INSTITUT D'ASTROPHYSIQUE DE PARIS
Unité mixte de recherche 7095 · CNRS - Sorbonne Université

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Gravitational waves: from theory to discoveries

Irina Dvorkin

Institut d'Astrophysique de Paris

Sorbonne Université

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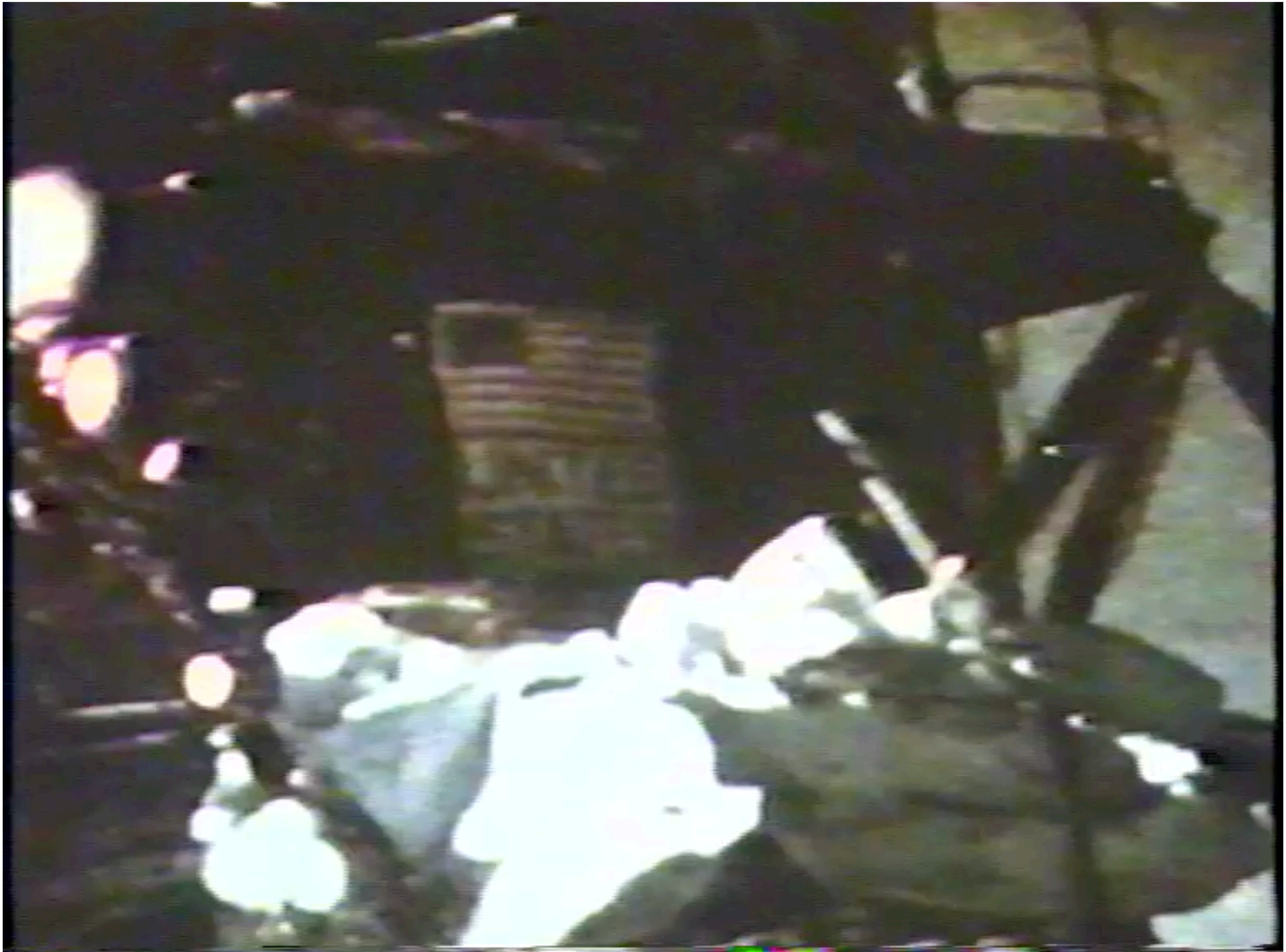
Outline

- GW theory
  - **Introduction to General Relativity, Einstein equation**
  - Black holes
  - Gravitational waves
- GW sources
  - Formation of stellar-mass compact binaries
  - LIGO/Virgo observations and binary black hole populations
  - Formation of massive compact binaries
  - Other transient sources
  - Continuous sources and stochastic backgrounds



Appolo 15, 1971

The equivalence principle

$$\sum \overrightarrow{F} = m_{iner} \overrightarrow{a}$$

$$\overrightarrow{F}_g = m_{grav} \overrightarrow{g}$$

Weak equivalence principle (universality of free fall):

- *Inertial* and *gravitational* masses are in identical ratio for all bodies

-> The trajectory of a point mass in a gravitational field depends only on its initial position and velocity, and is independent of its composition and structure

Strong equivalence principle:

- The outcome of any local experiment (gravitational or not) in a *freely falling laboratory* is independent of the velocity of the laboratory and its location in spacetime.

The metric in flat spacetime

Minkowski (flat) space-time

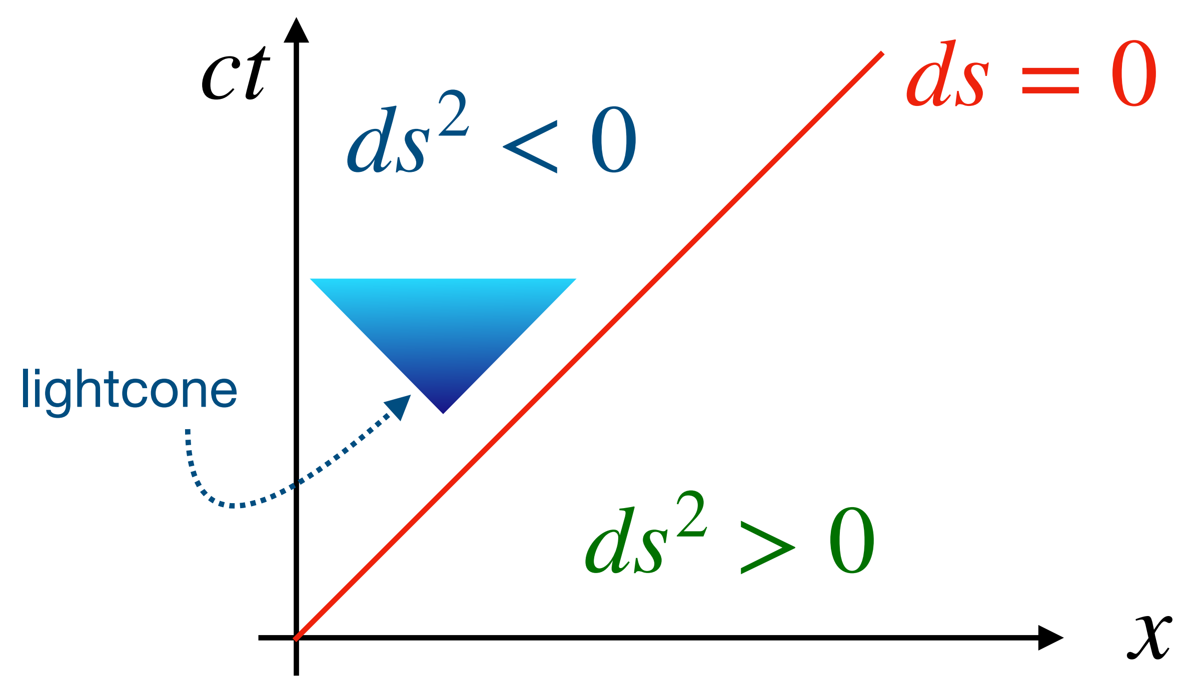
$$ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2$$

Signature $(-,+,+,+)$

conventions may vary!

Metric tensor $\eta_{\alpha\beta}$ so that: $ds^2 = \eta_{\alpha\beta}dx^\alpha dx^\beta$ with $\alpha = 0,1,2,3$

$$\eta_{\alpha\beta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$



The metric: change of coordinates

Coordinate transformation from x^μ (in which the metric is flat) to y^α using $x^\alpha(y^\mu)$

$$dx^\alpha = \frac{\partial x^\alpha}{\partial y^\mu} dy^\mu$$

Minkowski metric $ds^2 = \eta_{\alpha\beta} dx^\alpha dx^\beta$

Curved metric $$ds^2 = g_{\alpha\beta} dy^\alpha dy^\beta$$ with $$g_{\alpha\beta} = \frac{\partial x^\mu}{\partial y^\alpha} \frac{\partial x^\nu}{\partial y^\beta} \eta_{\mu\nu}$$

The metric is symmetric and non-singular

Examples (space only):

Flat space in curved coordinates: $$ds^2 = dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2 \right)$$

Curved space: $$ds^2 = dr^2 + \sin^2 r \left(d\theta^2 + \sin^2\theta d\phi^2 \right)$$

Vectors, tensors and the dual space

Metric $g_{\alpha\beta}$

Inverse metric $g^{\alpha\beta}$ so as: $g^{\alpha\beta} g_{\alpha\beta} = \delta_{\alpha}^{\beta}$

Vector transformation under coordinate change

$$V^{\alpha}(y) = \frac{\partial y^{\alpha}}{\partial x^{\mu}} V^{\mu}(x)$$

Vector in the dual space transformation

$$V_{\alpha}(y) = \frac{\partial x^{\mu}}{\partial y^{\alpha}} V_{\mu}(x)$$

Tensor transformation

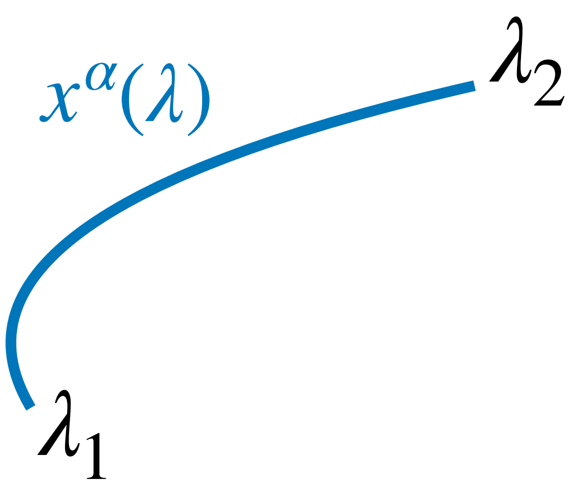
$$T_{\alpha}^{\ \beta\gamma}(y) = \frac{\partial x^{\mu}}{\partial y^{\alpha}} \frac{\partial y^{\beta}}{\partial x^{\nu}} \frac{\partial y^{\gamma}}{\partial x^{\sigma}} T_{\mu}^{\ \nu\sigma}(x)$$

Index raising and lowering using the metric tensor

$$V_{\alpha} = g_{\alpha\mu} V^{\mu}$$

The geodesic equation

Find the "shortest" path in spacetime $$d\ell = \sqrt{g_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta} d\lambda$$



$$\boxed{\ddot{x}^\mu + \Gamma^\mu{}_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta = 0}$$

n (=number of dimensions) 2nd order non-linear diff. eqs.

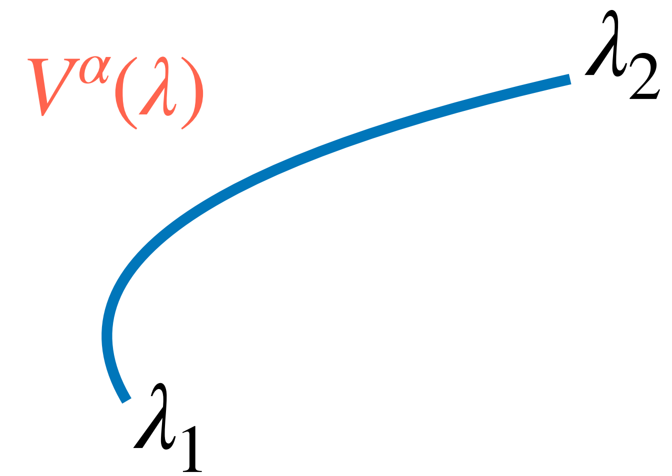
Christoffel symbols: $$\boxed{\Gamma^\mu{}_{\alpha\beta} = \frac{1}{2} g^{\mu\kappa}\left(g_{\alpha\kappa,\beta} + g_{\beta\kappa,\alpha} - g_{\alpha\beta,\kappa}\right)}$$

Christoffel symbols depend on the metric and its first derivatives $$g_{\alpha\beta,\gamma} = \frac{\partial g_{\alpha\beta}}{\partial x^\gamma}$$

In flat space-time all metric derivatives vanish: $$\Gamma^\mu{}_{\alpha\beta} = 0 \;\rightarrow\; \ddot{x}^\mu = 0$$

Covariant derivative

Vector (or tensor) derivatives along the curve



$$\frac{DV^\mu}{D\lambda} = \frac{dV^\mu}{d\lambda} + \Gamma^\mu{}_{\alpha\beta} V^\alpha \frac{dx^\beta}{d\lambda}$$

For example: the affine parameter is the proper time of the particle τ

The acceleration:

$$\frac{Du^\mu}{D\lambda} = \frac{du^\mu}{d\tau} + \Gamma^\mu{}_{\alpha\beta} u^\alpha \frac{dx^\beta}{d\tau} = \frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu{}_{\alpha\beta} \frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = \frac{f^\mu}{m}$$

If not external forces are present the particle moves along a geodesic

The equivalence principle and local flatness

Strong equivalence principle:

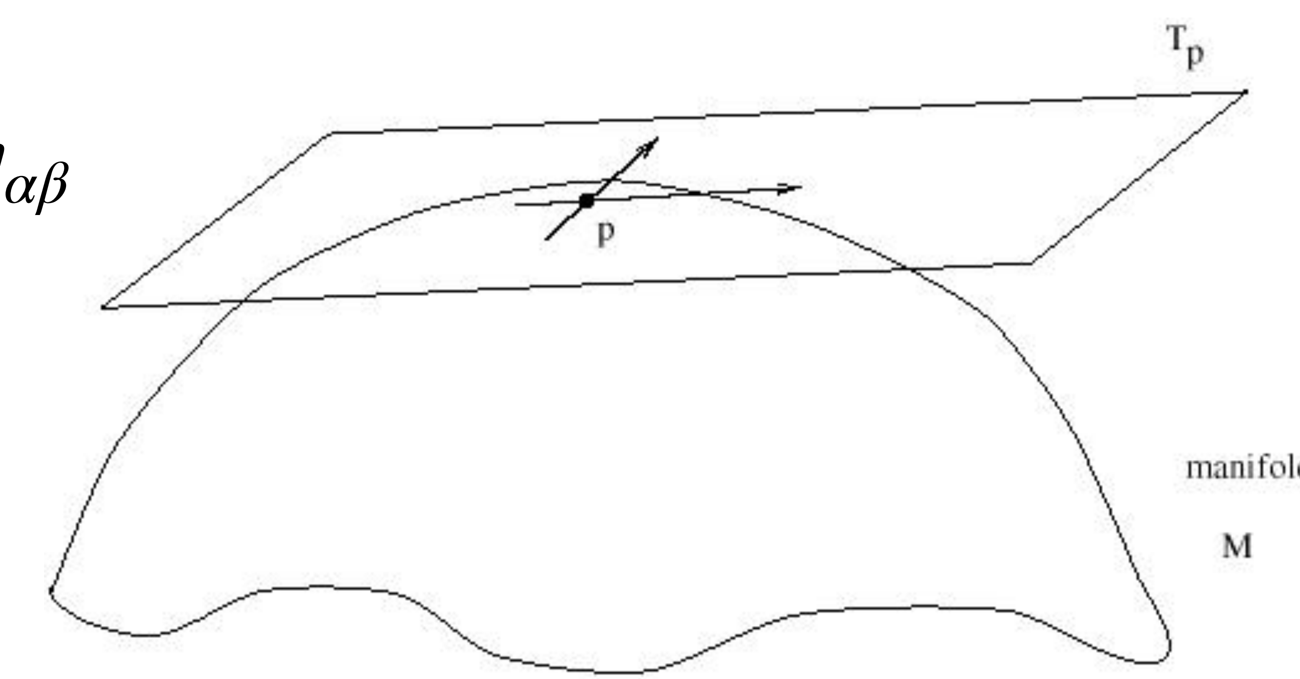
- The outcome of any local experiment (gravitational or not) in a *freely falling laboratory* is independent of the velocity of the laboratory and its location in spacetime.

Assume a general (curved) space-time, described by the metric $g_{\alpha\beta}(x)$

At any point P we can find coordinates y such that the metric is locally flat in these coordinates, and its first derivatives vanish locally

$$g'_{\alpha\beta}(y=P) = g_{\mu\nu}\frac{\partial x^\mu}{\partial y^\alpha}\frac{\partial x^\nu}{\partial y^\beta}(y=P) = \eta_{\alpha\beta}$$

$$\frac{\partial g'_{\alpha\beta}}{\partial y^\gamma}(y=P) = 0$$



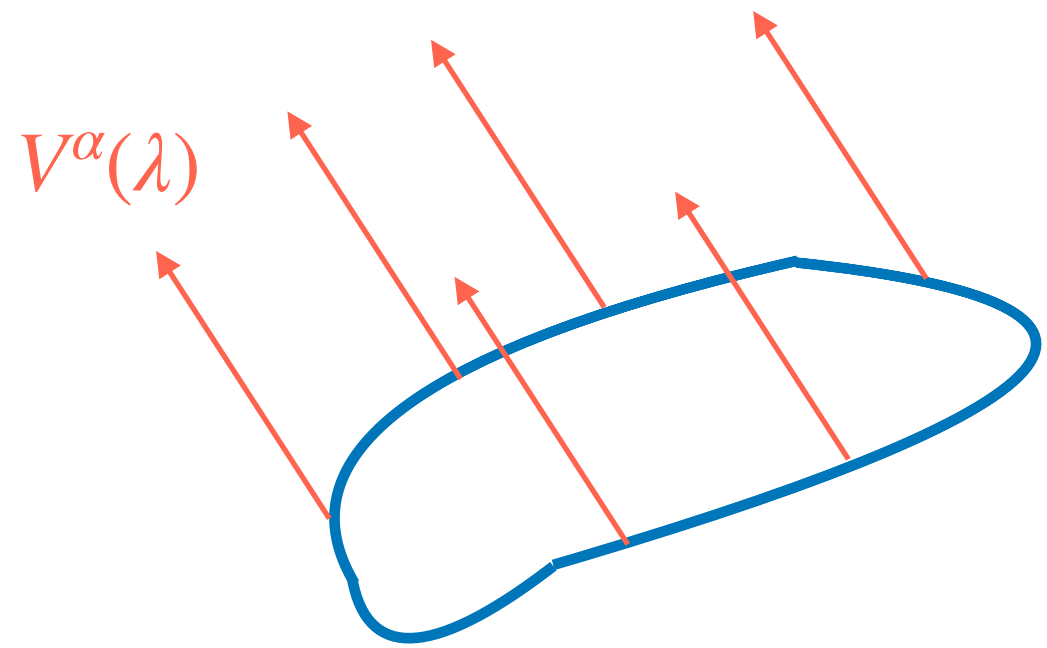
Carroll, gr-qc/9712019

The equivalence principle and local flatness (contd.)

How to express any physical law in a curved spacetime:

1. Make a coordinate transformation to a locally flat (free falling) system
2. Write down your law in the locally flat system (use covariant form)
3. Transform back to the original coordinate system

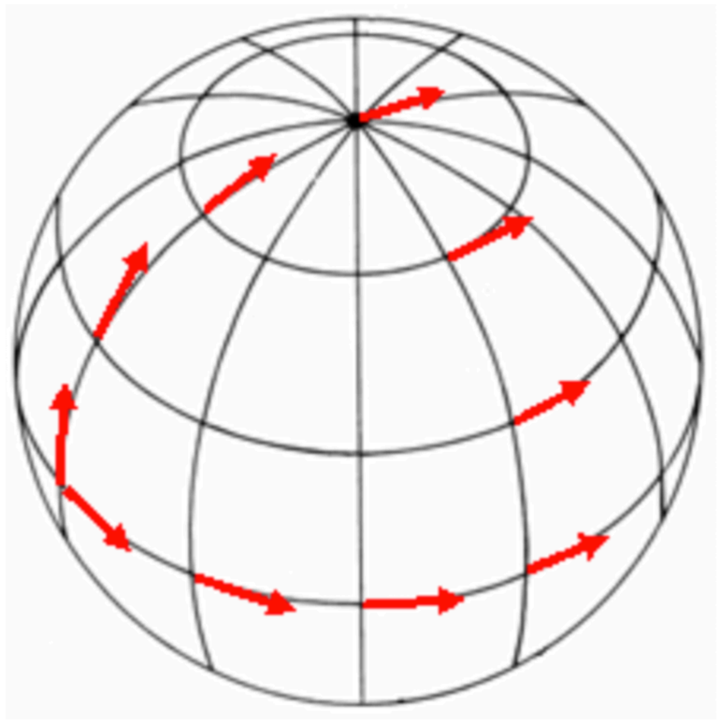
Parallel transport along a curve



Does a vector, translated parallel to itself along a curve, returns to itself?

$$\Delta V^\alpha = R^\alpha{}_{\beta\mu\nu} V^\mu dx^\mu dx^\nu$$

Riemann tensor (dimensions = 1/length^2)



$$R^\alpha{}_{\beta\mu\nu} = \Gamma^\alpha{}_{\beta\nu,\mu} - \Gamma^\alpha{}_{\beta\mu,\nu} + \Gamma^\alpha{}_{\sigma\mu}\Gamma^\sigma{}_{\beta\nu} - \Gamma^\alpha{}_{\sigma\nu}\Gamma^\sigma{}_{\beta\mu}$$

conventions may vary!

Riemann tensor vanishes iff the spacetime is flat

Riemann tensor = combinations of second derivatives of the metric

Geodesic deviation

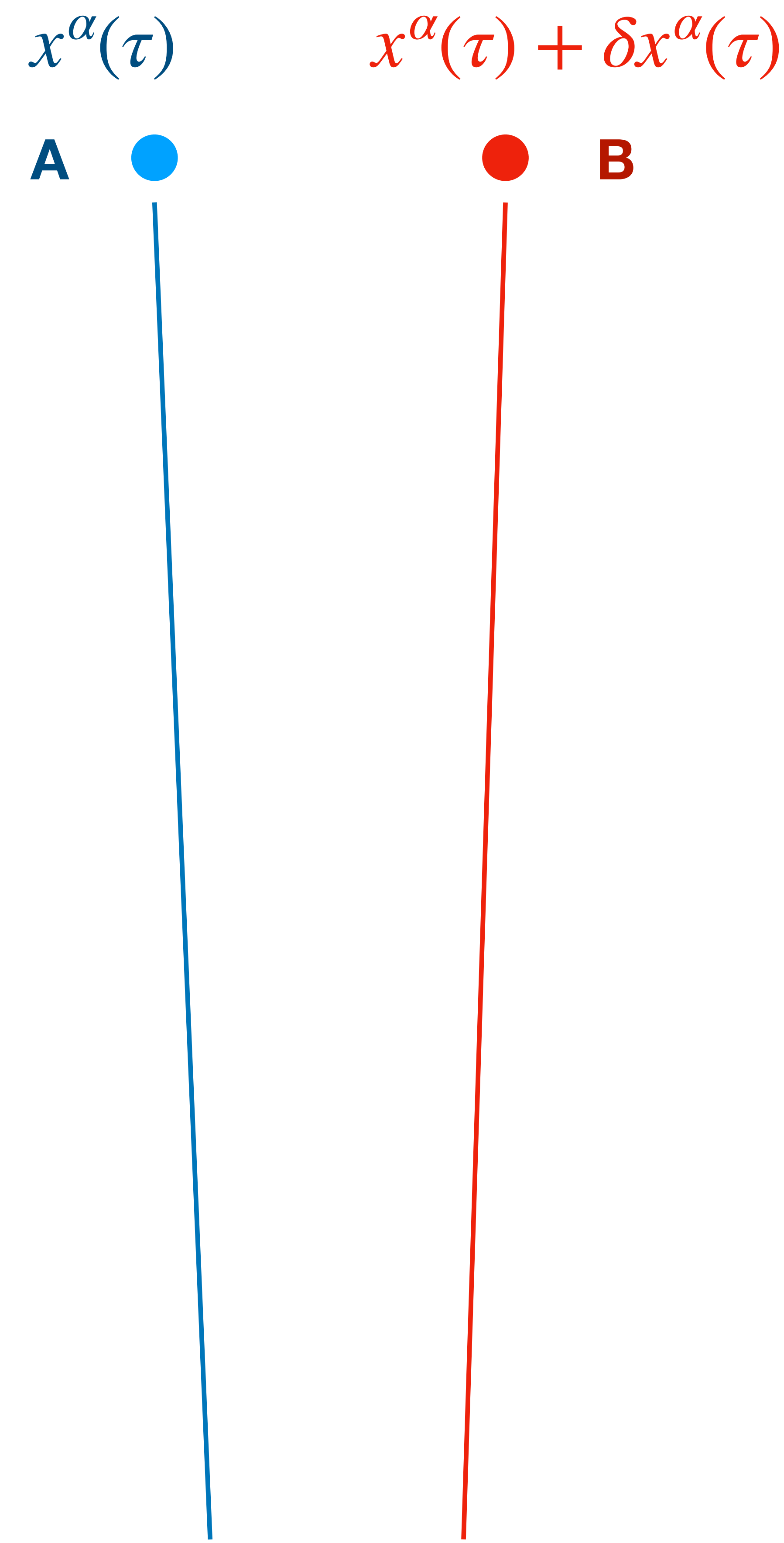
A and B follow two (slightly different) geodesics

$$\ddot{x}^\alpha = -\,\Gamma^\alpha_{\mu\nu}(x)\dot{x}^\mu \dot{x}^\nu$$

$$\ddot{x}^\alpha + \delta\ddot{x}^\alpha = -\,\Gamma^\alpha_{\mu\nu}(x+\delta x)(\dot{x}^\mu + \delta\dot{x}^\mu)(\dot{x}^\nu + \delta\dot{x}^\nu)$$

$$\frac{D^2 \delta x^\alpha}{D\tau^2} = R^\alpha{}_{\mu\nu\gamma}\,\dot{x}^\mu\,\dot{x}^\nu\,\delta x^\gamma$$

The world lines are not parallel in a curved spacetime: non-local effect
Tidal forces: the trajectories of neighboring particles diverge



Curvature, Ricci tensor and Ricci scalar

Ricci tensor $R_{\beta\gamma} = R^{\alpha}{}_{\beta\alpha\gamma}$ (is symmetric) **conventions may vary!**

Ricci scalar $R = R^{\alpha}{}_{\alpha}$ Example: *R*=2/*r*^2 on a 2-sphere of radius *r*

Riemann tensor is highly symmetric. In n dimensions it has $\frac{n^2(n^2-1)}{12}$ independent components.

n=1: Riemann tensor has no independent components. **No curvature!**

n=2: Riemann tensor has **1** independent component, determined by the **Ricci scalar**

n=3: Riemann tensor has **6** independent components, **same as Ricci tensor**

n=4: Riemann tensor has **20** independent components, more than Ricci tensor

Bianchi identities $R_{\alpha\beta\mu\nu;\lambda} + R_{\alpha\beta\lambda\mu;\nu} + R_{\alpha\beta\nu\lambda;\mu} = 0$

Stress-energy tensor

4-velocity $$u^\alpha = \frac{dx^\alpha}{c\,d\tau}$$

$$T^{\alpha\beta} = (\rho c^2 + P)u^\alpha u^\beta + P g^{\alpha\beta}$$

equation of state $$P = P(\rho)$$

symmetry $$T^{\alpha\beta} = T^{\beta\alpha}$$

energy-momentum conservation $$\frac{DT^{\alpha\beta}}{Dx^\beta} = 0$$

Perfect fluid, comoving reference frame

Energy density

Energy flux

$$T^{\alpha\beta} = \begin{pmatrix} \rho c^2 & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix}$$

Momentum density

Momentum flux

Einstein equation

In Newtonian theory: $\vec{g} = -\vec{\nabla}\phi$ where $\nabla^2\phi = 4\pi G\rho$

$\phi \to g_{\alpha\beta}$

In curved spacetime: $\rho \to T_{\alpha\beta}$ $Op(g_{\alpha\beta}) = T_{\alpha\beta}$

$\nabla^2 \to Op$ up to 2nd order derivatives

Matter tells spacetime how to curve, and spacetime tells matter how to move

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G}{c^4}T_{\mu\nu}$$

Einstein equation: gauge freedom

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu}$$

10 equations for 10 components of the metric?

Einstein equations covariant under coordinate transformation (4 equations)

10 equations = 6 evolution equations + 4 constraints from Bianchi identities

= 6 metric components + 4 gauge equations

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Schwarzschild solution

Assumptions: spherical symmetry, vacuum outside of the source

$$ds^2 = -\left(1 - \frac{2GM}{rc^2}\right) dt^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2 + r^2 d\Omega^2$$

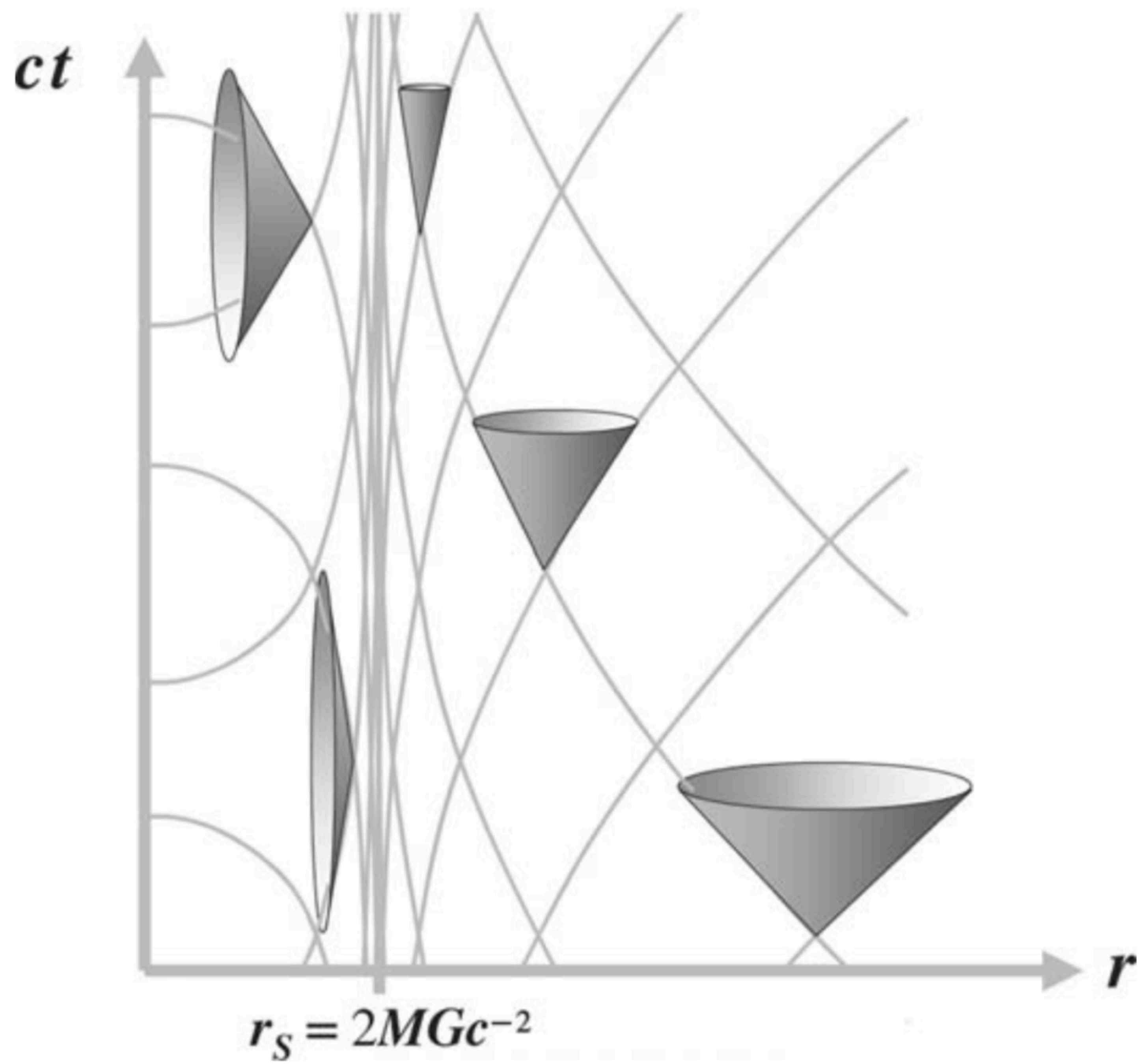
Parameter *M* to ensure weak field limit: $g_{00} = -1 - 2\frac{\phi_N}{c^2}$ with $\phi_N = -\frac{GM}{r}$

$r \to 0$ True singularity $R_{\alpha\beta\gamma\delta} \propto \frac{M}{r^3}$

$r = \frac{2GM}{c^2} \equiv R_S$ Coordinate singularity

Schwarzschild solution

$$ds^2 = -\left(1 - \frac{2GM}{rc^2}\right) dt^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2 + r^2 d\Omega^2$$



From: Raphael Ferraro, Introduction to Special and General Relativity

Tolman-Oppenheimer-Volkoff equations

Einstein equations inside a star with mass profile m and pressure profile P

$$ds^2 = -\, e^{2\Phi} dt^2 + e^{2\Lambda} dr^2 + r^2 d\Omega^2$$

with the parametrization

$$e^{2\Lambda} = \frac{1}{1 - 2m(r)/r}$$

Einstein equations become:

$$\frac{dm(r)c^2}{dr} = 4\pi r^2 e$$

$$\frac{dP}{dr} = -\, \frac{(e+P)(mc^2 + 4\pi r^3 P)}{r(r - 2Gm(r)/c^2)}$$

equation of state $P = P(\rho)$

Tolman-Oppenheimer-Volkoff equations: uniform density

Uniform density, pressure vanishes at radius R $\quad P(R) = 0$

$$P(r) = e_0 \left(\frac{1 - \left(1 - \frac{2GMr^2}{R^3c^2}\right)^{1/2} - \left(1 - \frac{2GM}{Rc^2}\right)^{1/2}}{3\left(1 - \frac{2GM}{Rc^2}\right)^{1/2} - \left(1 - \frac{2GMr^2}{R^3c^2}\right)^{1/2}} \right)$$

$$R = \sqrt{\frac{3}{8\pi e_0}\left(1 - \frac{(e_0 + P_c)^2}{(e_0 + 3P_c)^2}\right)}$$

$$P_c = e_0 \left(\frac{1 - \left(1 - \frac{2GM}{Rc^2}\right)^{1/2}}{3\left(1 - \frac{2GM}{Rc^2}\right)^{1/2} - 1} \right)$$

For $\quad GM/Rc^2 \to 4/9$

$$R < \frac{9}{8} R_S$$

Central pressure diverges

$$P_c \to \infty$$

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Linearized theory

Solve Einstein's equations assuming a small perturbation of the flat spacetime

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu}$$

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$ with $$|h_{\mu\nu}| \ll 1$$

Linearized equation in Lorenz gauge

$$\left(-\frac{\partial^2}{c^2 \partial t^2} + \nabla^2 \right) \bar{h}_{\alpha\beta} = -\frac{16\pi G}{c^4} T_{\alpha\beta}$$

Wave equation

Outside the source

$$\left(-\frac{\partial^2}{c^2\partial t^2} + \nabla^2\right)\bar{h}_{\alpha\beta} = 0$$

The perturbation travels with the speed of light

Plane wave solution

$$\boxed{\bar{h}_{\alpha\beta} = A_{\alpha\beta}\exp(ik_\gamma x^\gamma)}$$

Lorenz gauge

$$\bar{h}_{\mu\nu}{}^{,\nu} = 0 \rightarrow A_{\mu\nu}k^\nu = 0$$

orthogonal to propagation vector

Any solution is a superposition of plane waves

How many degrees of freedom?

The metric perturbation can be decomposed into 4 scalars, 2 transverse vectors, and a transverse trace-free tensor

Take wavevector in the z direction

h_{00} is a scalar under spatial rotations

1 d.o.f = scalar

$$h_{\mu\nu} = \begin{pmatrix} h_{00} & h_{01} & h_{02} & h_{03} \\ h_{10} & h_{11} & h_{12} & h_{13} \\ h_{20} & h_{21} & h_{22} & h_{23} \\ h_{30} & h_{31} & h_{32} & h_{33} \end{pmatrix}$$

h_{0i} is a 3-vector

$$\vec{h}_{0i} = \vec{\nabla}\Phi + \vec{\nabla} \times \vec{V}$$

3 d.o.f = divergence + transverse vector

h_{ij} contains trace + scalar $\partial^i \partial^j h_{ij}$ + transverse vector + traceless transverse tensor

6 d.o.f = divergence + trace + transverse vector + TT tensor

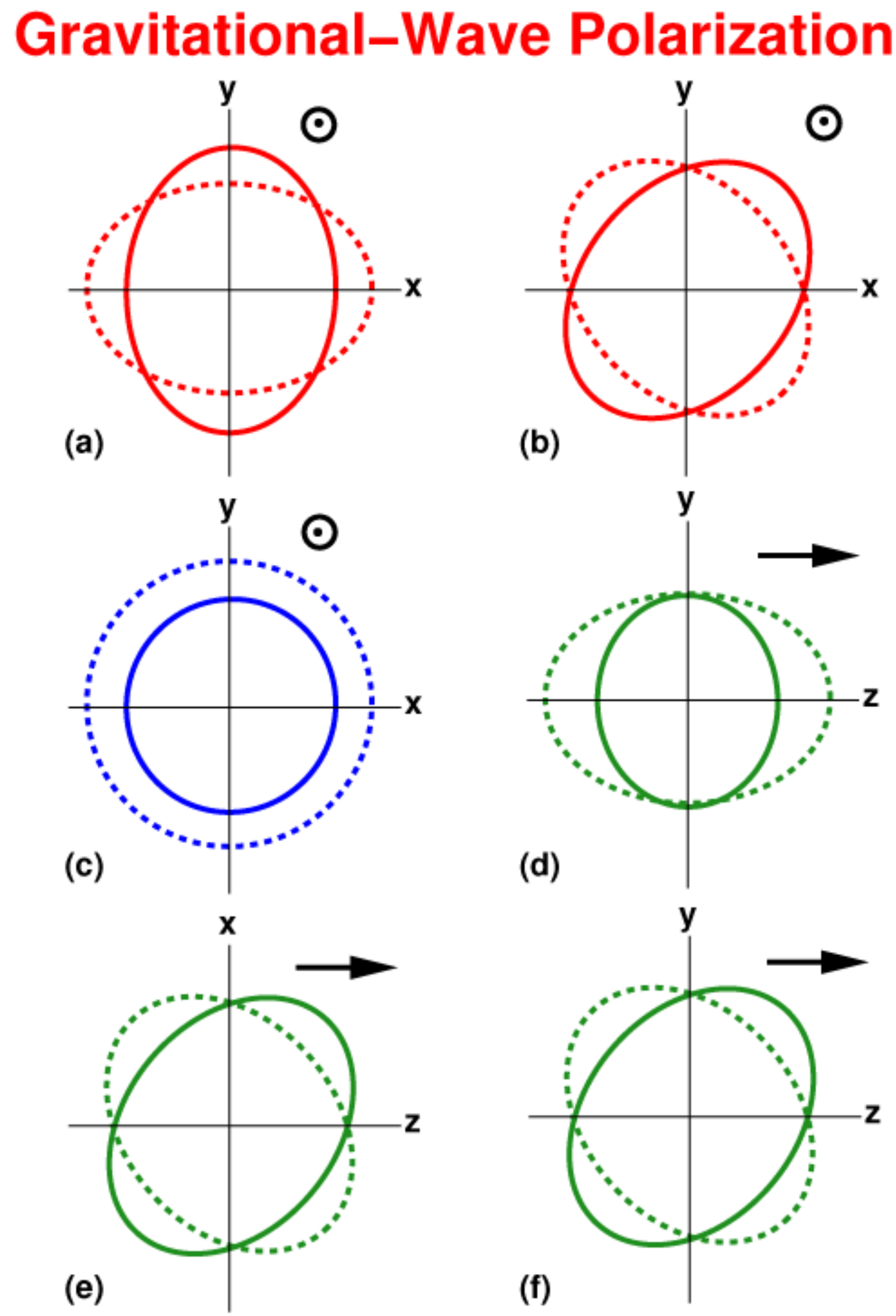
Polarizations (most general case!)

The metric perturbation can be decomposed into 4 scalars, 2 transverse vectors, and a transverse trace-free tensor

But only 2 scalar, 1 transverse vector and the TT tensor are invariant to coordinate transformations -> 6 d.o.f.

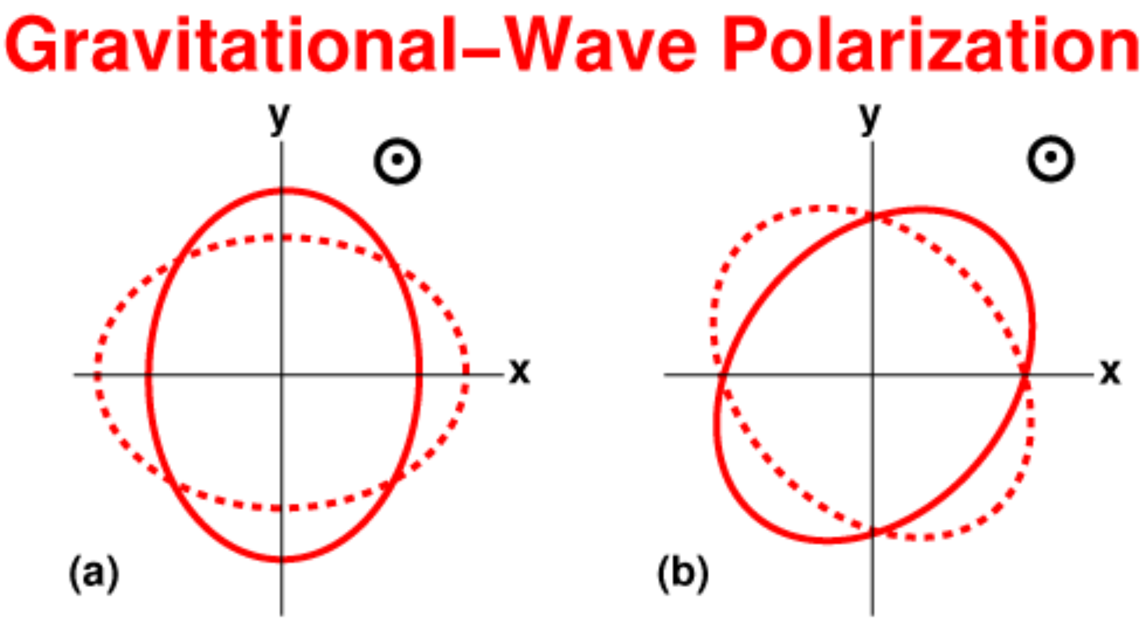
$$h_+, h_\times, h_b, h_l, h_x, h_y$$

$$h_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_b + h_+ & h_\times & h_x \\ 0 & h_\times & h_b - h_+ & h_y \\ 0 & h_x & h_y & h_l \end{pmatrix}$$



C. Will, in Living Reviews in Relativity

Polarizations in GR



In GR, out of the 6 remaining Einstein equations, 4 are constraint equations (no second-order time derivatives)

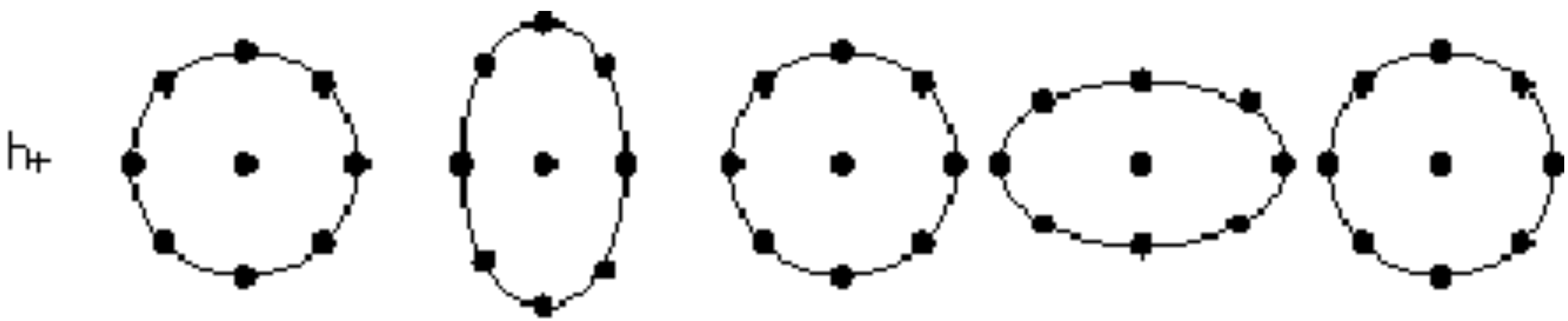
Only 2 equations are evolution equations -> 2 d.o.f. $h_+, h_\times$

$$h_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Plus polarization

$$h_{\mu\nu}(t - z/c) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & 0 & 0 \\ 0 & 0 & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cdot \cos\left(\omega(t - z/c)\right)$$

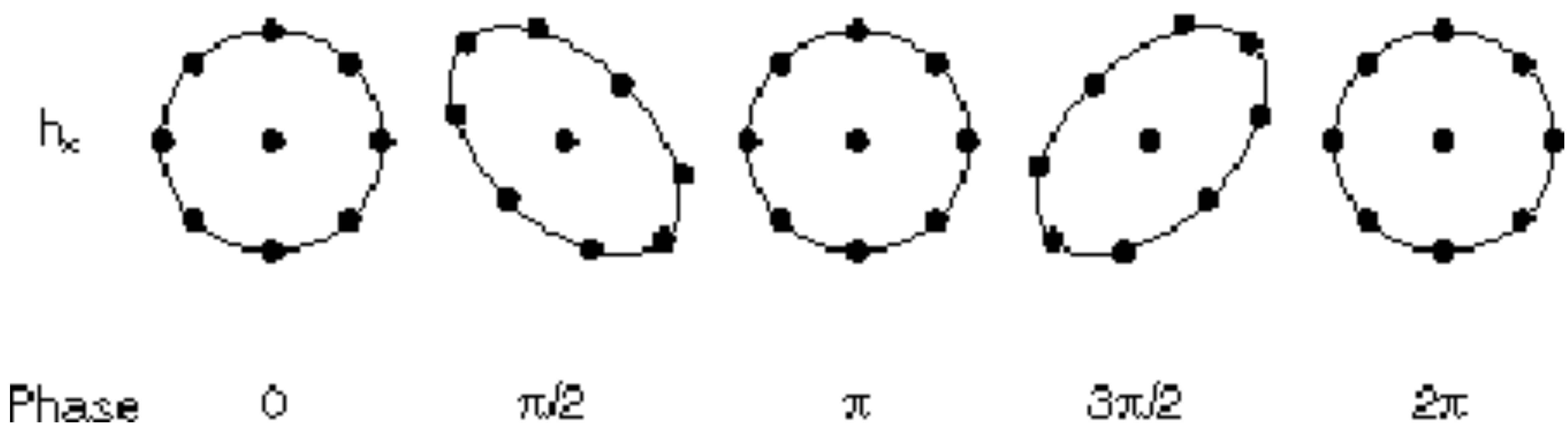
$$ds^2 = -\,c^2 dt^2 + dz^2 + \left(1 + h_+ \cos[\omega(t - z/c)]\right) dx^2 + \left(1 - h_+ \cos[\omega(t - z/c)]\right) dy^2$$

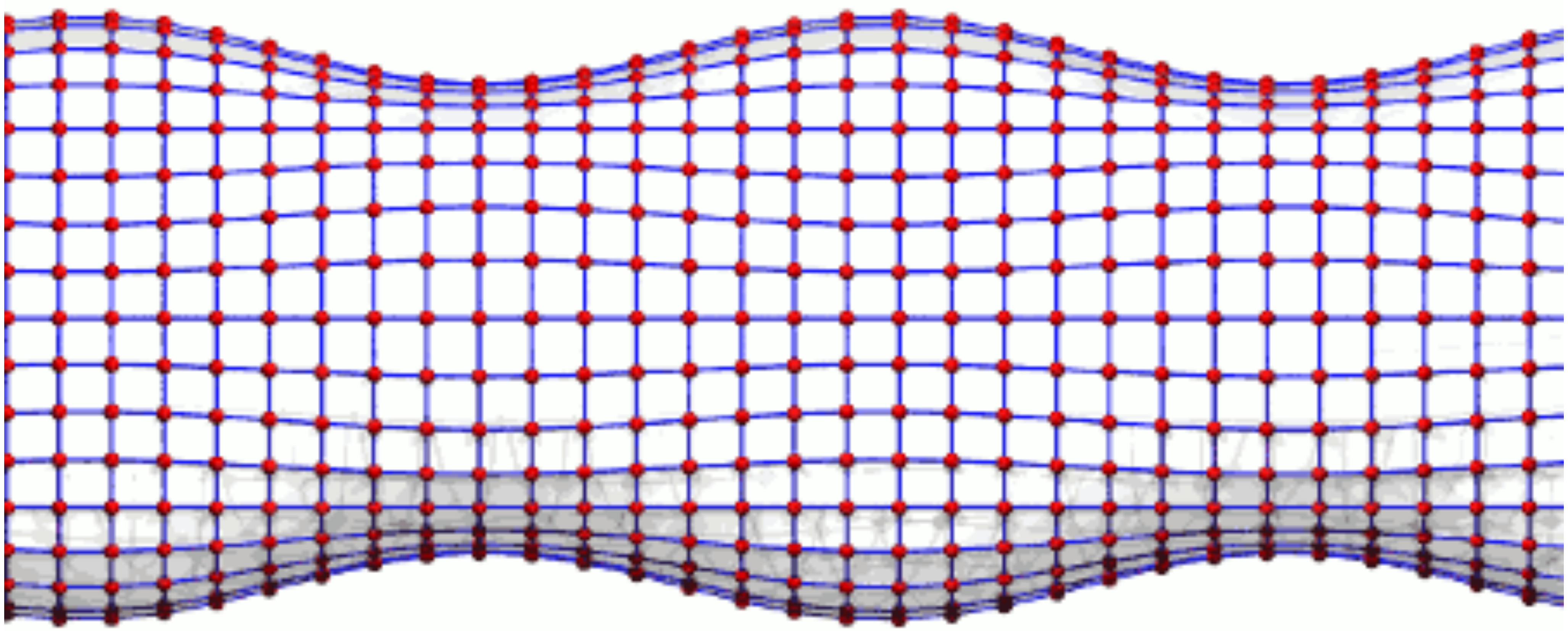


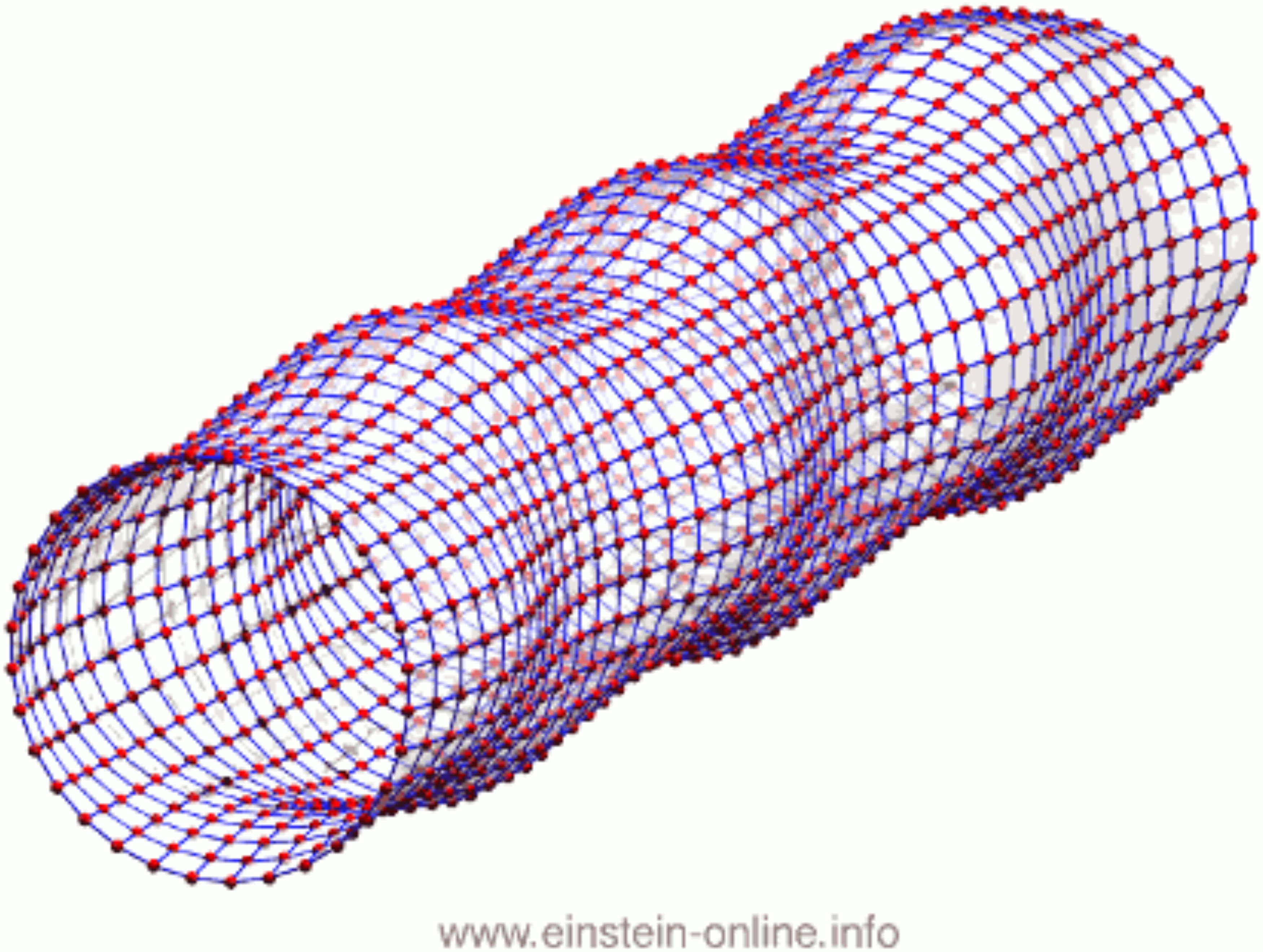
Cross polarization

$$h_{\mu\nu}(t - z/c) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & h_\times & 0 \\ 0 & h_\times & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cdot \cos\left(\omega(t - z/c)\right)$$

$$ds^2 = -c^2dt^2 + dz^2 + dx^2 + dy^2 + 2\left(1 + h_\times \cos[\omega(t - z/c)]\right) dxdy$$



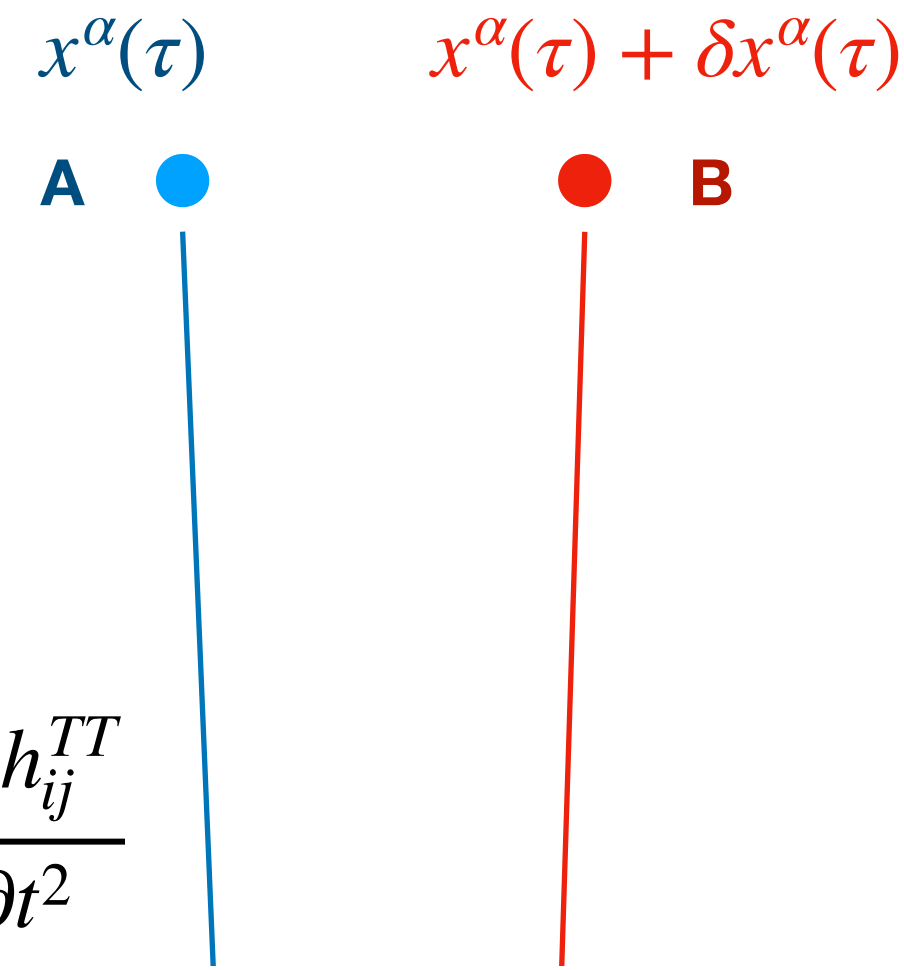



www.einstein-online.info

GW effect on test masses

Geodesic deviation:

$$\frac{D^2\delta x^\alpha}{D\tau^2} = R^\alpha{}_{\mu\nu\gamma}\,\dot{x}^\mu\,\dot{x}^\nu\,\delta x^\gamma$$



Take the curve parameters as the time coordinate:

$$\frac{D^2\delta x^i}{Dt^2} = R^i{}_{00\gamma}\,\delta x^\gamma \qquad \text{with} \qquad R^i{}_{00\gamma} = \frac{1}{2}\frac{\partial^2 h_{ij}^{TT}}{\partial t^2}$$

To linear order:

$$\boxed{\delta x^i(t) = \delta x^i(0) + \frac{1}{2}\left(h_{ij}^{TT}(t) - h_{ij}^{TT}(0)\right)\delta x^j(0)}$$

$$\boxed{\frac{\Delta x}{x} \simeq h}$$

Sources of GW

$$\left(-\frac{\partial^2}{c^2\partial t^2}+\nabla^2\right)\bar{h}_{\alpha\beta}=-\frac{16\pi G}{c^4}T_{\alpha\beta}$$

Matter field is characterized by its multipole moments:

$M=\int \rho d^3x$ Mass monopole $h \propto \frac{GM}{rc^2}$ Mass conservation

$D^i=\int \rho x^i d^3x$ Mass dipole $h \propto \frac{G\dot{D}}{rc^3}$ Linear momentum conservation

$Q^{ij}=\int \rho x^i x^j d^3x$ Mass quadrupole $h \propto \frac{G\ddot{Q}}{rc^4}$

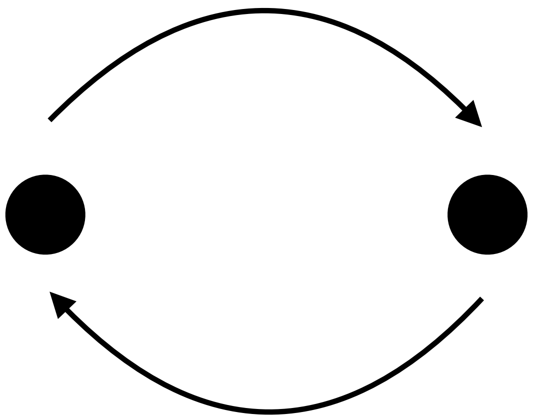
Angular momentum conservation

$L^i=\int \rho e^i_{jk}x^j v^k d^3x$ Angular momentum (first moment of mass current) $h \propto \frac{G\dot{L}}{rc^4}$

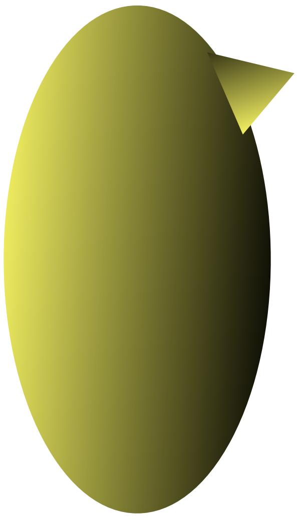
The quadrupole formula (TT gauge)

$$\bar{h}_{ij}^{TT}(t, \vec{x}) \simeq \frac{2G}{c^4 r} \ddot{Q}^{TT}(t - r/c)$$

Compact binaries (binary black holes, neutron stars…)



Non-spherical rotating stars

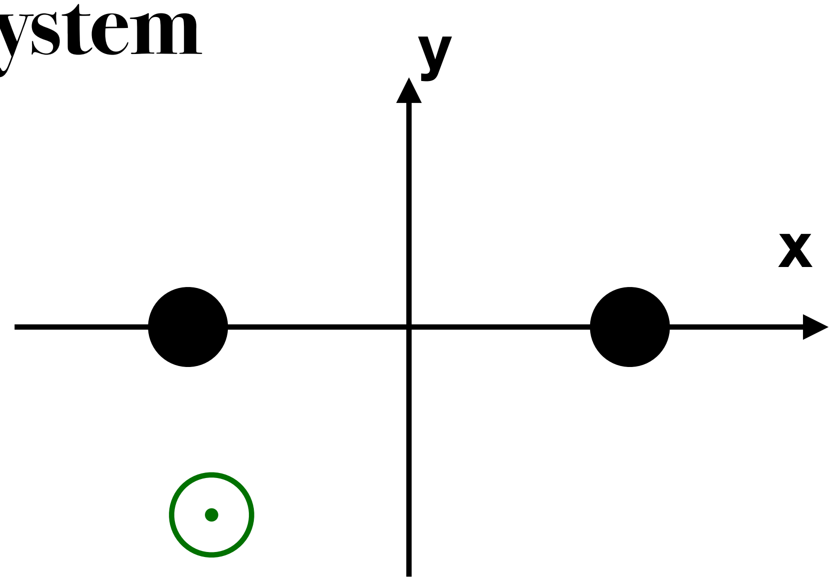


Stellar explosions

The quadrupole formula: binary system

Equal-mass circular binary in the x-y plane,

orbital frequency ω and initial separation a_0



$$Q_{xx} = \frac{1}{4} M a_0^2 \cos(2\omega t)$$

$$Q_{yy} = -\frac{1}{4} M a_0^2 \cos(2\omega t)$$

$$Q_{xy} = \frac{1}{4} M a_0^2 \sin(2\omega t)$$

$$x_1 = \frac{a_0}{2} \cos(\omega t)$$

$$y_1 = \frac{a_0}{2} \sin(\omega t)$$

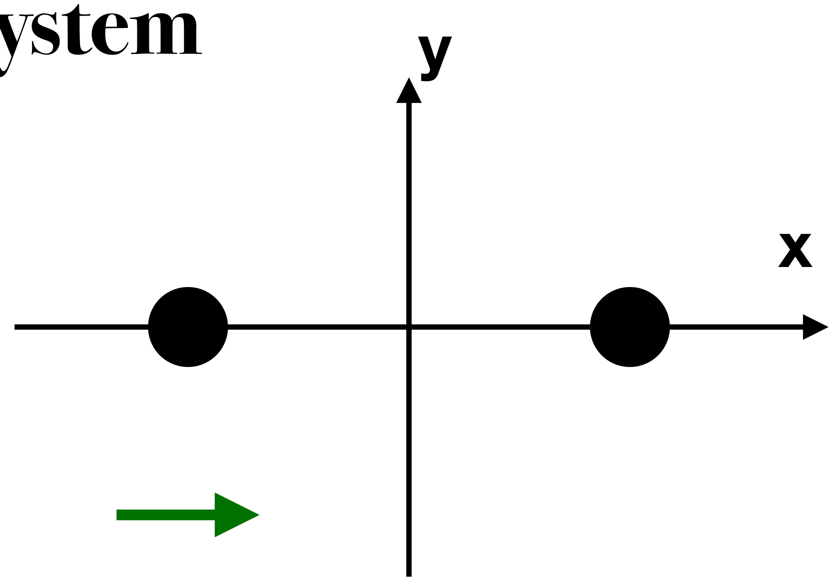
Radiation in the z direction

$$\bar{h}_{xx}^{TT} = -\bar{h}_{yy}^{TT} = -\frac{2GMa_0^2\omega^2}{c^2 r} \cos(2\omega(t - r/c))$$

$$\bar{h}_{xy}^{TT} = \frac{2GMa_0^2\omega^2}{c^2 r} \sin(2\omega(t - r/c))$$

Circular polarization

The quadrupole formula: binary system



Equal-mass circular binary in the x-y plane,

orbital frequency ω and initial separation a_0

$$Q_{xx} = \frac{1}{4} M a_0^2 \cos(2\omega t)$$

$$x_1 = \frac{a_0}{2}\cos(\omega t)$$

$$Q_{yy} = -\frac{1}{4} M a_0^2 \cos(2\omega t)$$

$$y_1 = \frac{a_0}{2}\sin(\omega t)$$

$$Q_{xy} = \frac{1}{4} M a_0^2 \sin(2\omega t)$$

Radiation in the x direction
$$\bar{h}_{yy}^{TT} = -\bar{h}_{zz}^{TT} = \frac{G M a_0^2 \omega^2}{2c^2 r}\cos(2\omega(t - r/c))$$

Linear polarization aligned with the orbital plane

Binary system: unequal masses

Kepler law:

$$\omega^2 = \frac{G(M_1 + M_2)}{a_0^3}$$

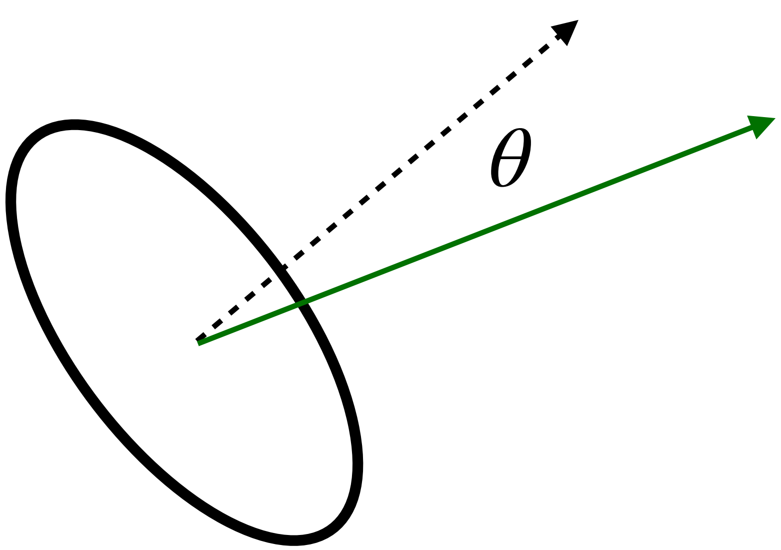
$$h_+ = \frac{4}{r}\left(\frac{GM_c}{c^2}\right)^{5/3}\left(\frac{\pi f_{GW}}{c}\right)^{2/3}\frac{1+\cos^2\theta}{2}$$

$$h_\times = \frac{4}{r}\left(\frac{GM_c}{c^2}\right)^{5/3}\left(\frac{\pi f_{GW}}{c}\right)^{2/3}\cos\theta$$

GW frequency:

$$f_{GW} = \frac{\omega}{\pi}$$

Orbital inclination: θ



Chirp mass: $$M_c = \frac{(M_1 M_2)^{3/5}}{(M_1 + M_2)^{1/5}}$$

GW energy flux and luminosity

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu}$$

Far from the source: the energy is due to GW

Weak field, developing to lowest order

$$t_{\mu\nu}^{TT} = \frac{c^2}{8\pi G} \left(R_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} \eta^{\delta\rho} R_{\lambda\rho} \right)$$

second order in *h*

Flux:

$$\frac{dE}{dtdA} = t_{03}^{TT} = -\frac{c^3}{16\pi G} \left\langle \dot{h}_+^2 + \dot{h}_\times^2 \right\rangle$$

$$\frac{dE}{dtdA} = -\frac{G}{8\pi c^5 r^2} \left\langle \dddot{Q}_{ij} \dddot{Q}^{ij} \right\rangle$$

Luminosity:

$$L_{GW} = -\frac{dE}{dt} = \frac{G}{5c^5} \left\langle \dddot{Q}_{ij} \dddot{Q}^{ij} \right\rangle$$

Luminosity of GW (quadrupole approximation)

$$L_{GW} = -\frac{dE}{dt} = \frac{G}{5c^5}\left\langle \dddot{Q}_{ij}\dddot{Q}^{ij}\right\rangle$$

But $\frac{G}{c^5}$ is extremely small...

Assume a compact object $\frac{GM}{Rc^2} \sim 1$

Mass quadrupoles and its derivatives:

$$Q \sim MR^2 \qquad \ddot{Q} \sim Mv^2 \sim E_{kin} \qquad \dddot{Q} \sim \frac{E_{kin}}{\tau} \sim \frac{E_{kin}}{R/v} \sim \frac{Mv^2}{R/v}$$

$$L_{GW} \sim \frac{G}{c^5}\dddot{Q}^2 \sim \frac{c^5}{G}\left(\frac{GM}{Rc^2}\right)^2\left(\frac{v}{c}\right)^6$$

Thankfully $\frac{c^5}{G}$ is extremely large!

Orbital evolution of a compact binary system

Energy is lost to GW, the orbit shrinks:

$$\frac{da}{dt} = -\frac{64}{5}\frac{G^3\mu m^2}{c^5 a^3}$$

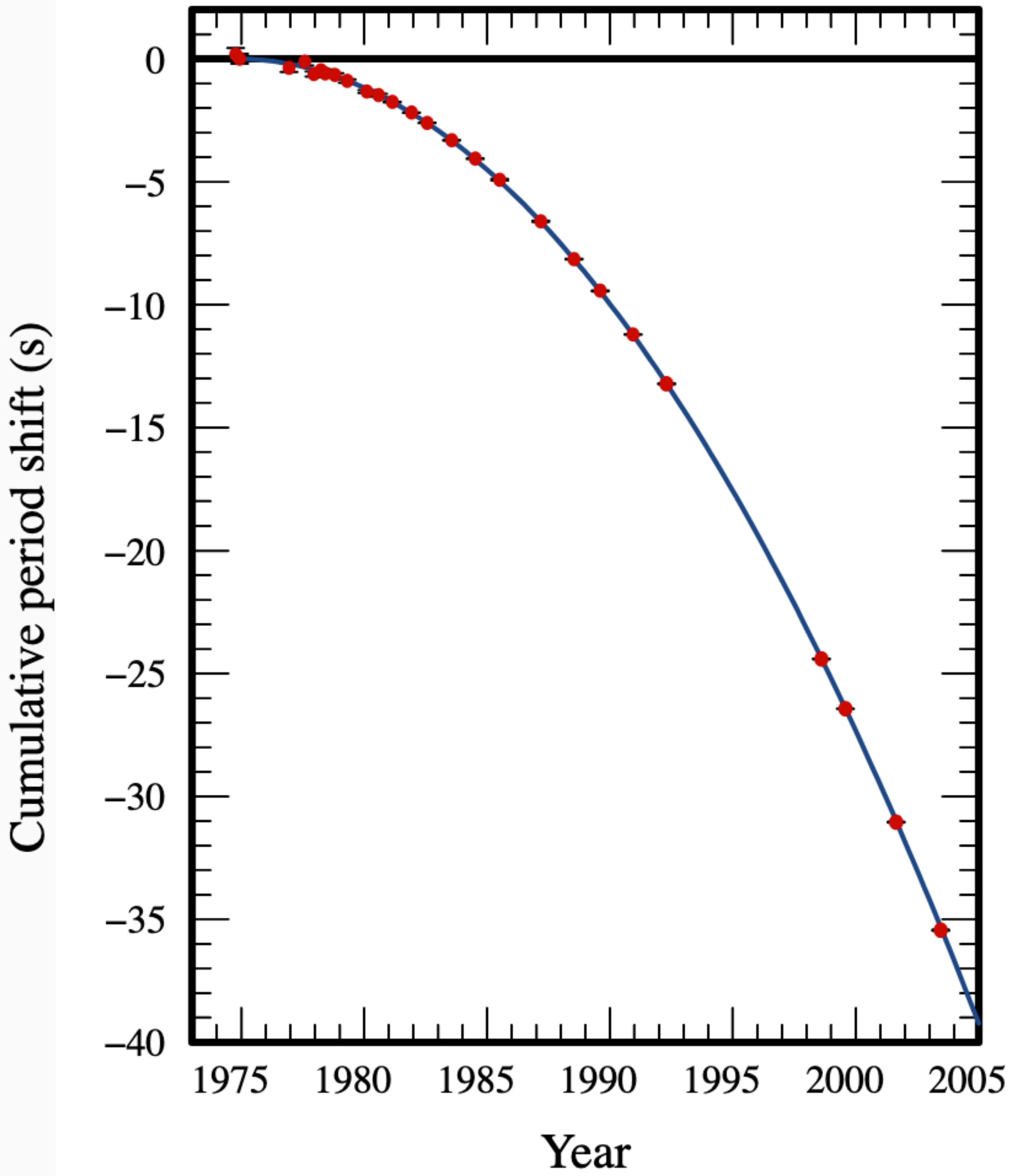
Coalescence time for a circular binary

$$t_{coal} = a_0^4 \cdot \frac{5}{256}\frac{c^5}{G^3}\frac{1}{\mu m^2}$$

Reduced mass $\mu = \frac{M_1 M_2}{M_1 + M_2}$

Total mass $m = M_1 + M_2$

Hulse-Taylor pulsar



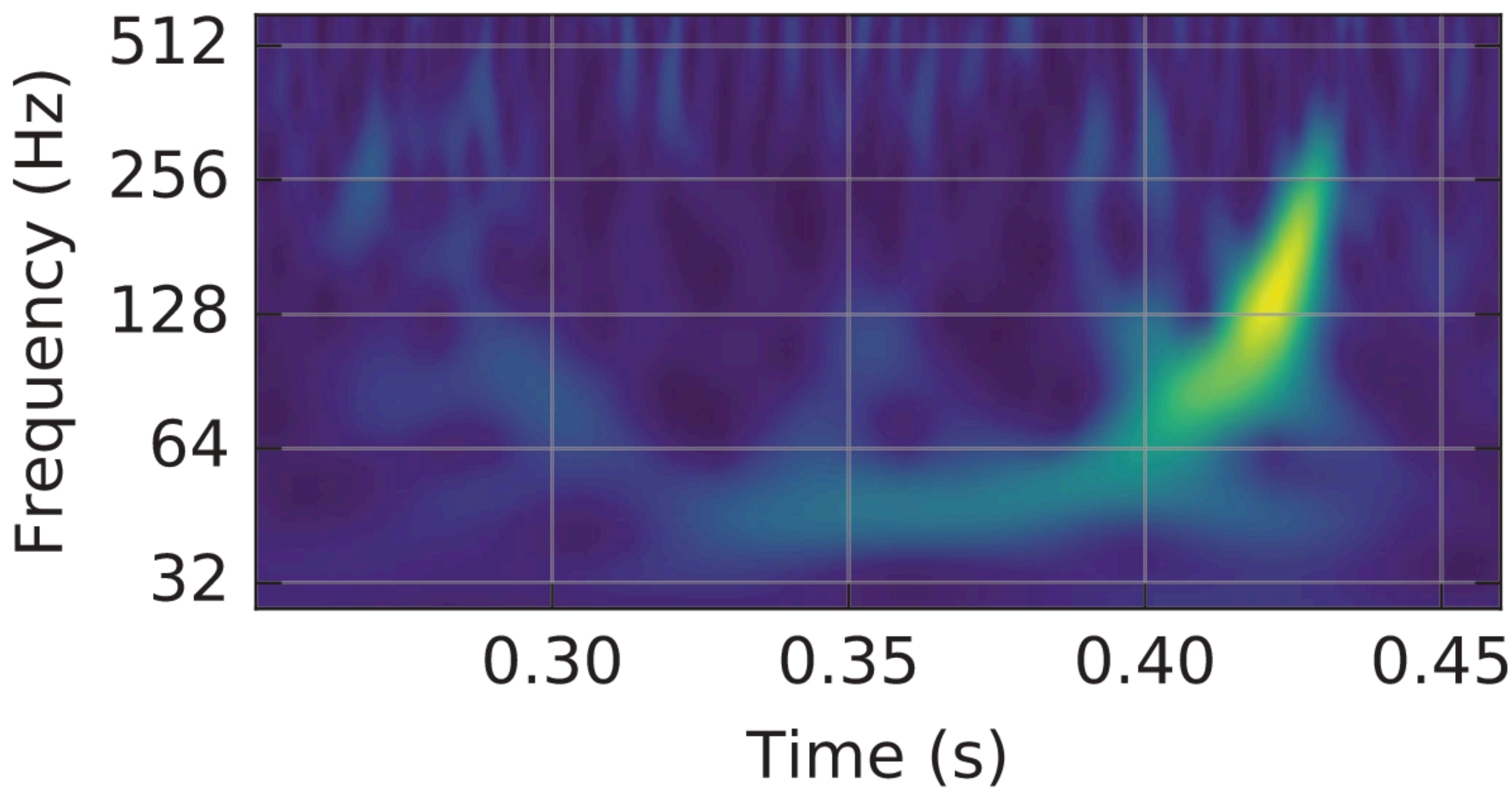
GW150914: frequency chirp

From orbital evolution:

$$f_{GW}(t) = \frac{1}{\pi}\left(\frac{GM_c}{c^3}\right)^{-5/8}\left(\frac{5}{256}\frac{1}{(\tau_{coal} - t)}\right)$$

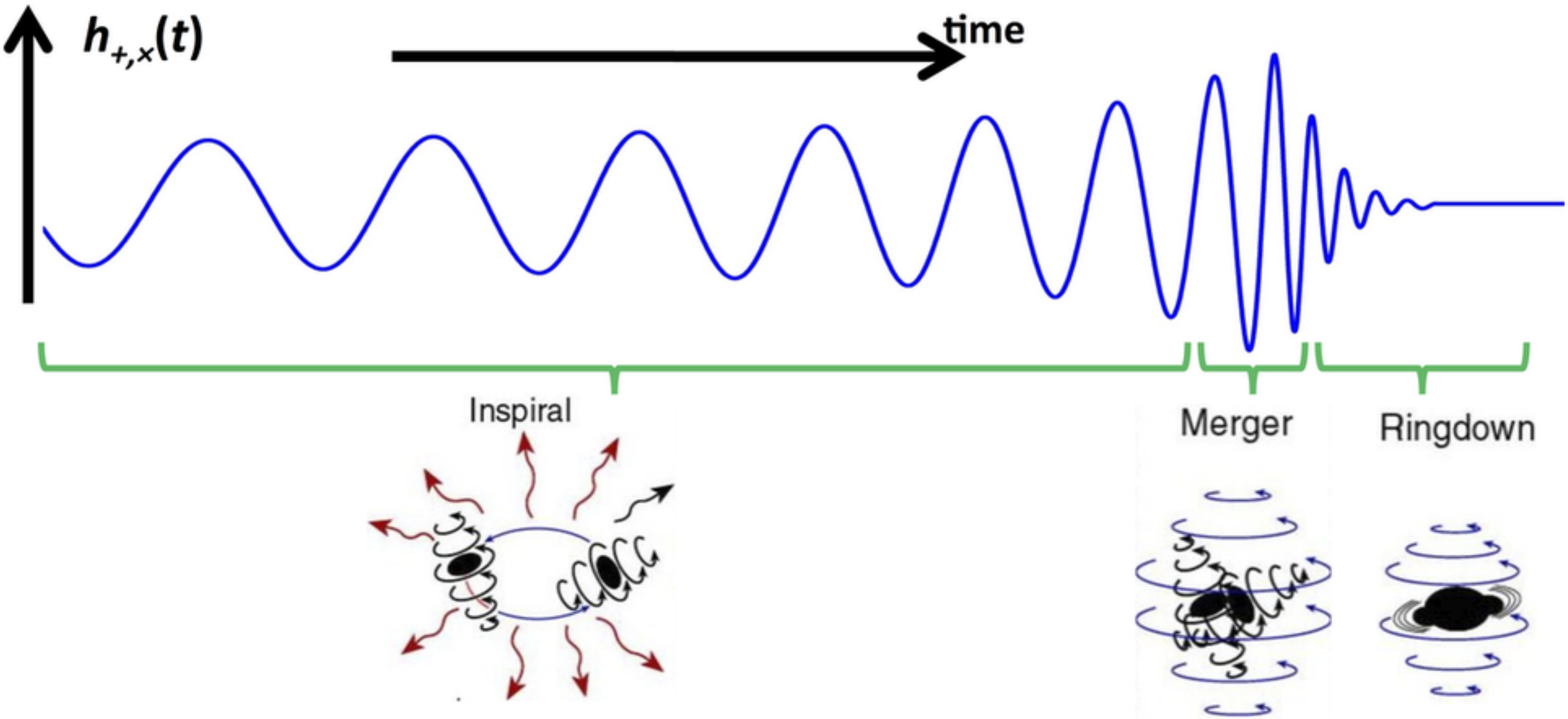
Frequency chirp:

$$\dot{f}_{GW}(t) = \frac{96}{5}\pi^{8/3}\left(\frac{GM_c}{c^3}\right)^{5/3} f_{GW}^{11/3}$$



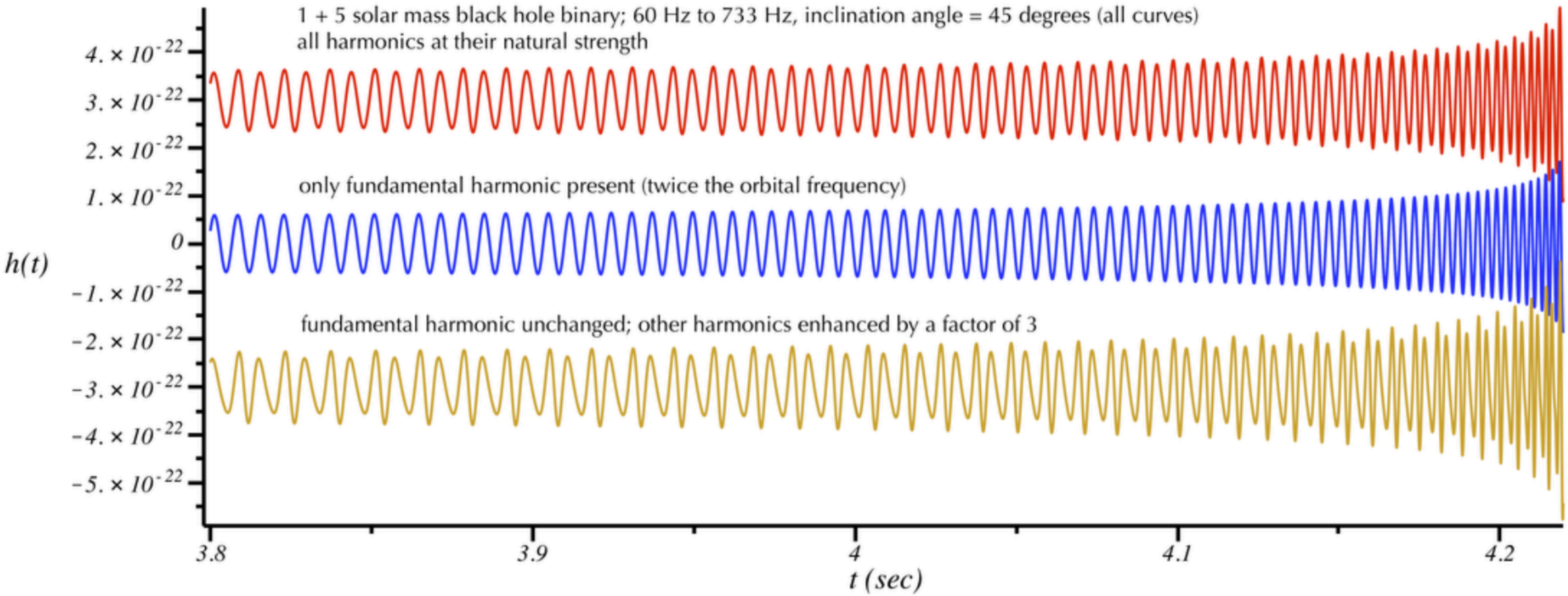
Abbott et al. (LVC) 2016

Compact binary merger



Beyond the quadrupole

- Quadrupole $f_{GW} = 2 \cdot f_{orbit}$
- Octupole $3 \cdot f_{orbit}$
- … $n \cdot f_{orbit}$



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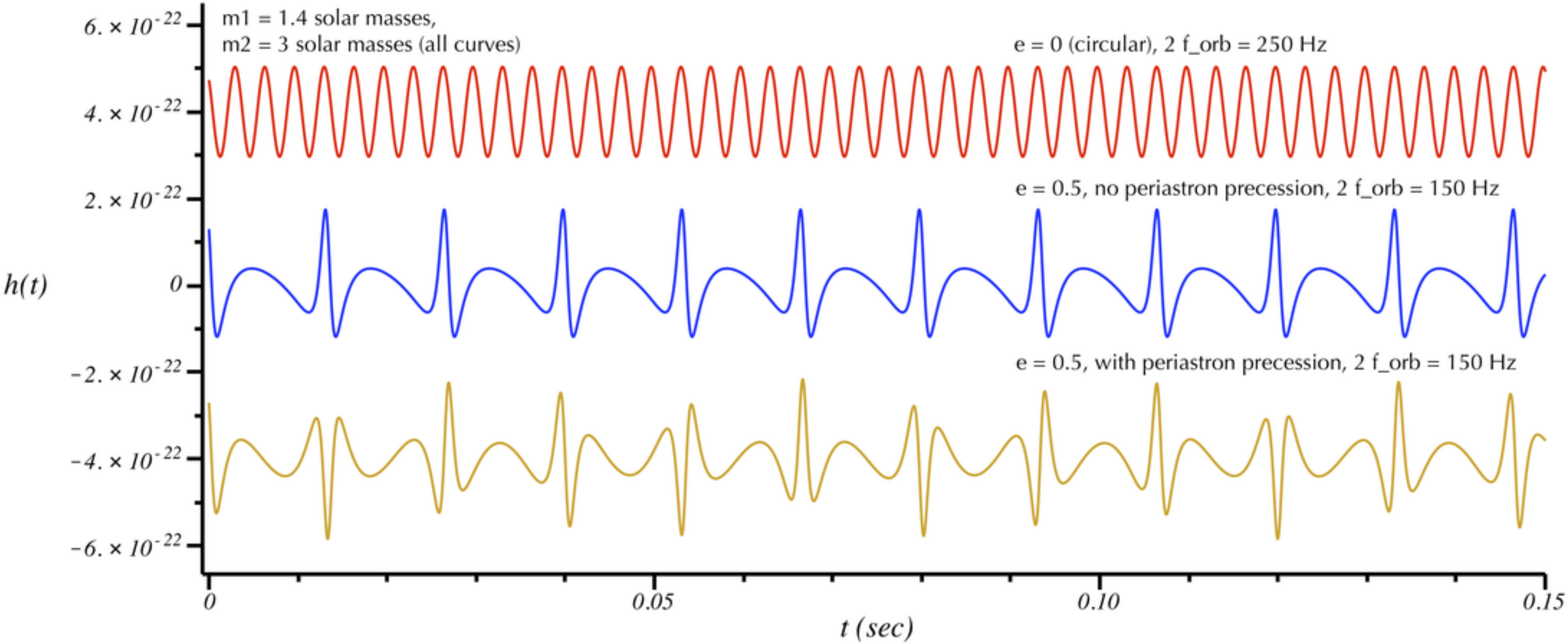
Eccentric orbits



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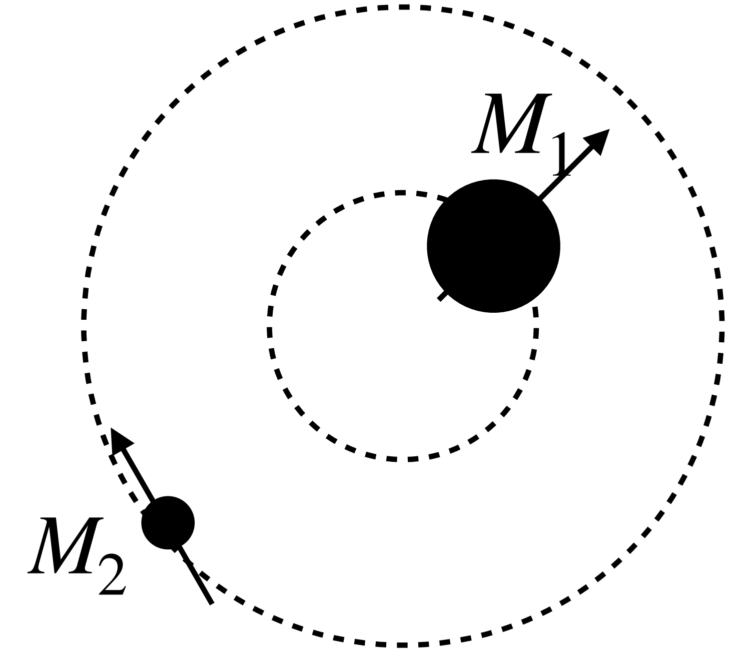
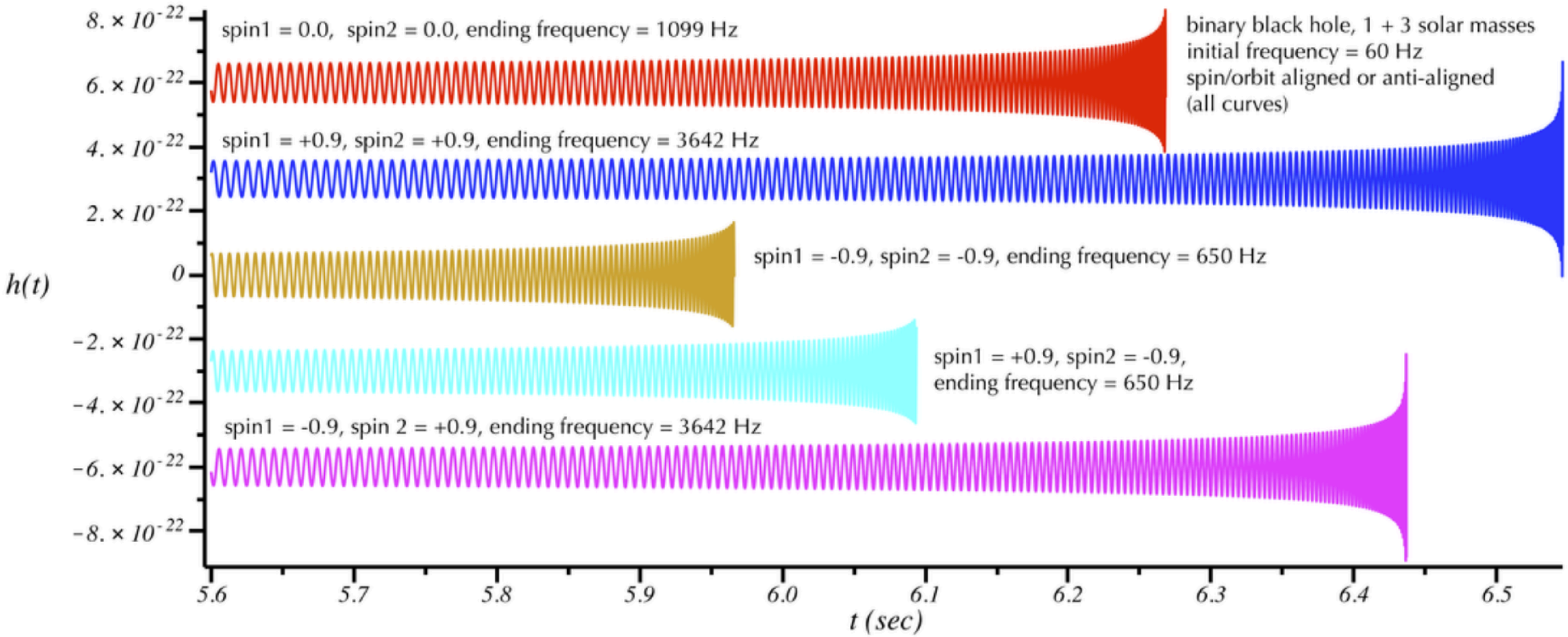
Spinning black holes



$1M_\odot + 3M_\odot$



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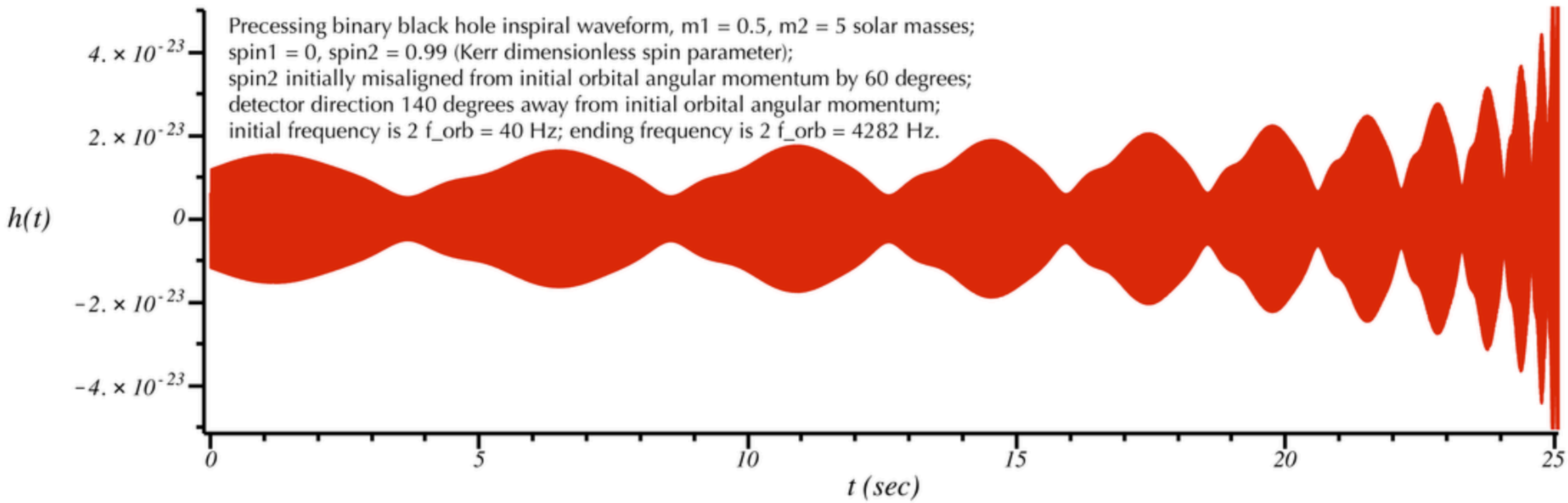
Precessing black holes

$0.5M_\odot + 5M_\odot$



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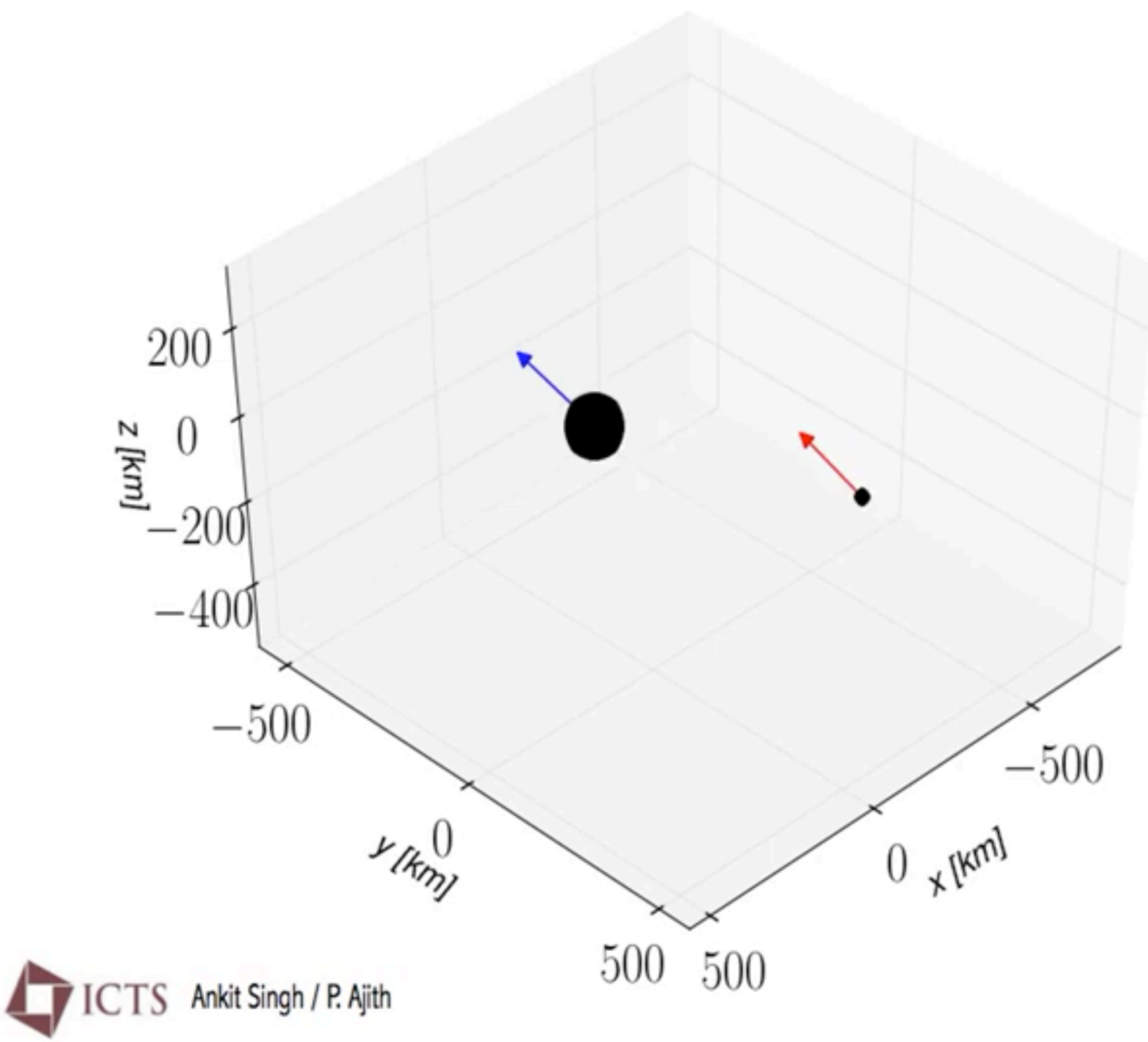
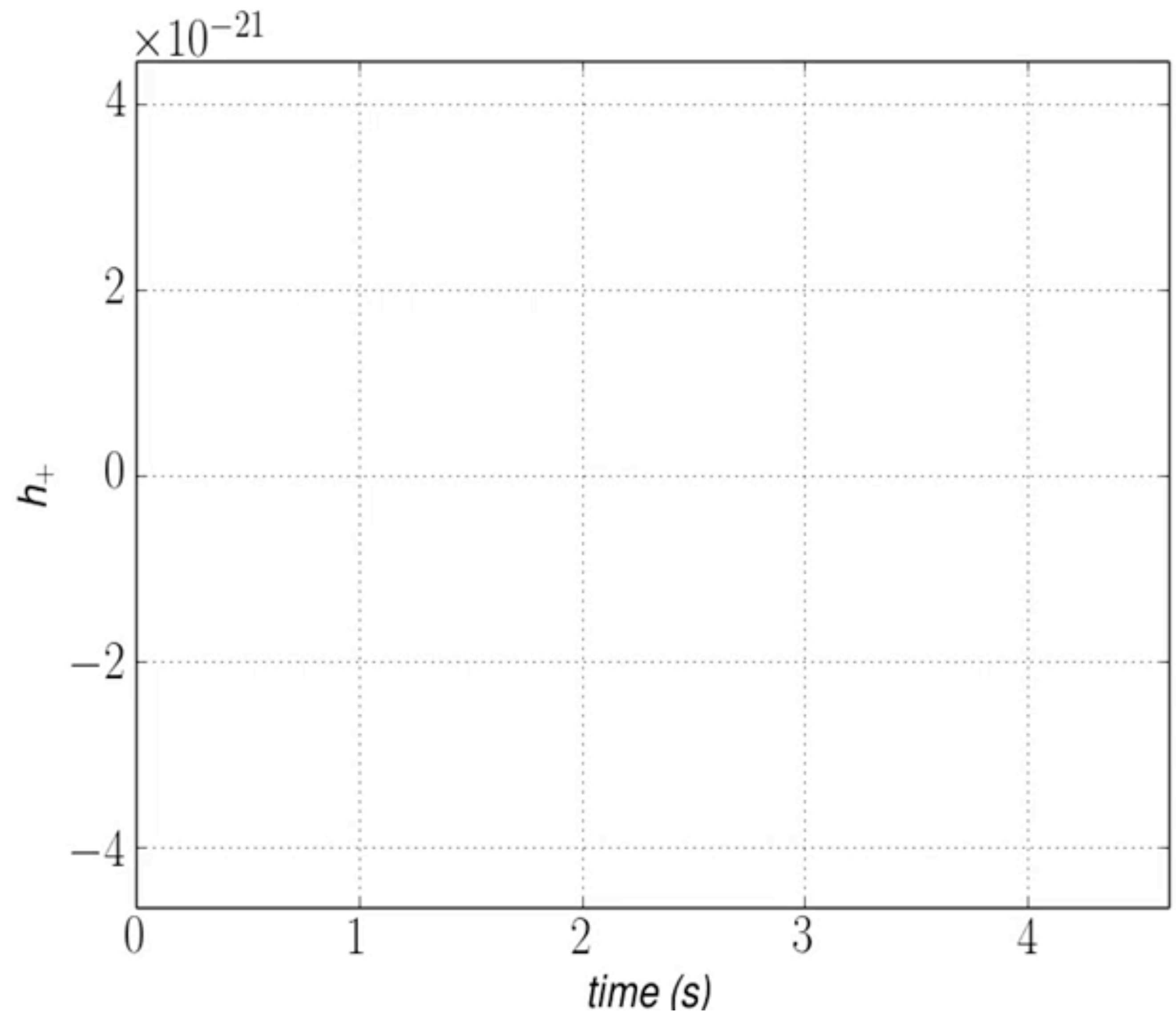
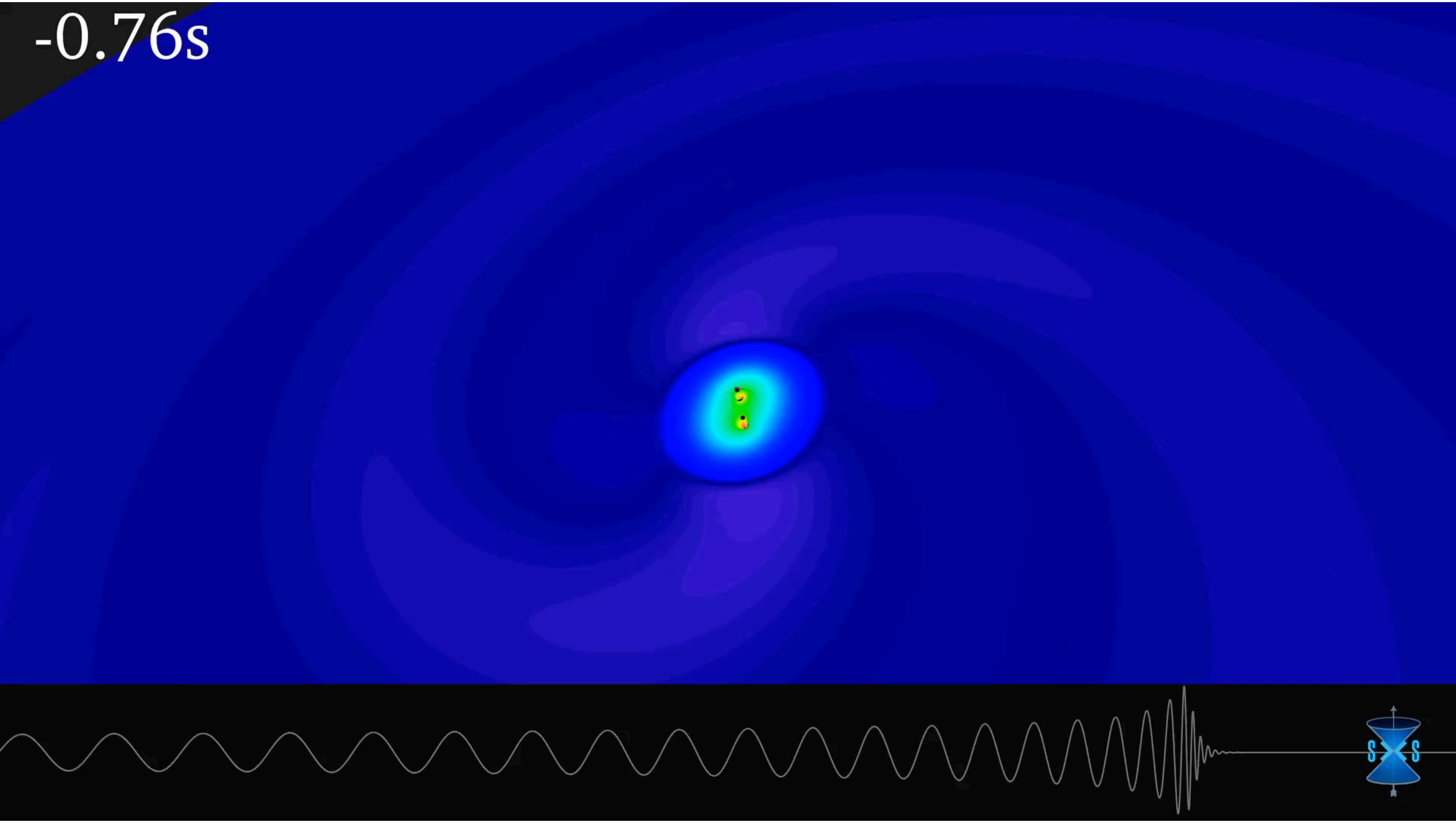
Precessing black holes



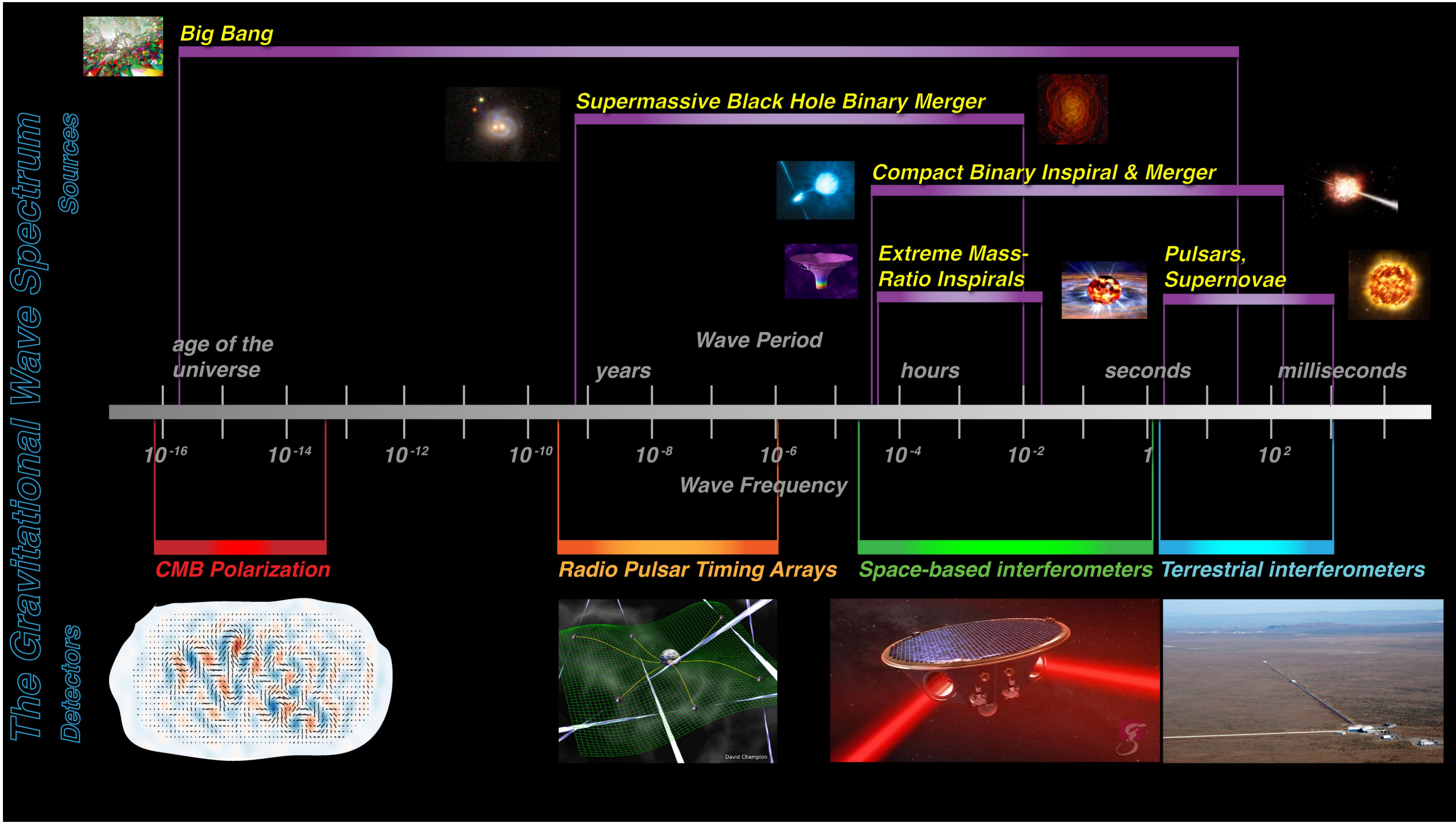


GW150914 : simulation of the signal



Credit: SXS

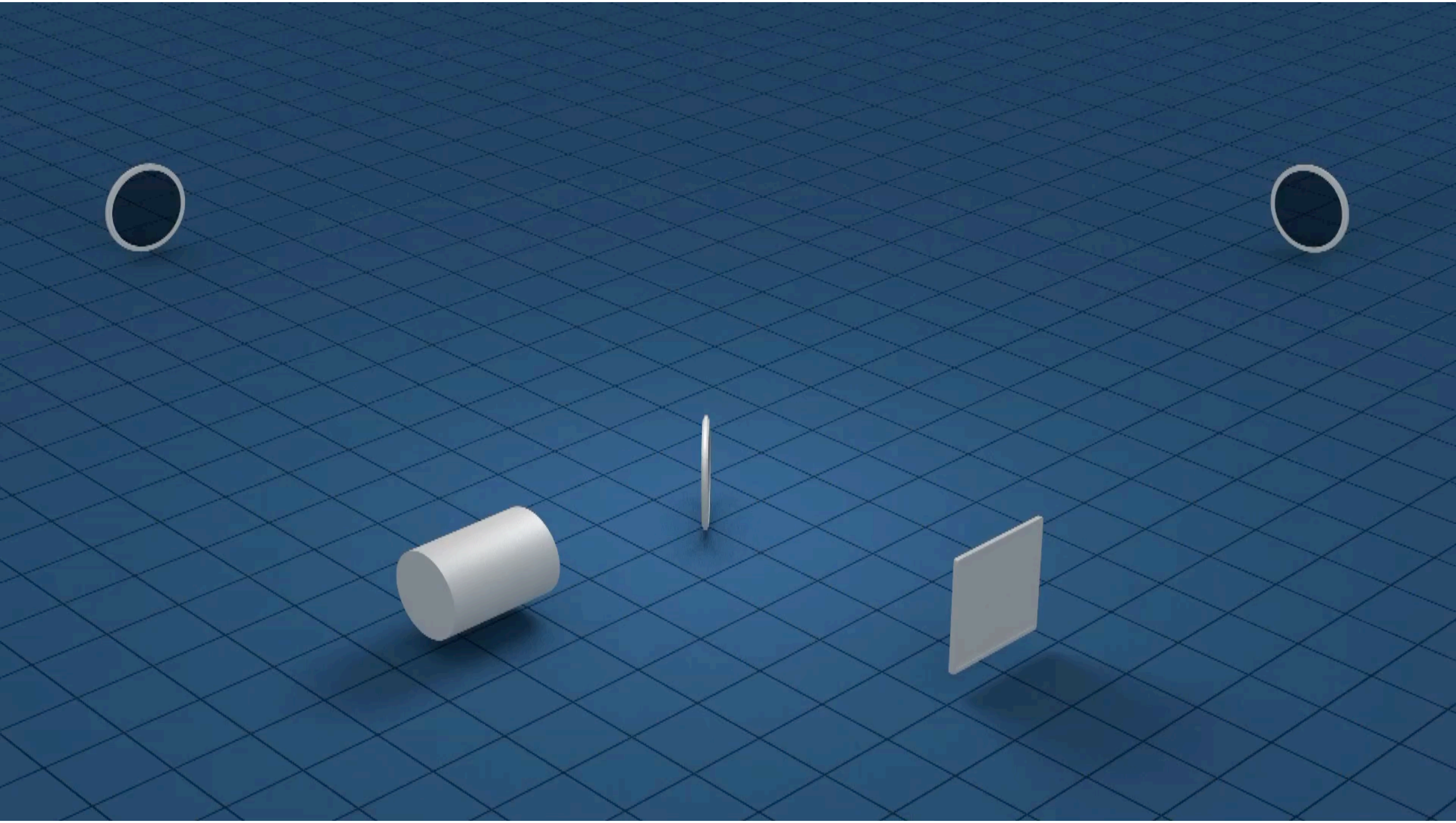
Overview of GW sources



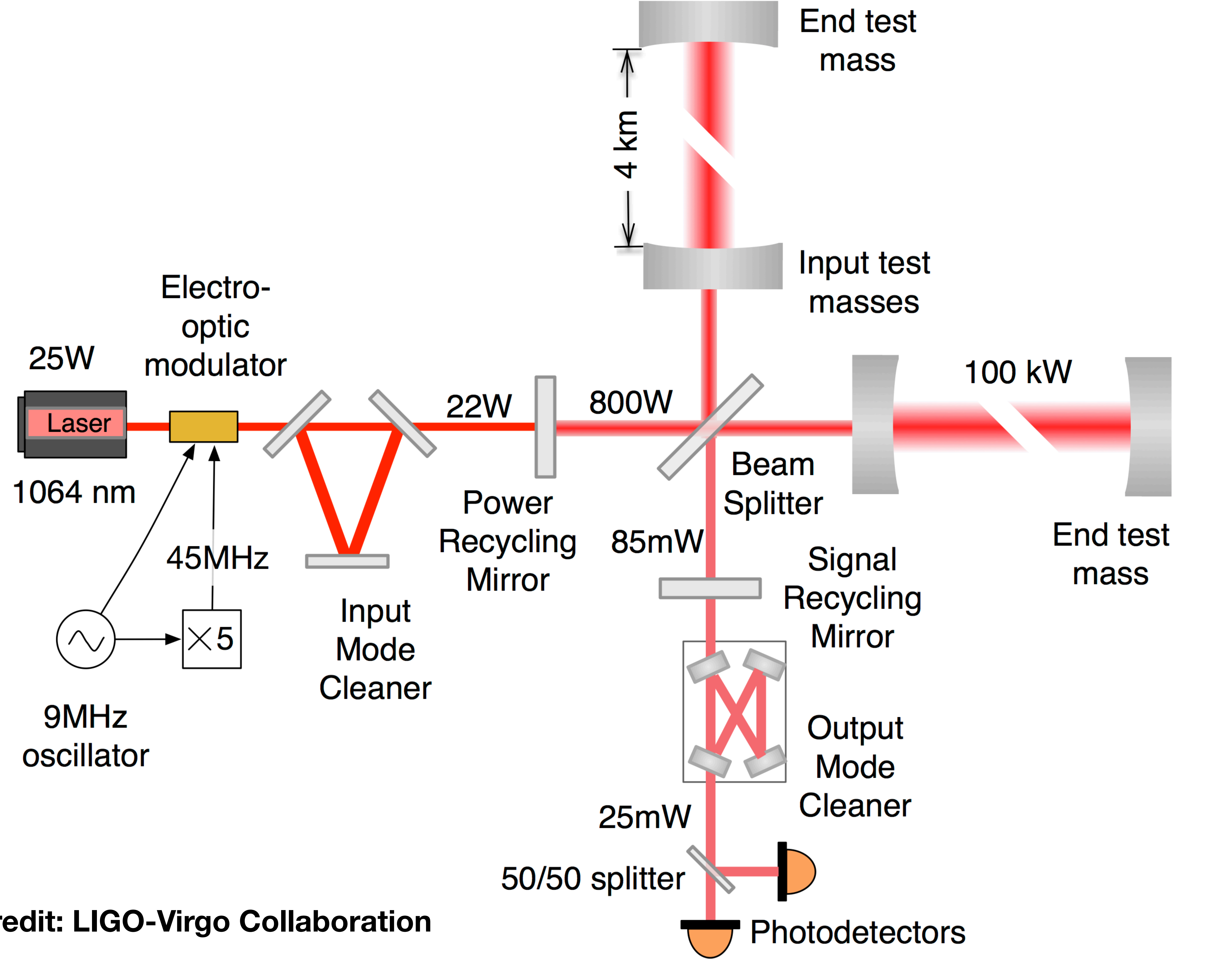
Gravitational-wave observatories

- LIGO (Hanford+Livingston, USA)
- Virgo (Italy)
- Kagra (Japan)

Gravitational-wave observatories: interferometry



Animation created by T. Pyle, Caltech/MIT/LIGO Lab



Credit: LIGO-Virgo Collaboration

Gravitational-wave observatories

- LIGO (Hanford+Livingston, USA)
- Virgo (Italy)
- Kagra (Japan)
- Pulsar Timing Arrays (radio telescopes; Europe+USA+Australia)
- *LIGO-India*
- *Einstein Telescope (Europe) / Cosmic Explorer (USA)*
- *LISA (space! ESA+NASA)*