Recent Progress in Waveform Modelling using Perturbation Theory

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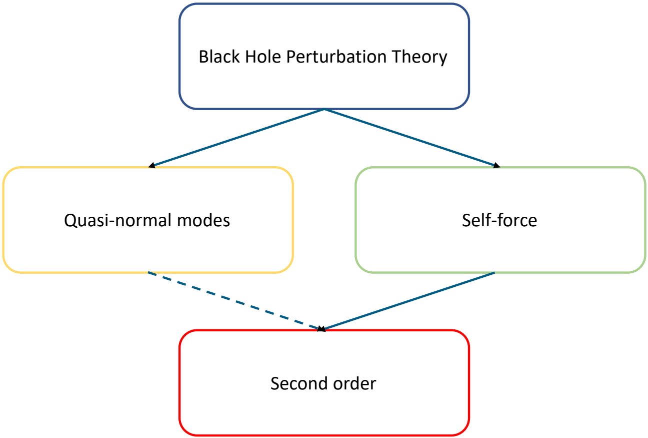
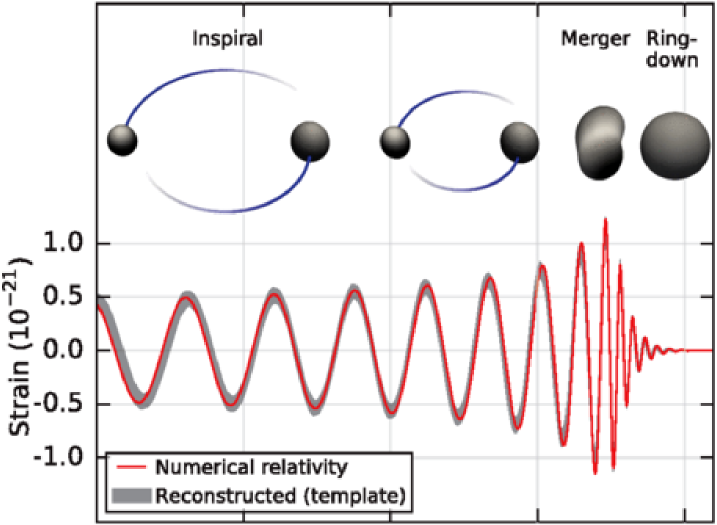
University of Nottingham

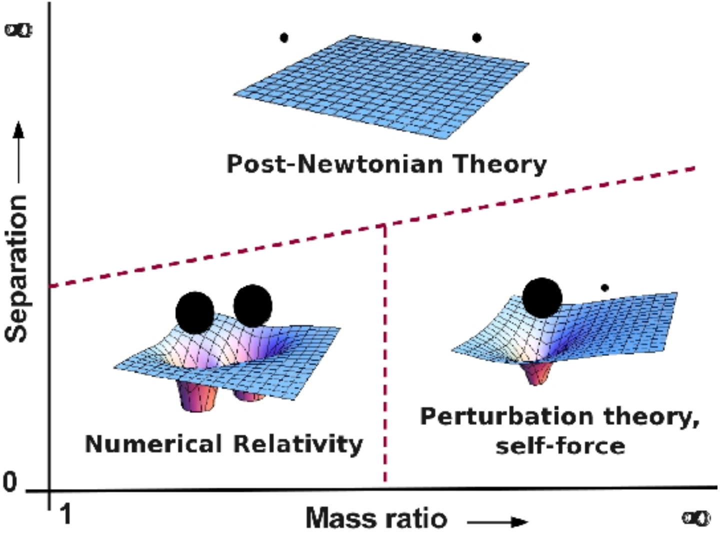
11th October 2022

Overview

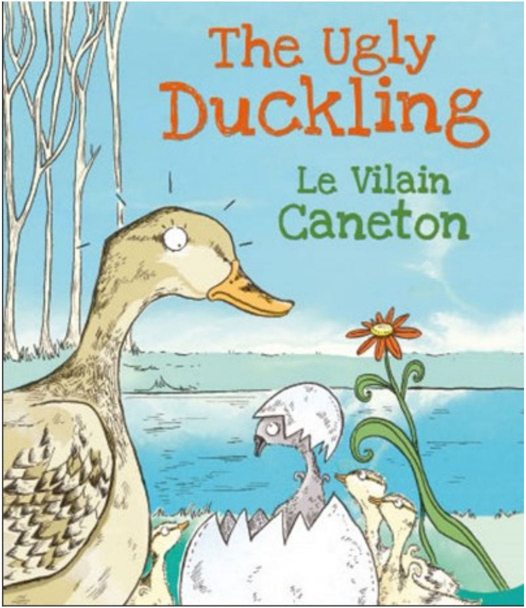

Black Hole Perturbation Theory
Quasi-normal modes
Self-force
Second order



[B.P. Abbott et al. Phys. Rev. Lett., 119(16):161101, 2017]



[L. Barack, DOI:10.1007/978-3-319-06349-2 Chap. 6 (2014).]


The Ugly
Duckling
Le Vilain
Caneton

Black Hole Perturbation theory basics

- Want to solve $G_{ab}[\mathbf{g}_{cd}] = T_{ab}$

Black Hole Perturbation theory basics

- Want to solve $G_{ab}[\mathbf{g}_{cd}] = T_{ab}$
- **But**, G_{ab} is non-linear in $\mathbf{g}_{cd}$.

Black Hole Perturbation theory basics

- $\mathbf{g}_{ab} = g^{(0)}_{ab} + \varepsilon h^{(1)}_{ab} + \varepsilon^2 h^{(2)}_{ab} + ... + \varepsilon^n h^{(n)}_{ab} + ...$

  where $|\varepsilon| \ll 1$ and $G_{ab}[g^{(0)}_{cd}] = 0$

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- $G_{ab}[\mathbf{g}_{ab}] \Rightarrow \delta G_{ab}[\varepsilon h^{(1)}_{ab} + \varepsilon^2 h^{(2)}_{ab} + ...]$
  $+\delta^2 G_{ab}[\varepsilon h^{(1)}_{ab} + \varepsilon^2 h^{(2)}_{ab} + ..., \varepsilon h^{(1)}_{ab} + \varepsilon^2 h^{(2)}_{ab} + ...] + ...$

Black Hole Perturbation theory basics

$$\varepsilon \delta G_{ab}[h_{ab}^{(1)}] + \varepsilon^2 \delta G_{ab}[h_{ab}^{(2)}] + \varepsilon^2 \delta^2 G_{ab}[h_{ab}^{(1)}, h_{ab}^{(1)}] \\ = \varepsilon T_{ab}^{(1)} + \varepsilon^2 T_{ab}^{(2)} + \mathcal{O}(\varepsilon^3),$$

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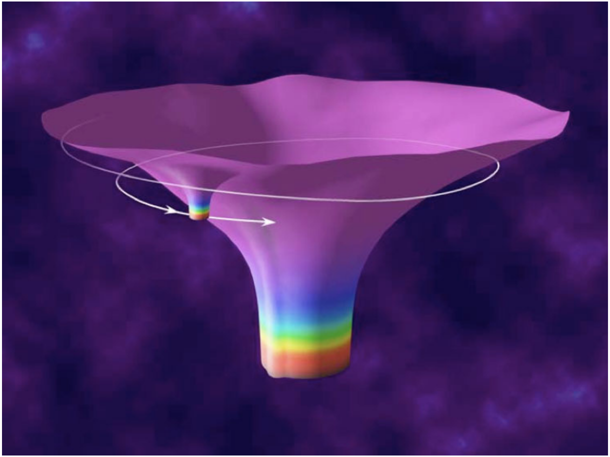
$$\Rightarrow \varepsilon\delta G_{ab}[h_{ab}^{(1)}] = \varepsilon T_{ab}^{(1)}$$

$$\varepsilon^2\delta G_{ab}[h_{ab}^{(2)}] = \varepsilon^2 T_{ab}^{(2)} - \varepsilon^2\delta^2 G_{ab}[h_{ab}^{(1)}, h_{ab}^{(1)}]$$

$$\varepsilon^3\delta G_{ab}[h_{ab}^{(3)}] = \ldots$$

Approximating the spacetime

$$\mathbf{g}_{ab} = g^{(0)}_{ab} + \varepsilon h^{(1)}_{ab} + \varepsilon^2 h^{(2)}_{ab} + \mathcal{O}(\varepsilon^3)$$

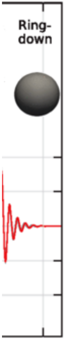


[NASA website]

Quasi-normal modes of black holes

- Solutions?

  Vacuum, $\varepsilon\delta G[h_{ab}^{(1)}] = 0$



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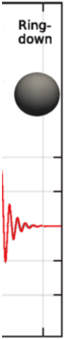

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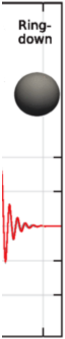

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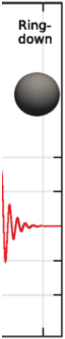

Quasi-normal modes of black holes

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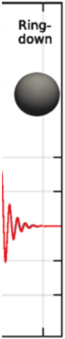

$\Rightarrow$ Gravitational waves $\sim e^{-i\omega t}$

$\Rightarrow$ Dissipating energy $\Rightarrow \mathrm{Im}[\omega] \neq 0$

$\Rightarrow$ Discrete set of quasi-normal mode frequencies $\omega_{n,l,m}$

[Chandrasekhar & Detweiler, https://doi.org/10.1098/rspa.1975.0112]

Are second-order Quasi-Normal modes Significant?

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$\Rightarrow$ Homogeneous solutions: $\omega = \omega_{n,l,m}$

$\Rightarrow$ Particular solutions $\omega = \omega_{n_1,l_1,m_1} + \omega_{n_2,l_2,m_2}$

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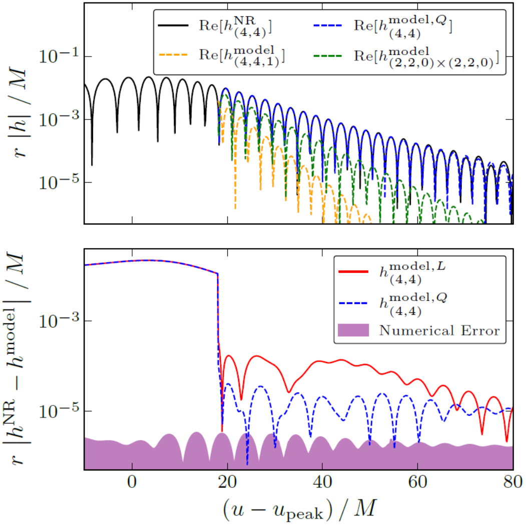
Some of these unique frequencies will be detectable!

[Cheung, et al.arXiv:2208.07374.] [Mitman, et al. arXiv:2208.07380.] [Lagos and Hui. arXiv:2208.07379.]

How important are the particular solutions



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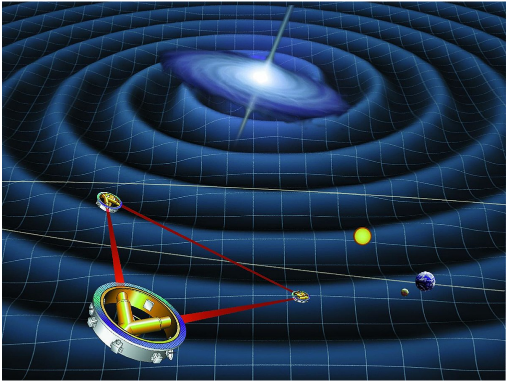
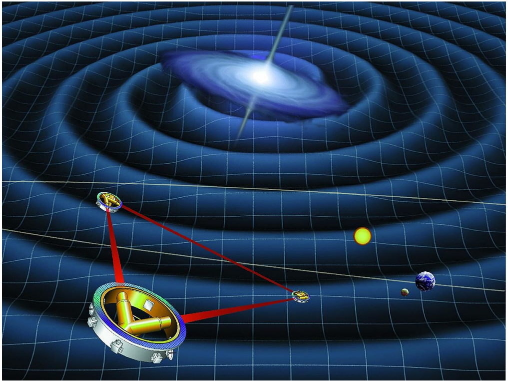
Extreme-Mass-Ratio Inspirals



[Source: NASA, http://lisa.jpl.nasa.gov/gallery/lisa-waves.html.]

LISA will be sensitive to Extreme-Mass-Ratio Inspirals

Extreme-Mass-Ratio Inspirals



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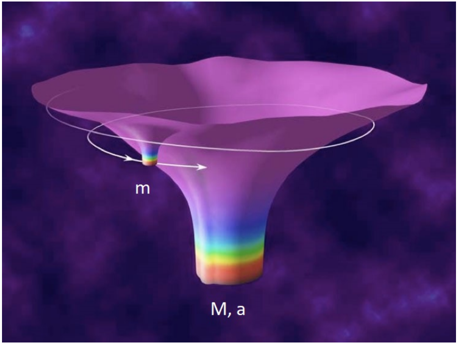
LISA will be sensitive to Extreme-Mass-Ratio Inspirals

Multi-year long measurements with space-filling orbits:

- **Precise** measurements
- Potential world leading tests of General Relativity

Perterbation theory for Extreme-Mass-Ratio Inspirals



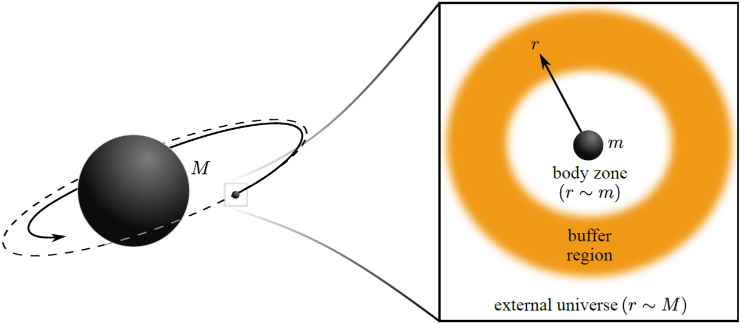
[Source: NASA website]

Expansion Parameter:
$\varepsilon \sim \frac{m}{M}$

Binary mechanics:
Gravitational **self-force**

$$ma^{\alpha} = \varepsilon F^{\mu}_{(1)diss}[h^{(1)}_{ab}] + \varepsilon^2 F^{\mu}_{(1)cons}[h^{(1)}_{ab}] + \varepsilon^2 F^{\mu}_{(2)diss}[h^{(2)}_{ab}] + \mathcal{O}(\varepsilon^3)$$

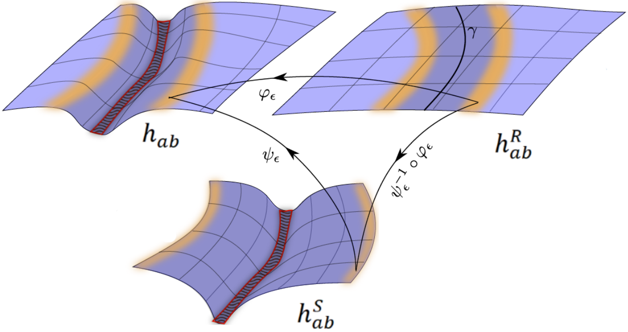
Matched asymptotic expansions



[Barack & Pound. Reports on Progress in Physics 82.1 (2018): 016904.]

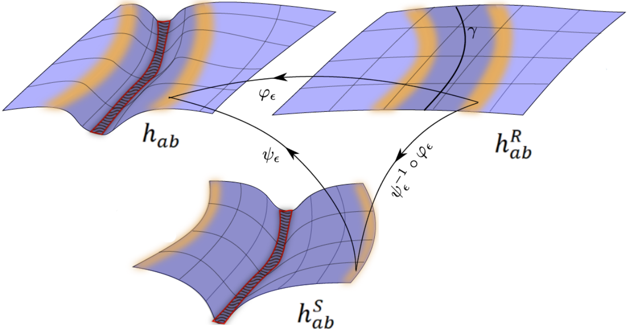
- $T^{(1)}_{ab}$ is effectively a point mass

Singular fields in self-force



[Pound. Springer, Cham, 2015. 399-486.]

Singular fields in self-force



[Pound. Springer, Cham, 2015. 399-486.]

The regular part of the field ($h^R_{ab} = h_{ab} - h^S_{ab}$) provides the self-force ($F^\mu[h^R_{ab}]$)

[Detweiler & Whiting. PRD 67.2 (2003): 024025.]

Challenges at second order

Solving $\varepsilon^2 \delta G_{ab}[h^{(2)}_{ab}] = \varepsilon^2 T^{(2)}_{ab} - \varepsilon^2 \delta^2 G_{ab}[h^{(1)}_{ab}, h^{(1)}_{ab}]$

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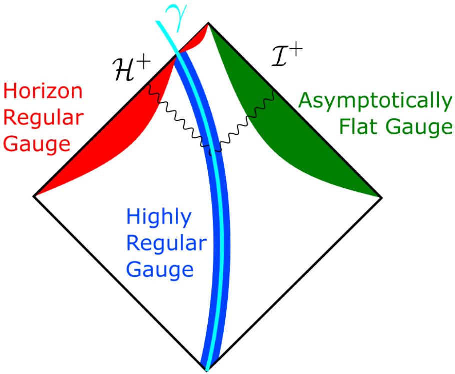
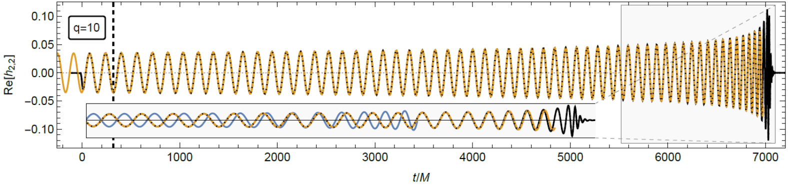


Second-order self-force waveforms (Schwarzschild)



[Wardell, et al. arXiv:2112.12265.]

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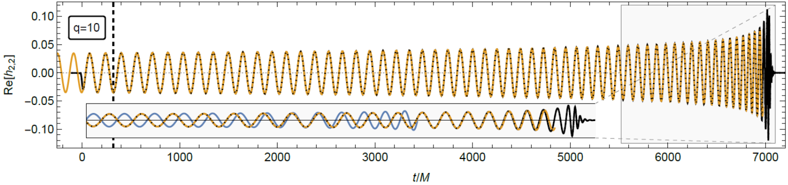

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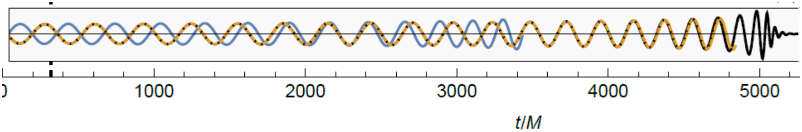

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The future for self-force waveforms

- Extension to Kerr

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Thank you for listening (email: a.r.c.spiers@nottingham.ac.uk)