\right]^*_L \left[Y_{l_{x\,j}}(\hat{x}_j) \otimes Y_{l_{y\,j}}(\hat{y}_j)\right]_L \, \mathrm{d}\Omega_i \, \mathrm{d}x_i \mathrm{d}y_i .$$

- Reduces to a 3-dimensional integral by analytic integration over some angles:

$$I = \iint \mathrm{d}x_i \mathrm{d}y_i \int_{-1}^{1} h^{(L)}(x_i, y_i, u) \frac{x_i y_i}{x_j y_j} \, \hat{f}_\beta(x_i, y_i) \, \hat{f}_\alpha(x_j, y_j) \, \mathrm{d}u,$$

- I is then computed numerically with Gauss quadratures.
- Using Lagrange conditions → only one sum: Gauss-Legendre quadrature over u with $N_u \sim 15 - 20$.

Backup - The Faddeev-Merkuriev equations

- Additional corrections are required to impose boundary conditions for scattering including the Coulomb potential: $V_i = V_{s_i} + V_{C_i}$.
- Separation of the potential into a short-range and a long-range contribution[1]:

$$
\begin{aligned}
V_i^{(s)}(x_i, y_i) &= V_{s_i}(x_i) + V_{C_i}\,\chi(x_i, y_i) \\
V_i^{(l)}(x_i, y_i) &= V_{C_i}(x_i)\,\left[1 - \chi(x_i, y_i)\right] \\
\chi(x_i, y_i) &= \frac{2}{1 + \exp\left[\frac{(\mathrm{x}/\mathrm{x}_0)^{\mu}}{1+\mathrm{y}/\mathrm{y}_0}\right]}
\end{aligned}
$$

- Faddeev-Merkuriev equation for component i:

$$
\left(E - H_0 - V_i^{(s)} - V_i^{(l)} - V_j^{(l)} - V_k^{(l)}\right)\Phi_i(\boldsymbol{x}_i, \boldsymbol{y_i}) = V_i^{(s)}\left[\Phi_j(\boldsymbol{x}_j, \boldsymbol{y}_j) + \Phi_k(\boldsymbol{x}_k, \boldsymbol{y}_k)\right].
$$

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[1] S.P. Merkuriev. *Annals of physics* **130** (1980) 395

Backup - Antiprotonic atom decay

- A more formal view of the antiprotonic atom decay[1].

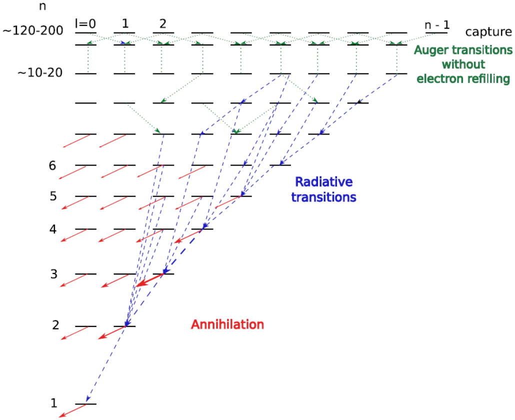

Figure: Scheme of the antiprotonic atom decay[1]

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[1] T. Aumann,*et al.* *The European physical journal A* **58** (2022) 88