Effective two-body approach to the three-body problem

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GDR Ondes Gravitationnelles

In collaboration with : Enrico Trincherini, Francesco Serra

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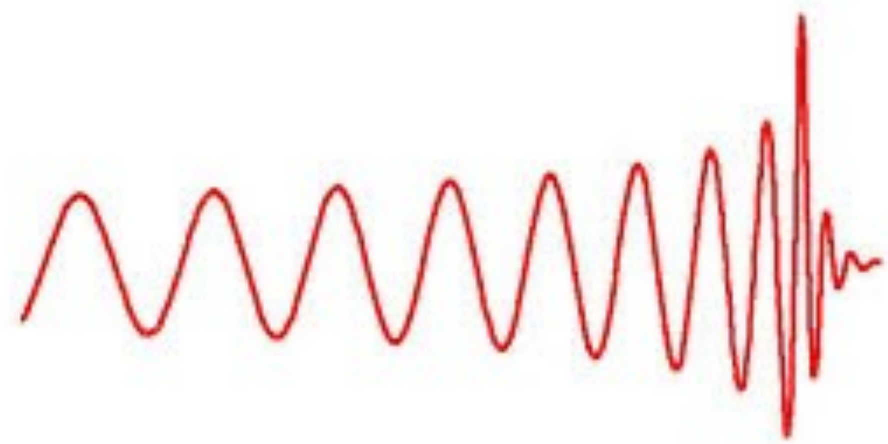
Istituto Nazionale di Fisica Nucleare

GRAVITATIONAL WAVES

Detection of GW so far beautifully corresponds to two-body systems



$$\Phi(f) = \phi_0 + 2\pi f t_0 + \sum_{k=0}^{7} \alpha_k f^{(k-5)/3}$$

$\alpha_k \leftarrow m_1, m_2, \chi_1, \chi_2$

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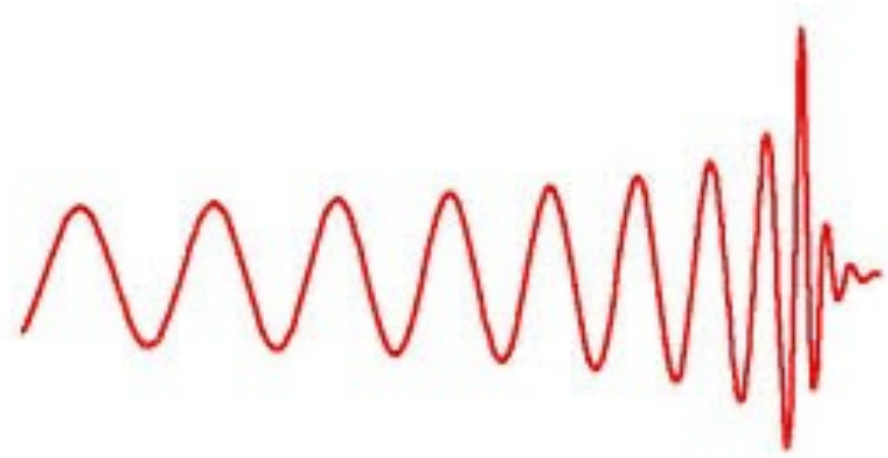

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m_1, m_2, χ_1, χ_2

If we ever detect a new feature in data, we have (as 19th century astronomers) two possible explanations:

- Modification of GR
- Perturbation by a third body (this talk)

GRAVITATIONAL WAVES

THIS QUESTION IS NOT PURELY ACADEMIC !

- 90% of low-mass binaries are expect to belong to a 'hierarchical' triple system

Tokovinin et al. 2006

- 'Migration traps' around SMBH at $R \sim 20 - 600 R_{\rm sch}$

Bellovary et al. 2015

Gravitational waves

This question is not purely academic !

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Priorities in gravitational waveform modelling for future space-borne detectors: vacuum accuracy or environment?

Lorenz Zwick,* Pedro R. Capelo and Lucio Mayer
Center for Theoretical Astrophysics and Cosmology, Institute for Computatio
Winterthurerstrasse 190, CH-8057 Zürich, Switzerland

Led by these considerations, we argue that *systematically including environmental effects* in waveform templates should take priority with respect to further increasing the accuracy of vacuum templates. If the goal is to maximise the science yield of future missions, the community could be better served by shifting the focus from the source of GWs to its surroundings.

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What is the influence of a third body on the waveform?

$\Longrightarrow$ We have to formulate the 3-body problem in GR and solve it perturbatively

RELATIVISTIC THREE-BODY PROBLEM

GR corrections at 1PN

EOM in post-Newtonian

$$\boldsymbol{a}_a = -\sum_{b\neq a}\frac{Gm_b\boldsymbol{x}_{ab}}{r_{ab}^3} + \frac{1}{c^2}\sum_{b\neq a}\frac{Gm_b\boldsymbol{x}_{ab}}{r_{ab}^3}\left[4\frac{Gm_b}{r_{ab}} + 5\frac{Gm_a}{r_{ab}} + \sum_{c\neq a,b}\frac{Gm_c}{r_{bc}} + 4\sum_{c\neq a,b}\frac{Gm_c}{r_{ac}} - \frac{1}{2}\sum_{c\neq a,b}\frac{Gm_c}{r_{bc}^3}(\boldsymbol{x}_{ab}\cdot\boldsymbol{x}_{bc}) - v_a^2 + 4\boldsymbol{v}_a\cdot\boldsymbol{v}_b - 2\boldsymbol{v}_b^2 + \frac{3}{2}(\boldsymbol{v}_b\cdot\boldsymbol{n}_{ab})^2\right] - \frac{7}{2c^2}\sum_{b\neq a}\frac{Gm_b}{r_{ab}}\sum_{c\neq a,b}\frac{Gm_c\boldsymbol{x}_{bc}}{r_{bc}^3} + \frac{1}{c^2}\sum_{b\neq a}\frac{Gm_b}{r_{ab}^3}\boldsymbol{x}_{ab}\cdot(4\boldsymbol{v}_a - 3\boldsymbol{v}_b)(\boldsymbol{v}_a - \boldsymbol{v}_b), \tag{3.1}$$

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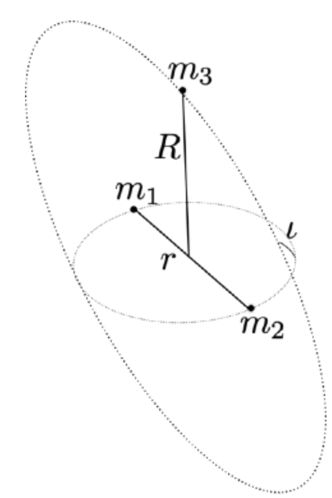
- No analytic waveform template
- Lack of physical intuition



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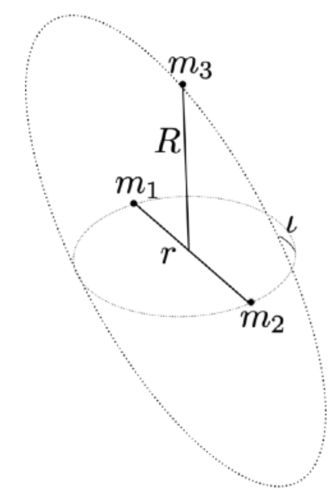
- Numerical evolution over long timescales difficult



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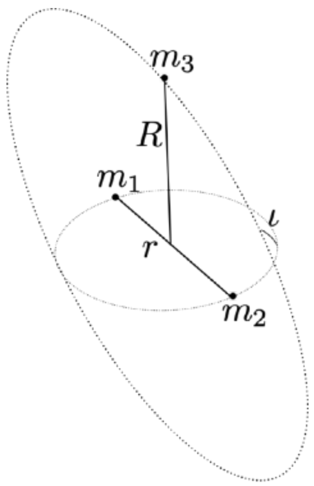

- No analytic waveform template
- Lack of physical intuition
- Numerical evolution over long timescales difficult
- Issues in the radiative sector

EFFECTIVE TWO-BODY

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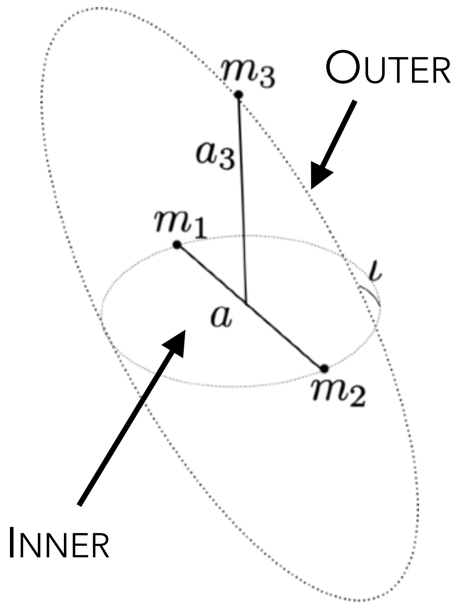

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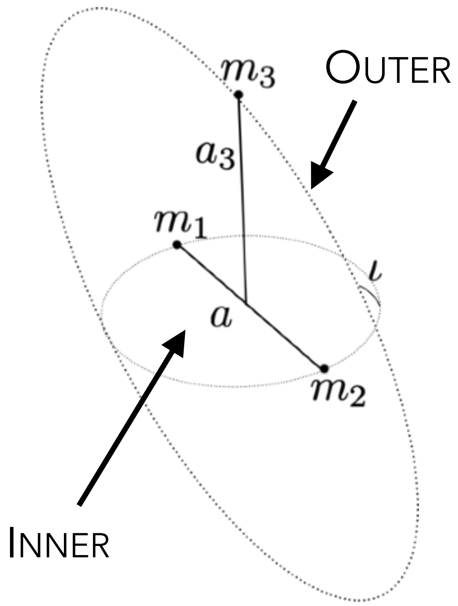
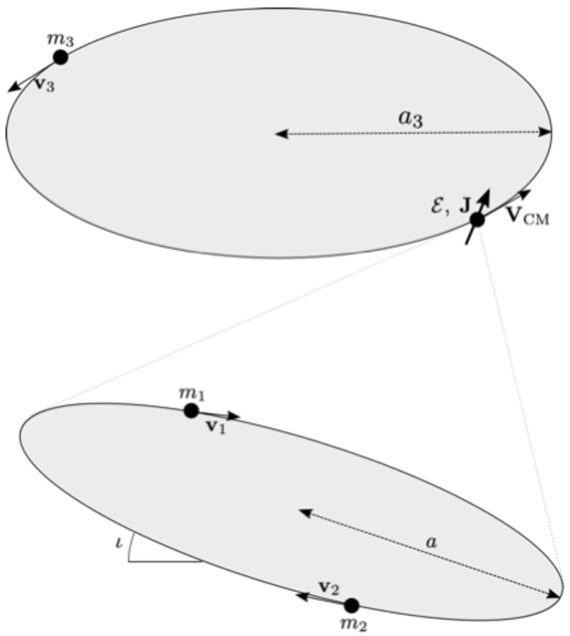


3-body motion = 2-body with spin!



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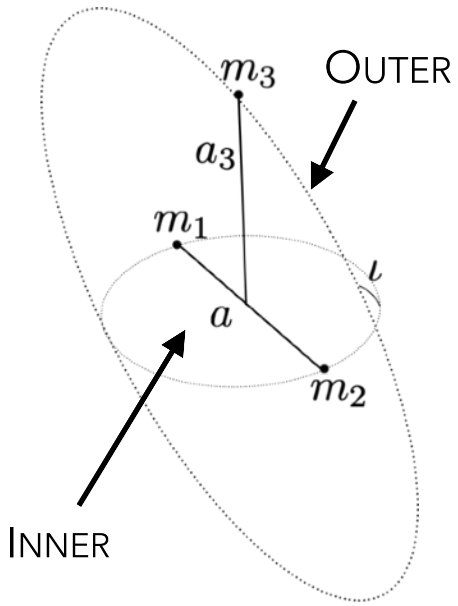

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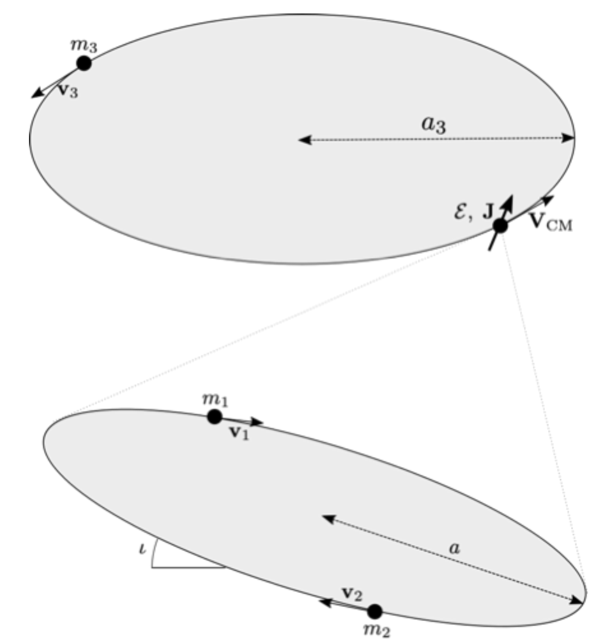

$$\mathscr{L}_{\rm full} = \sum_{A=1}^{3} - m_A \sqrt{-g_{\mu\nu} v_A^\mu v_A^\nu} \quad \Leftrightarrow \quad \mathscr{L}_{\rm EFT} = - \mathscr{E} \sqrt{-g_{\mu\nu} V^\mu_{\rm CM} V^\nu_{\rm CM}} + \frac{1}{2} J_{\mu\nu} \Omega^{\mu\nu} - m_3 \sqrt{-g_{\mu\nu} v_3^\mu v_3^\nu}$$

PROPER TIME

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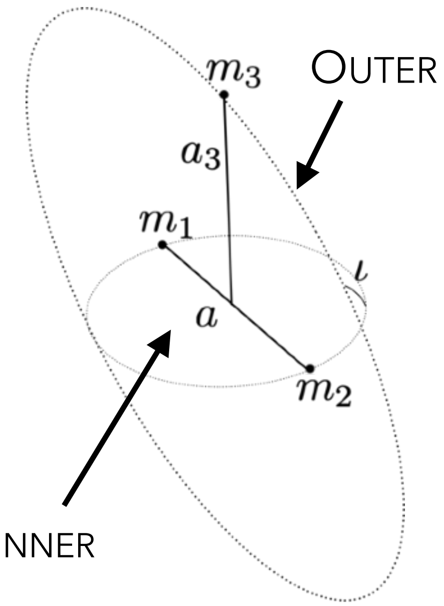

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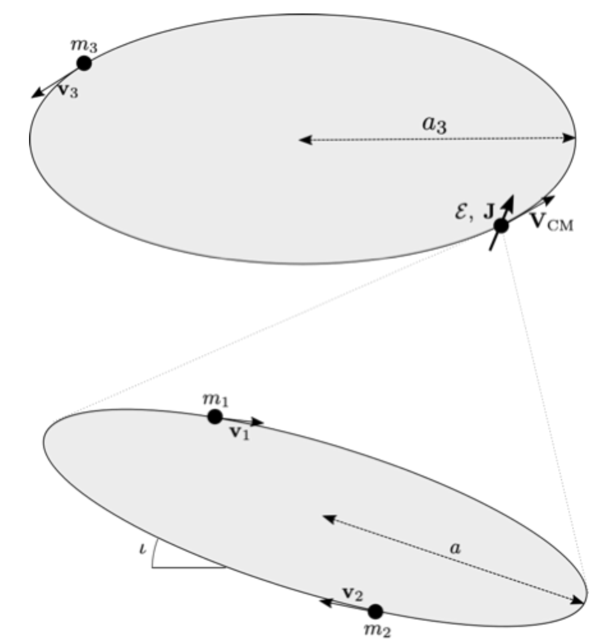

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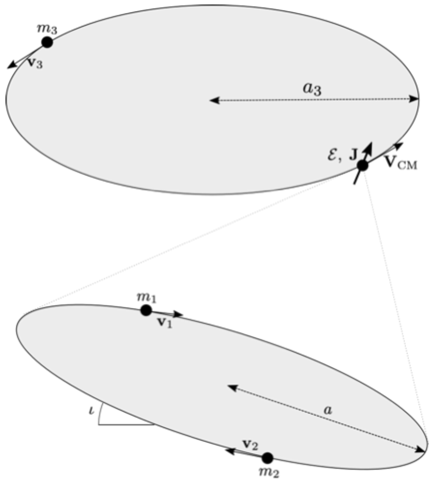
PROPER TIME

The equivalence principle fixes nearly everything!

$$\mathscr{E} = m - \frac{G_N m \mu}{2a}\,, \qquad J_{ij} = \epsilon_{ijk}J^k\,, \quad \Omega_{ij} = \epsilon_{ijk}\Omega^k\,, \qquad \mathbf{J} = \sqrt{G_N m a(1-e^2)}\;\hat{\mathbf{j}}\,, \qquad \mathbf{\Omega} = \hat{\mathbf{e}} \times \dot{\hat{\mathbf{e}}}$$

$\hat{\mathbf{e}} \equiv$ UNIT RUNGE-LENZ VECTOR



A SYSTEMATIC EXPANSION

AK, F. Serra, E. Trincherini 2021

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As in any EFT, the Lagrangian is organised with power-counting rules:

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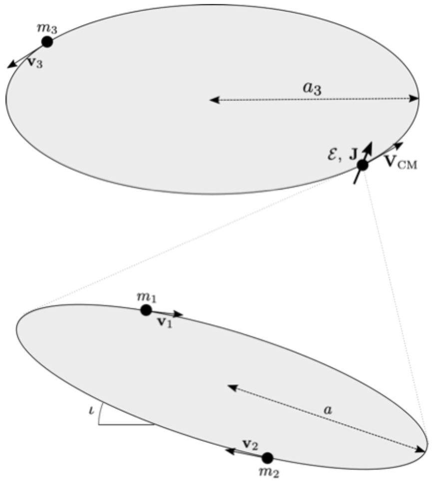

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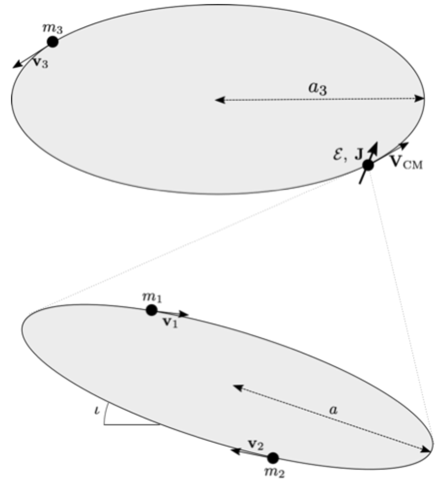
To get the EOM for the point-particles, one should 'integrate out' the gravitational field

$$\mathcal{H} = -\frac{Gm_1m_2}{2a} - 3m\frac{G^2m_1m_2}{a^2\sqrt{1-e^2}}$$

Hamiltonian of inner orbit $\varepsilon^{-1}v^0 + \varepsilon^{-1}v^2$



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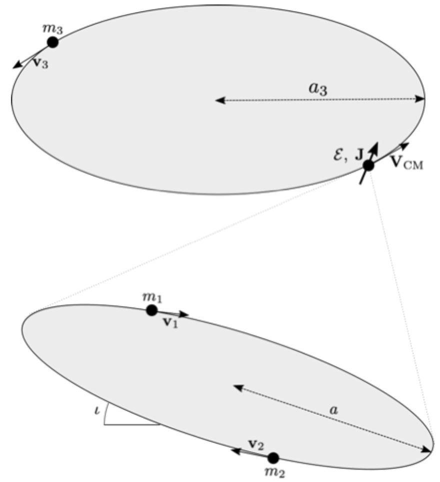

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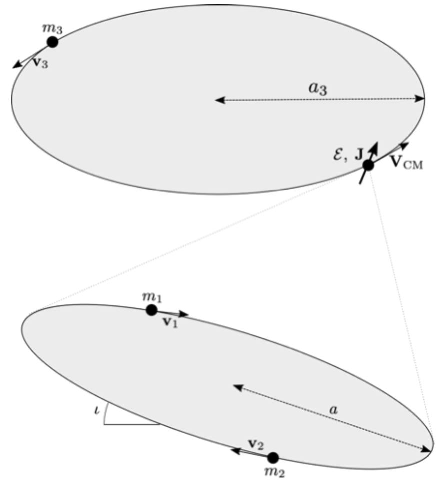
Spin-orbit coupling $\varepsilon^{3/2}v^2$



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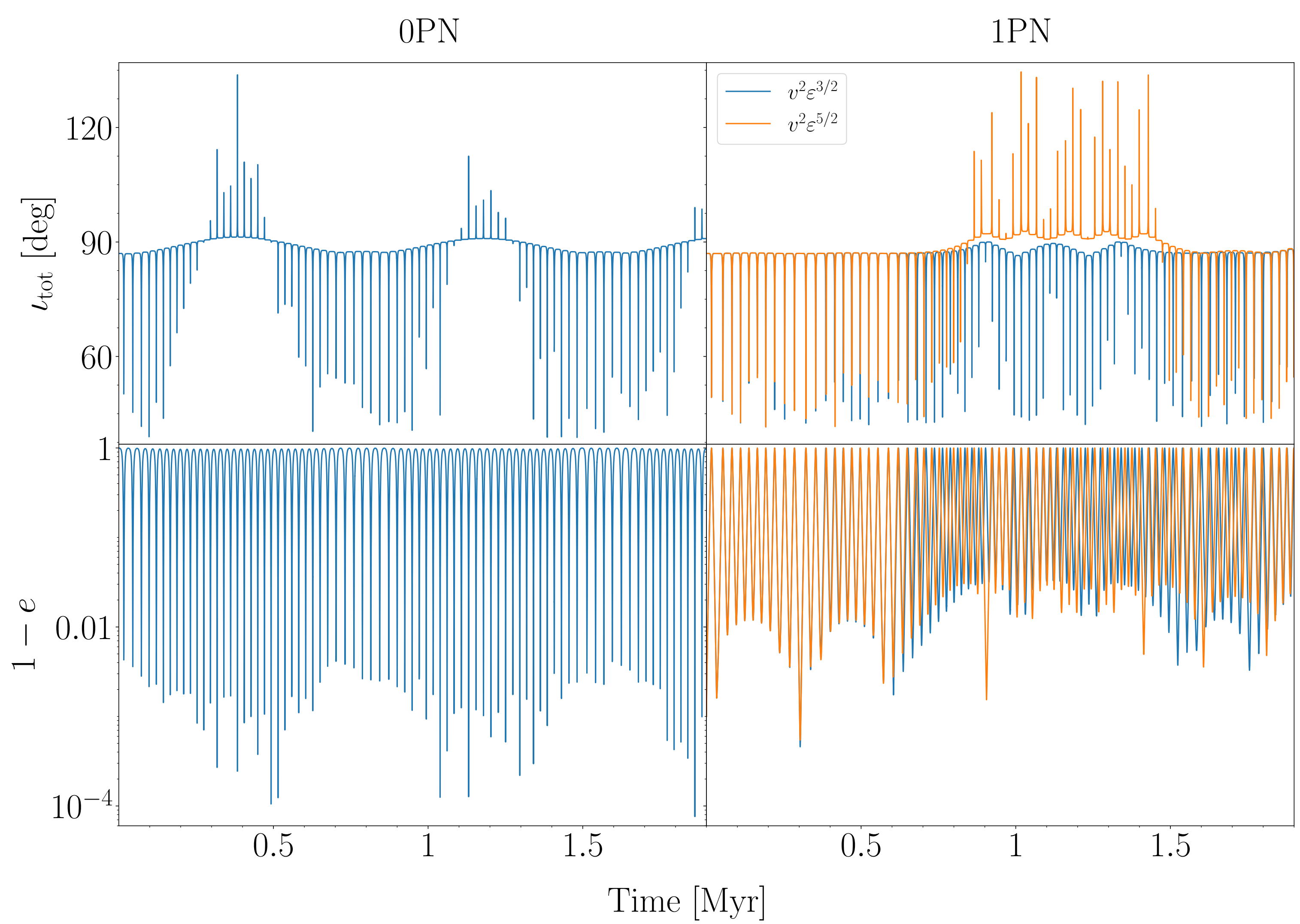
$$+\frac{4m+3m_3}{2m}\frac{G}{a_3^3(1-e_3^2)^{3/2}}\mathbf{J}\cdot\mathbf{J}_3$$ Spin-orbit coupling $\varepsilon^{3/2}v^2$

$$+\mathcal{H}_{\rm quad}$$ Quadrupolar coupling $\varepsilon^2v^0 + \varepsilon^2v^2$

Application

Long-term evolution of relativistic 3-body systems

$$\mathcal{H} = \mathcal{H}_{\text{inner}} + \mathcal{H}_{\text{outer}} + \mathcal{H}_{\varepsilon^{3/2}v^2} + \mathcal{H}_{\varepsilon^2 v^0} + \mathcal{H}_{\varepsilon^2 v^2} + \dots$$



$m_3 = m = 50\ M_\odot$

$a_3 = 350$ AU

$a = 5$ AU

$e_3 = 0.7$

AK, F. Serra, E. Trincherini (In prep.)

Conclusions

- Very rich phenomenology in the Newtonian 3-body problem, even more in the relativistic one…

- EFT formulation suited to precision computations

- Future work: more precise waveforms for 3-body problem

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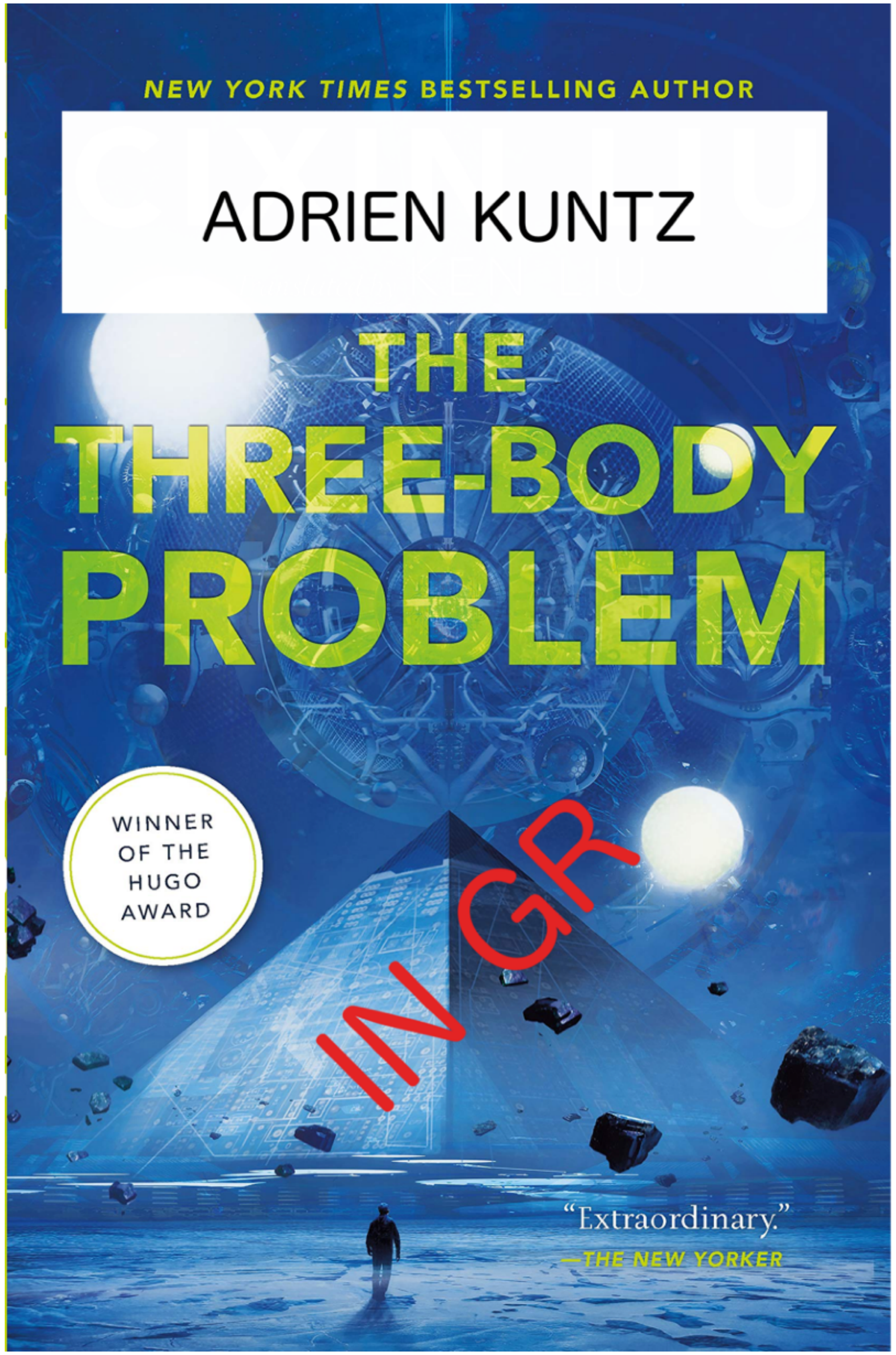
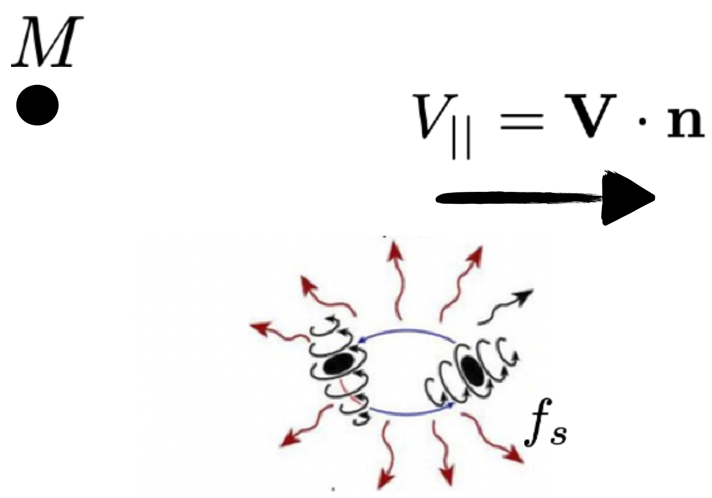


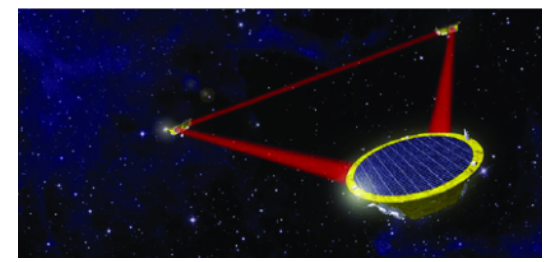
Application #2

Relativistic Doppler effect in waveforms

- Longitudinal Doppler effect: Randall Xianyu '18 Inayoshi et al. '17 Strokov et al. '17…



$$f_r = \frac{f_s}{1 + V_{||}}$$



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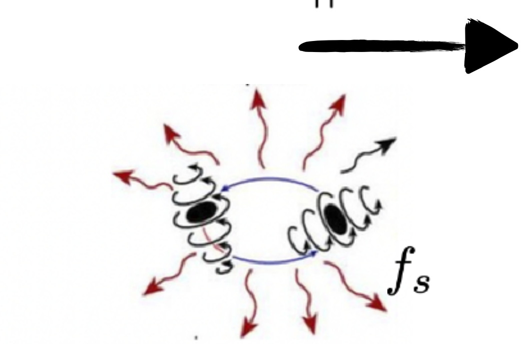
M

$$V_{||} = \mathbf{V} \cdot \mathbf{n}$$



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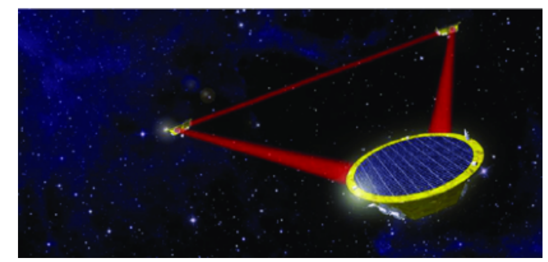

- Transverse Doppler effect: break degeneracies

$$f_r = \frac{f_s\sqrt{1 - V^2}}{1 + V_{||}}$$

AK, K. Leyde (In prep.)

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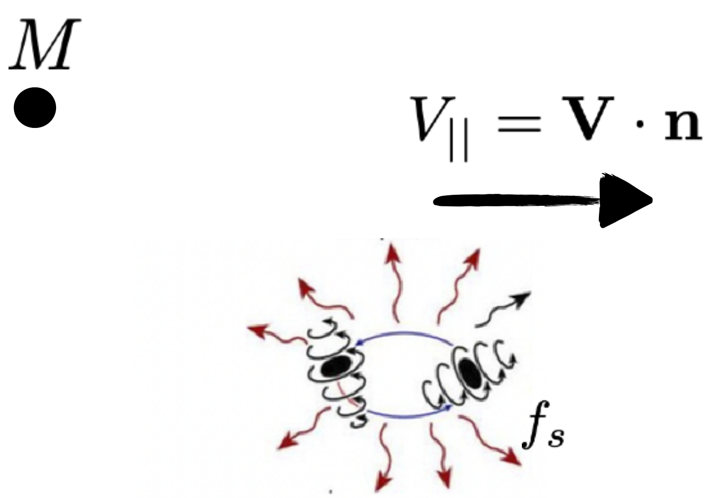
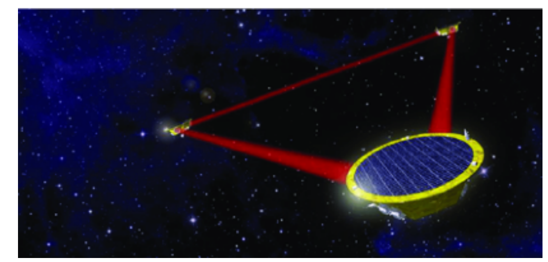


$$f_r = \frac{f_s}{1 + V_{||}}$$

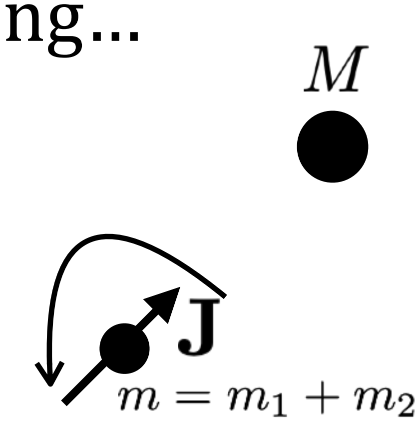


- Transverse Doppler effect: break degeneracies

$$f_r = \frac{f_s\sqrt{1 - V^2}}{1 + V_{||}}$$

AK, K. Leyde (In prep.)

- Higher order effects in waveforms like spin-orbit coupling…



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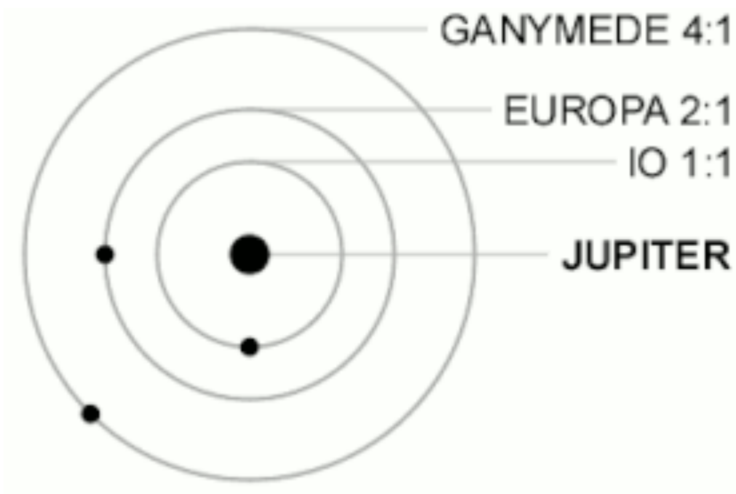
APPLICATION #3

NEW RESONANCES

- Resonances are a fascinating phenomenon of the 3-body problem



Application #3

New resonances

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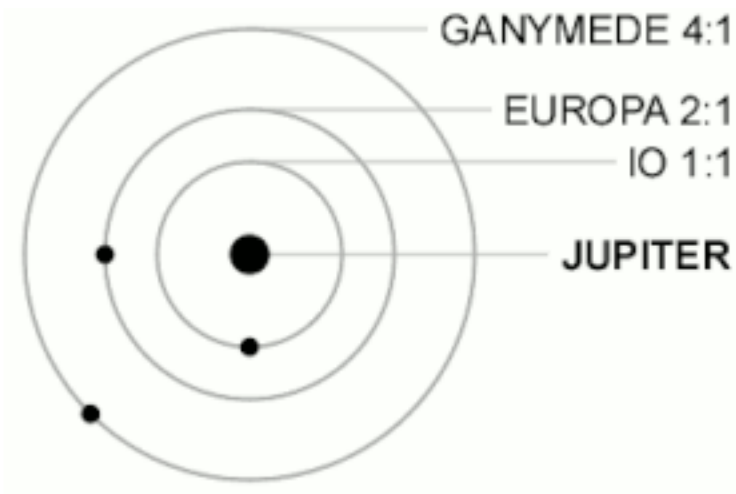
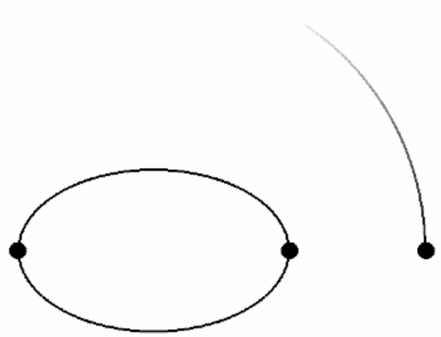


- When relativistic effects are included, there are other kinds of resonances



Perihelion angle

Outer orbit frequency

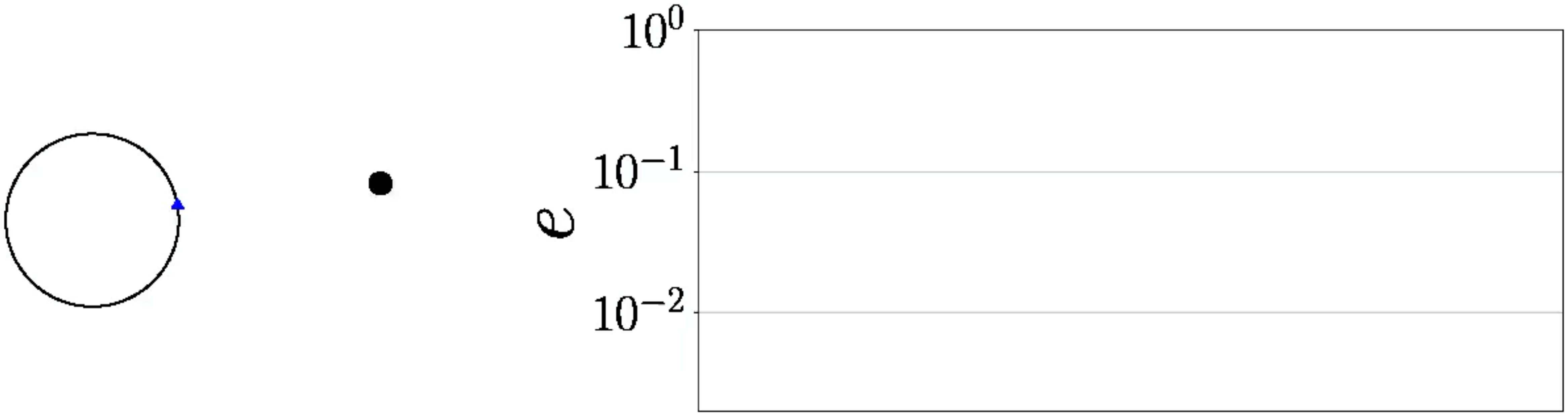
$$p\,\dot{\omega} + q\sqrt{\frac{GM}{a_3^3}} = 0$$

$$p, q \in \mathbb{Z}$$

Application #3

New resonances

$$a(t) = a_0 \left(1 - \frac{t}{t_{\rm RR}}\right)^{1/4}$$



AK 2021

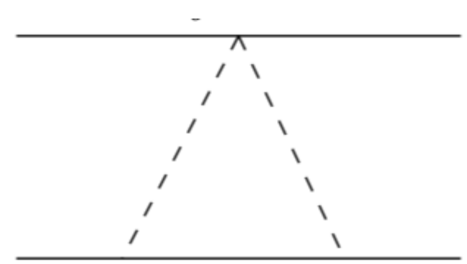
THE TWO-BODY PROBLEM IN GR

The two-body dynamics is encoded in the EFFECTIVE ACTION :

$$e^{iS_{\rm eff}[\mathbf{x}_1(t),\mathbf{x}_2(t)]} = \int \mathscr{D}h_{\mu\nu} e^{iS[\mathbf{x}_1(t),\mathbf{x}_2(t),h_{\mu\nu}]}$$

REAL PART: CONSERVATIVE

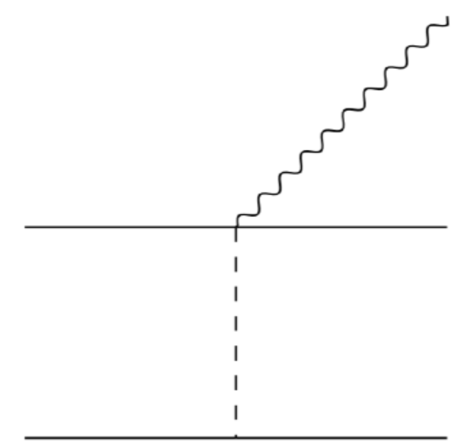
$$\Re(S_{\rm eff}) = \int dt L[\mathbf{x}_A, \mathbf{v}_A]$$



$$L \quad = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 + \frac{G m_1 m_2}{|\,\mathbf{x_1}(t) - \mathbf{x_2}(t)\,|} + L_{\rm 1PN} + \ldots$$

IMAGINARY PART: DISSIPATIVE

$$\Im(S_{eff}) = \frac{T}{2}\int dE d\Omega \frac{d^2\Gamma}{dE d\Omega}$$



$$P = \frac{G}{5}\left\langle \dddot{Q}^{kl}\dddot{Q}_{kl}\right\rangle + \ldots$$

HAMILTONIAN FOR PRECESSION RESONANCE

$$\mathcal{H} = -\frac{G_N m\mu}{2a} - \frac{G_N M \mu_3}{2a_3} - 3\mu \frac{G_N^2 m^2}{a^2\sqrt{1-e^2}} + \mathcal{H}_{\text{quad}}$$

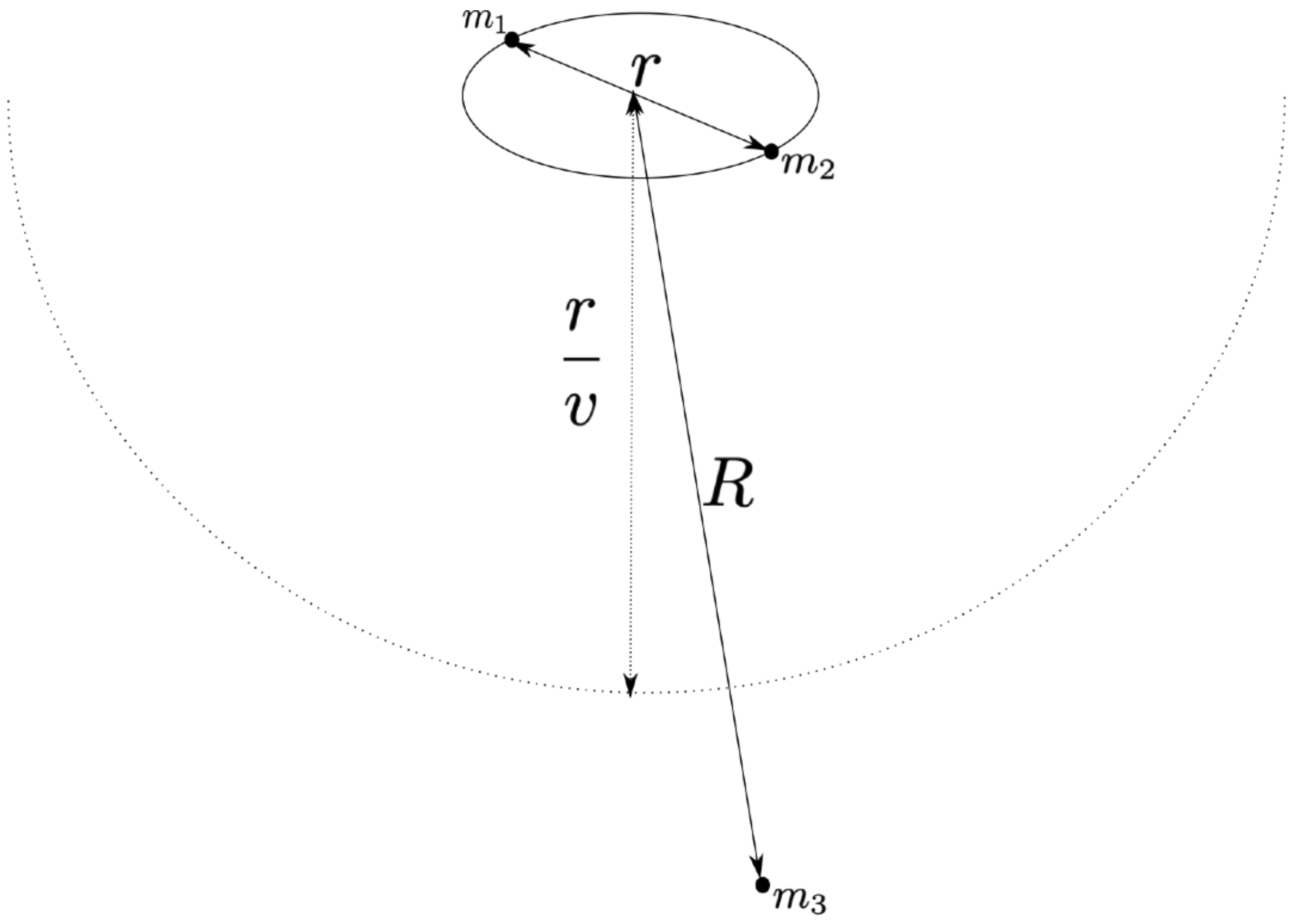
- One should NOT average over the outer orbit
  (this is why it was never studied up to now)

- Resonant terms in $\mathcal{H}_{\text{quad}} \supset \cos\left(2\omega - q\sqrt{GM/a_3^3}\,t\right)$

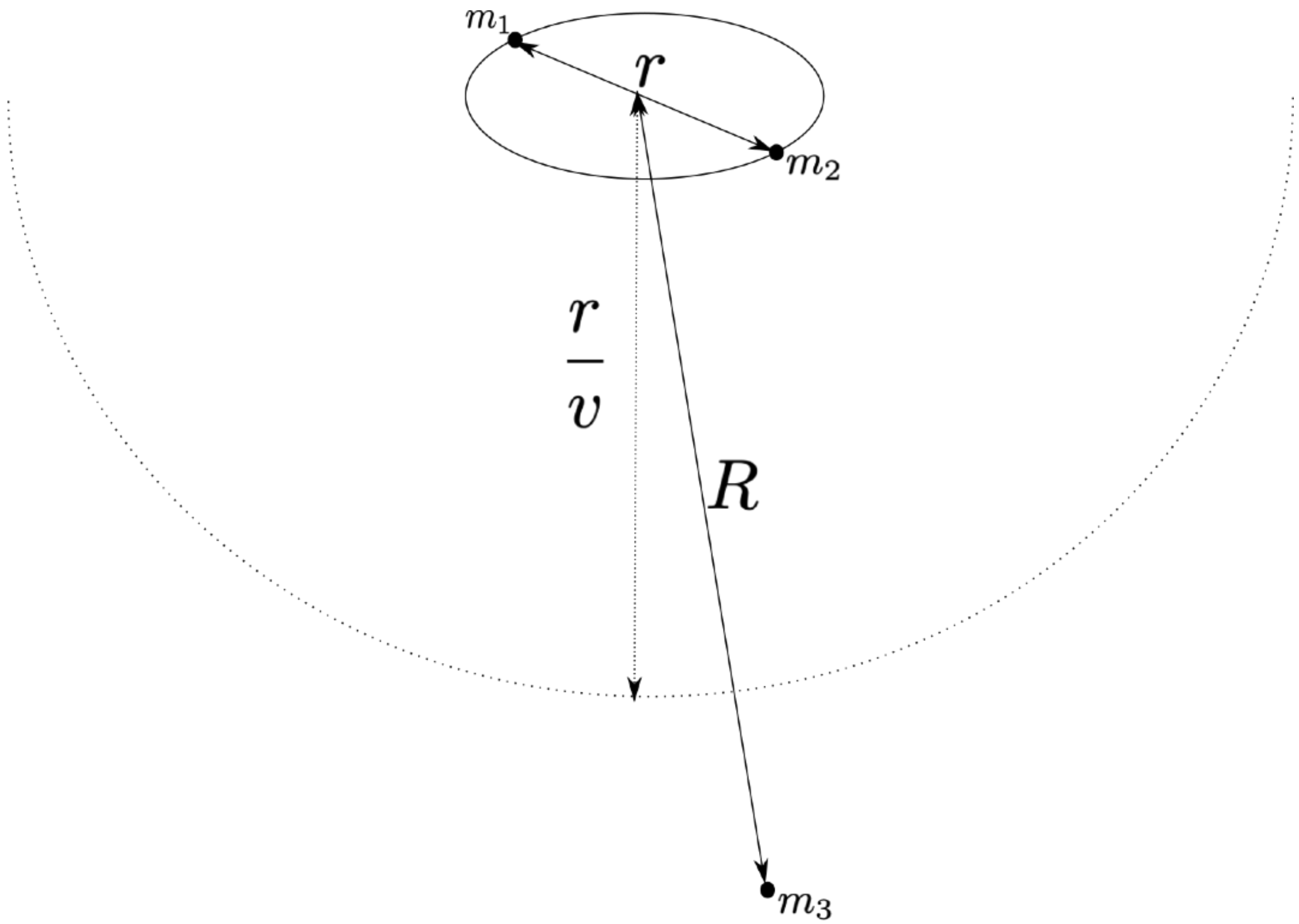
- Exchange of energy

- Radiation-reaction: $a(t) = a_0\left(1 - \frac{t}{t_{\text{RR}}}\right)^{1/4}$

ISSUES IN RADIATIVE SECTOR



ISSUES IN RADIATIVE SECTOR

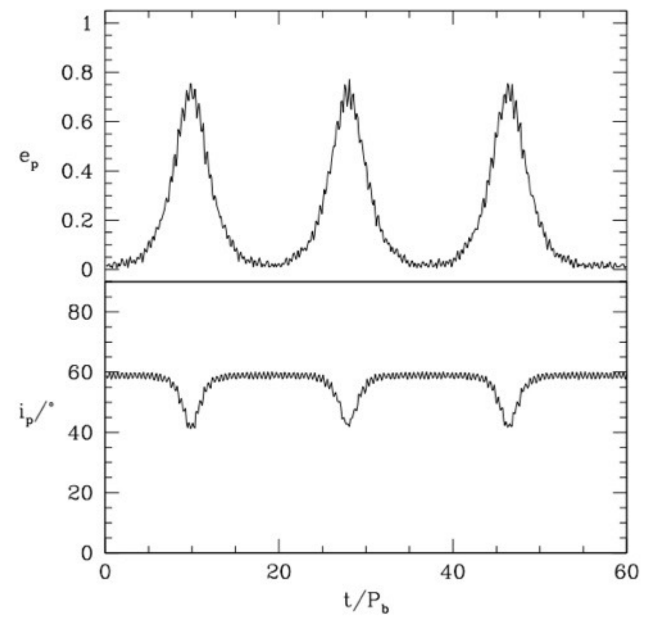


$$\mathcal{O}(v) \ll 1$$

$$Q^{ij}(t-r) \simeq Q^{ij}(t) - r\partial_t Q^{ij} + \dots$$

$$Q^{ij}(t-R) \neq Q^{ij}(t) - R\partial_t Q^{ij} + \dots$$

SUPPLEMENTARY SLIDE ON CROSS TERMS



$$e = \langle e \rangle + \delta e$$

Low-energy DOF, assumed constant in the averaging procedure

High-energy fluctuations, to be integrated out