Post-Newtonian waveform: a state of the art.

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Motivation: The gravitational waveform

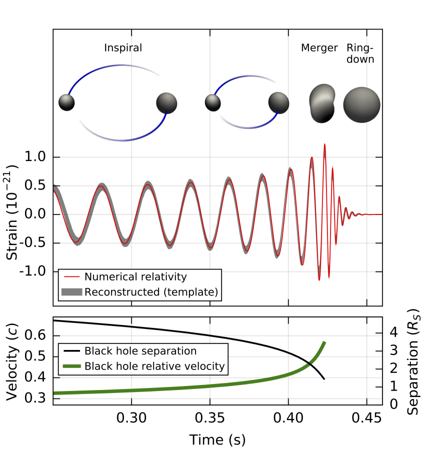


Figure from *LSC & Virgo*, PRL **116**(2016)061102.

The strain in a detector is given by

$$\frac{\Delta L}{L} = R\left[h(t)\, e^{i\phi(t)}\right],$$

with

- R the detector's response function,
- h the *amplitude* of the wave,
- ϕ the *phase* of the wave.

Present (and future) detectors are more sensible to the phase.

Motivation: The post-Newtonian approximation

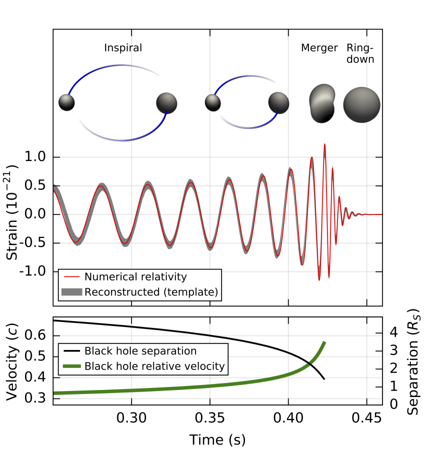
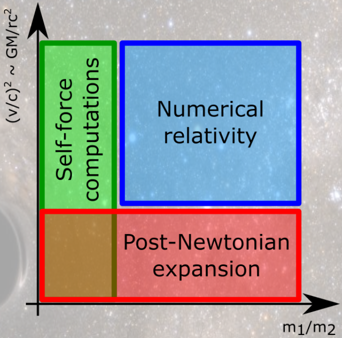


Figure from *LSC & Virgo*, PRL **116**(2016)061102.

For bound systems,



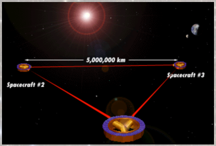
$\hookrightarrow$ Provide *analytical* results for the waveform, in the inspiral regime

$$\frac{v^2}{c^2} \sim \frac{G m_{\text{tot}}}{r c^2} \ll 1.$$

Motivation: Do we need high PN orders ?

⇒ The current accuracy is 3.5PN,
i.e. the $(v/c)^7$ correction beyond the Newtonian formulas.

- LIGO, Virgo and KAGRA are doing fine with this 3PN precision.



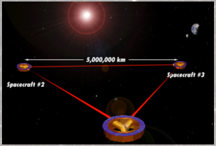
- What about LISA ?

↪ For a BHB of 20 $M_\odot$, 4PN corrections will induce a change of 10^{-15} in the phasing: $\sim 1/2\,000$ cycles over 2.10^{11} in the LISA band.

⇒ Stellar-mass binaries will only require 3PN waveforms[1] and most of them, only 2PN.

[1] A. Mangiagli, A. Klein, A. Sesana, E. Barausse & M. Colpi, PRD **99**(2019)064056.

Motivation: Do we need high PN orders ?



- But LISA will also see massive BHs !

$\hookrightarrow$ For a BHB of $2.10^6\ M_\odot$, 4PN corrections will induce a change of $\sim 10\%$ in the phasing.

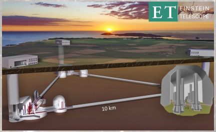
$\hookrightarrow$ For a BHB of $2.10^5\ M_\odot$, 4PN corrections will induce a change of $\sim 10^{-4}$ in the phasing.

$\Rightarrow$ 4PN is important for an accurate determination of the astrophysical parameters (masses, spins,...) of MBHB.

- What about the Einstein Telescope ?

$\rightsquigarrow$ ET will be very sensitive of the late inspiral, where high PN corrections hits !

$\rightsquigarrow$ "Beyond GR" compact objects differ from GR ones at 5PN, and may be detectable by ET.



$\Rightarrow$ 4PN (and 5PN ?) will be crucial to distinguish between usual BH/NS and more exotic objects.

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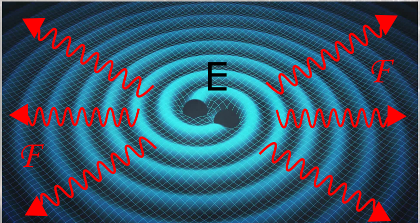
How to compute the waveform: The balance equation

- Conservation of energy for an isolated system

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\mathcal{F}.$$



Courtesy to Swinburne Astronomy Productions

$\hookrightarrow$ E is the “conserved” energy,

$\hookrightarrow$ $\mathcal{F}$ is the gravitational flux.

$\Rightarrow$ On (quasi-)circular orbits, both E and $\mathcal{F}$ depend only on the orbital frequency ω

$\hookrightarrow$ A simple integration gives the gravitational phase $\phi = \int \mathrm{d}t\, \omega$.

$\Rightarrow$ On elliptic orbits, expansions in eccentricity are usually performed.[2]

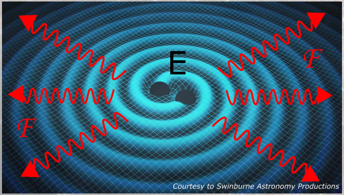
[2] G. Cho, S. Tanay, A. Gopakumar & H. M. Lee, PRD **105**(2022)064010.

How to compute the waveform: The energy

- Balance equation

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\mathcal{F}.$$

$\hookrightarrow$ E is the "conserved" energy.



- At 4PN, E has been computed by three different approaches:
  - ✓ *ADM*-Hamiltonian: full 4PN order, up to one *ambiguity* parameter,[3]
  - ✓ Fokker Lagrangian[4] and Effective Field Theories[5]: full 4PN order.
- At 5PN, E is known up to 2 coefficients.[6]
- At 6PN, some sectors of E have already been computed.[7]

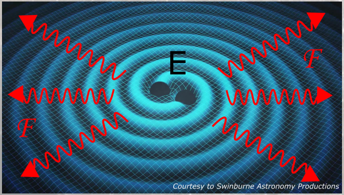
[3] T. Damour, P. Jaranowski & G. Schäfer, PRD **93**(2016)084014.
[4] T. Marchand, L. Bernard, L. Blanchet & G. Faye, PRD **97**(2018)044023.
[5] S. Foffa, R. Porto, I. Rothstein & R. Sturani, PRD **100**(2019)024048.
[6] D. Bini, T. Damour & A. Geralico, PRD **102**(2020)024062.
[7] D. Bini, T. Damour & A. Geralico, PRD **102**(2020)024061.

How to compute the waveform: The flux

- Balance equation

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\mathcal{F}.$$

$\hookrightarrow$ $\mathcal{F}$ is the gravitational flux.



Einstein's *Quadrupole formula* is extended to[8]

$$\mathcal{F} = \sum_{\ell \geq 2} \frac{G}{c^{2\ell+1}} \left[a_\ell \dot{U}_L \dot{U}_L + \frac{b_\ell}{c^2} \dot{V}_L \dot{V}_L \right]$$

with U_L (V_L) the *radiative* mass (current) moments.

- $\mathcal{F}$ is known up to 3.5PN order,[9]
- The 4.5PN coefficient for circular orbits has been computed.[10]

[8] K. Thorne, Rev.Mod.Phys. **52**(1980)299.
[9] L. Blanchet, Liv.Rev.Relat **17**(2014)2.
[10] T. Marchand, L. Blanchet & G. Faye, CQG **33**(2016)244003.

How to compute the waveform: The modes

- The amplitude of the wave is projected on spherical harmonics

$$h_+ - ih_\times = \sum h^{\ell m}\, Y^{\ell,m}_{-2}(\theta,\varphi)$$

- We extract the modes $h^{\ell m}$ from the TT projection of the metric at spatial infinity

$$h_{+,\times} = \frac{4G}{R}\, e^{ij}_{+,\times} \mathcal{P}^{TT}_{ijkl} \sum_p \left[\frac{N_{P-2}}{c^{2+p}\, p!}\, U_{klP-2} + \frac{2p\, \epsilon_{kab} N_{bP-2}}{c^{3+p}\,(p+1)!}\, V_{laP-2} \right] + o\left(\frac{1}{R}\right)$$

↪ The knowledge of the radiative moments U_L and V_L gives both the flux and the modes.

Computing the waveform: State of the art

- For point-like particles
  - ↪ The state-of-the art is 3.5PN for the phase[11] and the modes,[12]
  - ↪ Using EFT, confirmation of the flux at 2PN[13] (3PN is on its way).
- For spinning particles
  - ↪ the spin–orbit phase is known at 3.5PN,[14]
  - ↪ the spin–spin phase is known at 4PN,[15]
  - ↪ the modes are known at 3.5PN.[16]
- For particles with tidal effects
  - ↪ The phase is known at 7.5PN order, *ie.* 2.5 relative PN,[17]
  - ↪ The spin-tidal interaction phase is known at 6.5PN order.[18]

[11] L. Blanchet, G. Faye, B. R. Iyer & B. Joguet, PRD **65**(2002)061501.
[12] L. Blanchet, G. Faye & B. R. Iyer, CQG **32**(2015)045016.
[13] A. K. Leibovich, N. T. Maia, I. Z. Rothstein & Z. Yang, PRD **101**(2020)084058.
[14] A. Bohe, S. Marsat & L. Blanchet, CQG **30**(2013)135009.
[15] G. Cho, R. A. Porto & Z. Yang, arXiv:2201.05138.
[16] Q. Henry, S. Marsat & M. Khalil, arXiv:2209.00374.
[17] Q. Henry, G. Faye & L. Blanchet, PRD **102**(2020)044033.
[18] T. Abdelsalhin, L. Gualtieri & P. Pani, PRD **98**(2018)104046.

Computing $\mathcal{F}$: General strategy

- We need the 4PN flux

$$\mathcal{F} = \frac{G}{5\,c^5}\left[\dot{U}_{ij}\,\dot{U}_{ij} + \frac{1}{c^2}\left(\frac{16}{9}\,\dot{V}_{ij}\,\dot{V}_{ij} + \frac{5}{189}\,\dot{U}_{ijk}\,\dot{U}_{ijk}\right) + \dots\right].$$

↪ The 3PN V_{ij} and U_{ijk} are known[19] as well as lower order moments.

⇒ We need the radiative mass quadrupole U_{ij} at 4PN.

$$h_{ij}^{\mathrm{TT}} = \frac{2G}{c^4 R}\,\mathcal{P}_{ijkl}^{\mathrm{TT}}\,U_{kl} + \mathcal{O}\left(\frac{1}{c^5}\right) + o\left(\frac{1}{R}\right).$$

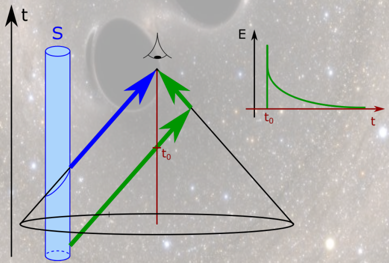
- From it, we will also obtain the 4PN dominant mode, h^{22}.

[19] Q. Henry, G. Faye & L. Blanchet, CQG **38**(2021)185004.

Computing $\mathcal{F}$: The radiative mass quadrupole

- U_{ij} parametrizes the waveform at spatial infinity : non-linearities during the wave propagation enter the game

$$U_{ij} = \frac{\mathrm{d}^2 \mathrm{I}_{ij}}{\mathrm{d}t^2} + \text{tail terms} + \text{memory terms} + \dots$$



↪ The *source* quadrupole I_{ij} describes the physics near the objects.

Computing $\mathcal{F}$: The source quadrupole

- The Newtonian mass moments are simply

$$\mathrm{I}_L^N = \int \mathrm{d}^3\vec{x}\, \rho(\vec{x})\, \hat{x}_L\,,$$

with $\hat{x}_L = \mathrm{STF}[x_L]$, *eg* $\hat{x}_{ij} = x_i x_j - r^2\delta_{ij}/3$.

$\Rightarrow$ The relativistic source mass moments are[20]

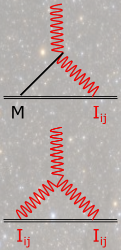
$$\mathrm{I}_L = \underset{B}{\mathrm{FP}} \int \mathrm{d}^3\vec{x} \left(\frac{r}{r_0}\right)^B \int_{-1}^{1} \mathrm{d}z \left\{ \delta_\ell\, \hat{x}_L\, \Sigma - \frac{\alpha_\ell\, \delta_{\ell+1}}{c^2} \hat{x}_{aL}\, \dot{\Sigma}_a + \frac{\beta_\ell\, \delta_{\ell+2}}{c^4} \hat{x}_{abL}\, \ddot{\Sigma}_{ab} \right\},$$

where the energy densities Σ, Σ_a, Σ_{ab} are evaluated in $(\vec{x}, t + zr/c)$ and PN-expanded and $\delta_\ell(z)$ takes into account the propagating nature of GR.

[20] L. Blanchet, CQG **15**(1998)1971.

Computing $\mathcal{F}$: The non-linearities

- The non-linearities are of three types :



↪ The "tail" type, that are created by the scattering of the waves onto the static curvature of the space-time,

↪ The "memory" type, that are created by the diffusion of the waves,

↪ The "gauge" type, coming from a diffeormorphism relating the source moments to the propagating modes.

⇒ The "tail-of-memory" appearing at 4PN is currently under investigation,[21]

⇒ The "gauge" interactions have been computed up to 4PN.[22]

[21] D. Trestini, FL & L. Blanchet, arXiv:2209.02719;
D. Trestini & L. Blanchet, *in prep.*

[22] L. Blanchet, G. Faye & FL, CQG **39**(2022)195003.

Computing $\mathcal{F}$: Dealing with divergences.

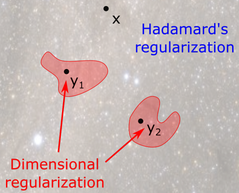
- During the computation of I_{ij}, two types of divergences strike

↪ UV divergences, due to the point-particle approximation.

$$\rho(\vec{x}) = m_1\,\delta(\vec{x}-\vec{y}_1) + m_2\,\delta(\vec{x}-\vec{y}_2)\,.$$

⇒ We have to use dimensional regularization
ie. compute I_{ij} in d-dimension,
and take $d \to 3$ at the end of the day.

⇝ In practice we compute I_{ij} in 3D,
and add the difference around the singularities.



- The full UV-regularized I_{ij} has been computed[23]

[23] T. Marchand, Q. Henry, FL, S. Marsat, G. Faye & L. Blanchet, CQG **37**(2020)21.

Computing $\mathcal{F}$: Dealing with divergences.

- During the computation, two types of divergences strike

$\hookrightarrow$ IR ones, due to the extension of the PN approximation in all space.

$$\int \mathrm{d}^3\vec{x}\, F(t - r/c) \xrightarrow[\text{PN}]{} \sum_p \frac{(-1)^p}{p!} \int \mathrm{d}^3\vec{x}\, r^p \, \frac{\mathrm{d}^p F(t)}{c^p \mathrm{d}t^p}$$

- Formalism defined with Hadamard regularization, up to 3.5PN,

$\hookrightarrow$ We need to use dimensional regularization at 4PN.

- The full IR-regularized I_{ij} has been computed,[24]

$\hookrightarrow$ some divergences $\propto \frac{1}{d-3}$ remains at 3 & 4PN !

$\Rightarrow$ We need to compute the non-linear terms to compensate them.[25]

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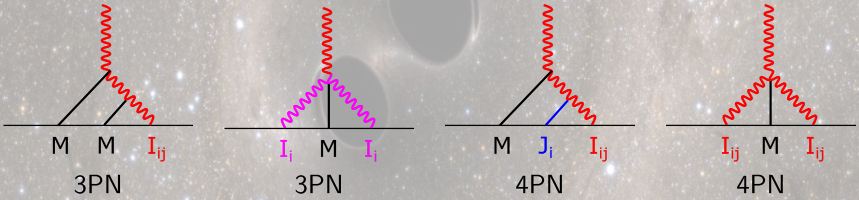
[24] FL, Q. Henry, L. Blanchet & G. Faye, CQG **39**(2022)115007.
[25] W. D. Goldberger & A. Ross, PRD **81**(2010)124015.

Computing $\mathcal{F}$: Dealing with divergences.

- The remaining divergences in I_{ij} have to cancel against those appearing during the computation of the non-linearities in U_{ij}.

$\hookrightarrow$ Those divergences arise from different interactions:



- We have thus computed the difference between Hadamard and dimensional regularization schemes of those interactions[26]

$\Rightarrow$ and found exactly what was needed to kill the divergence !

[26] FL, L. Blanchet, Q. Henry & G. Faye, CQG **39**(2022)115008.

Computing $\mathcal{F}$: Checking the result

We have two types of checks to test our final result

- "Internal" consistency checks

✓ All divergences have to cancel, by using the *same* coordinate shift as in the equations of motion,

↪ The scales associated with the Hadamard- and UV-regularization processes have to disappear.

- "External" consistency checks

↪ In the test-mass limit $m_2 \ll m_1$, we have to recover the known results for the phase[27] and the mode.[28]

↪ New possibility: comparing with second-order self-force results.[29]

[27] T. Tanaka, H. Tagoshi & M. Sasaki, Prog.Theor.Phys. **96**(1996)1087.
[28] R. Fujita & B. R. Iyer, PRD **82**(2010)044051.
[29] B. Wardell, A. Pound, N. Warburton, J. Miller, L. Durkan & A. Le Tiec, arXiv:2112.12265.

Summary and perspectives

- Computing the gravitational phase at 4PN is important for LISA, and will be crucial for ET.

↪ The only missing piece for the non-spinning waveform is the Hadamard regularized tail-of-memory, currently under investigation.

- What next ?

↪ The 5PN energy is practically known, discussions are ongoing,

↪ Is it realistic to compute the 5PN flux with current methods ?

↪ Inclusion of other physical and/or environmental effects ? For example magnetic fields in white dwarf binaries[30] ?

[30] A. Bourgoin, C. Le Poncin-Lafitte, S. Mathis & M.-C. Angonin, arXiv:2201.03226.


Thank you for your attention
φ_{4PN}
We are almost there...
... stay tuned !!