METRIC RECONSTRUCTION FOR NON-RADIATIVE SPACETIMES

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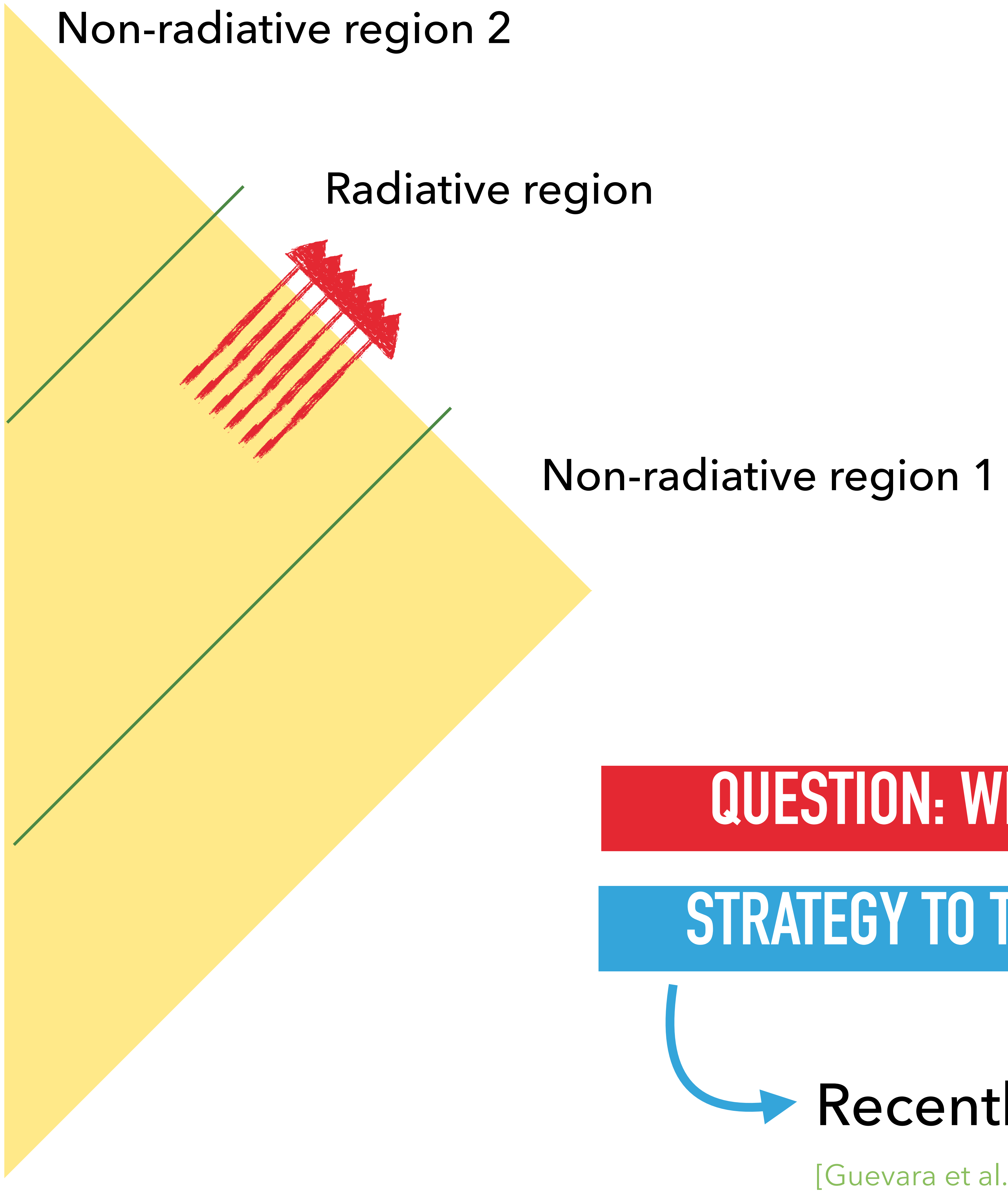
GdR Ondes Gravitationnelles @ IJCLab, Orsay

Based on ArXiv 2206:XXXX with Geoffrey Compère and Ali Seraj

on ArXiv 2010:10000 + work in progress with Luc Blanchet, Geoffrey Compère, Guillaume Faye, Ali Seraj

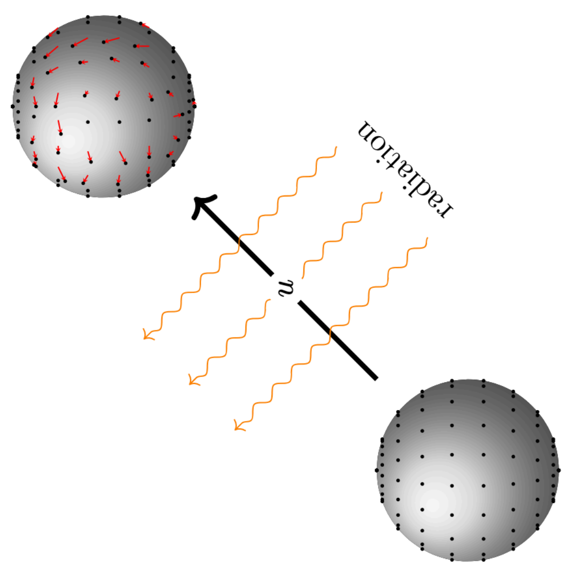
MOTIVATION



Non-radiative regions 1 and 2 differ from each other

Gravitational ``vacuum” is degenerate

E.g., supertranslations label gravitational vacua



$$\delta_T C_{ab} = -\,2D_{\langle a}D_{b\rangle}T(\theta,\phi)$$

QUESTION: WHAT CHARACTERISE COMPLETELY NON-RADIATIVE STATES?

STRATEGY TO TAKE: GO DEEPER IN THE INFRARED STRUCTURE OF GRAVITY

Recently discovered $Lw_{1+\infty}$ algebra

[Guevara et al., 2103.03961], [Strominger, 2105.14346], [Freidel-Pranzetti-Raclariu, 2112.15573]

BONDI GAUGE AND METRIC

Bondi coordinates: $\{u, r, \theta^a\}$

Bondi gauge: $g_{rr} = 0 = g_{ra}$ and $\partial_r \det\left(r^{-2} g_{ab}\right) = 0$

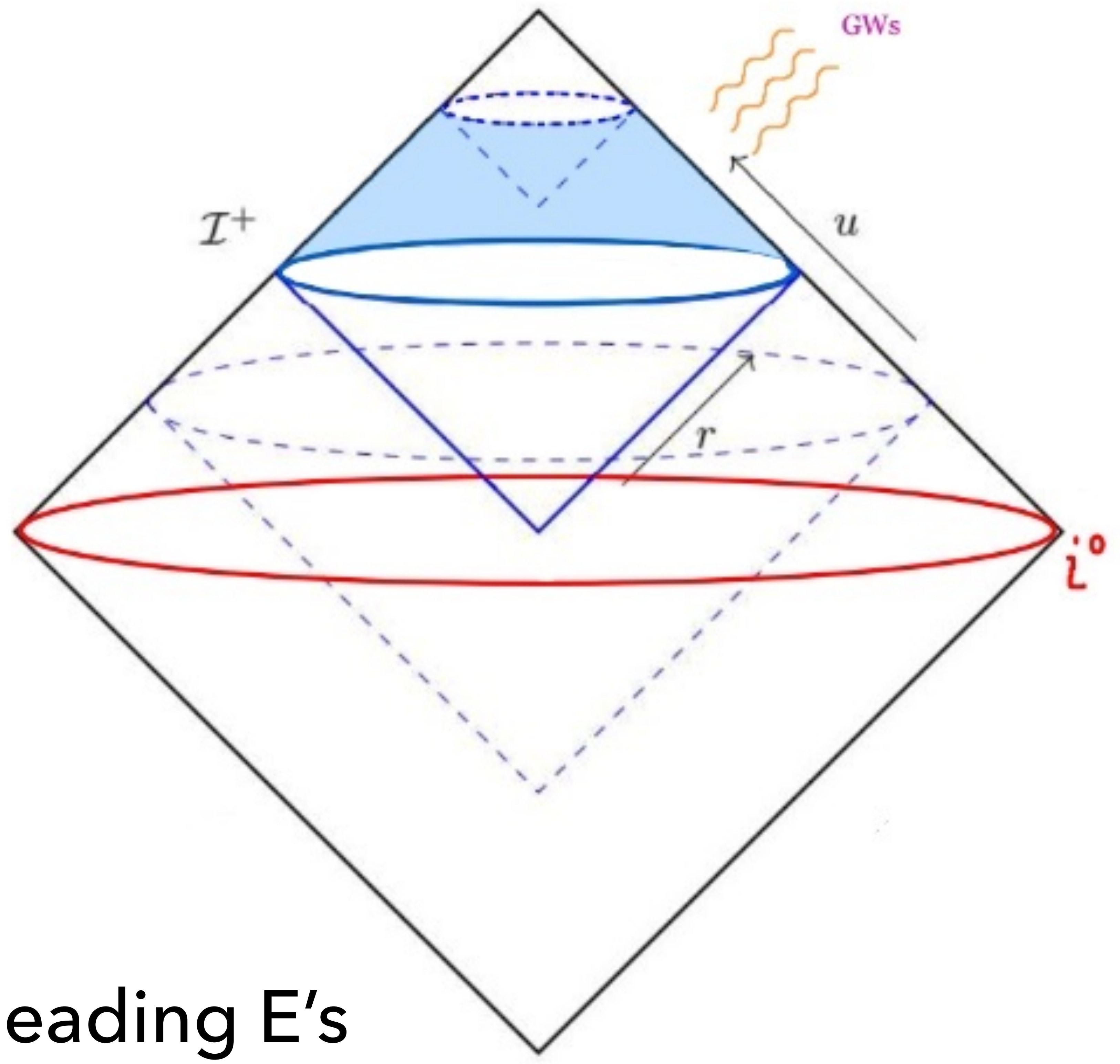
Bondi metric: $ds^2 = -\, e^{2\beta}\left(F du^2 + 2 du dr\right) + g_{ab}\left(d\theta^a - \frac{U^a}{r^2}du\right)\left(d\theta^b - \frac{U^b}{r^2}du\right)$

Asymptotic expansion: [Bondi-van der Burg-Metzner, 1962], [Sachs, 1962], [...], [Grant-Nichols, 2109.03832]

$$F = 1 - \frac{2Gm}{r} + \mathcal{O}(r^{-2})$$

$$g_{ab} = r^2\sqrt{1 + \frac{\mathscr{C}_{cd}\mathscr{C}^{cd}}{2r^2}}\gamma_{ab} + rG\mathscr{C}_{ab} \qquad \mathscr{C}_{ab} = C_{ab} + \sum_{n=2}^{+\infty} r^{-n} \underset{(n)}{E}_{ab}$$

$$g_{ab}U^b = G\left\{\frac{1}{2}D^b C_{ab} + \frac{1}{r}\left[\frac{2}{3}N_a - \frac{1}{16}\partial_u\left(C_{cd}C^{cd}\right)\right] + \mathcal{O}(r^{-2})\right\}$$



BONDI FIELDS: mass & angular momentum aspects, shear and sub-leading E's

LOCAL FLUX-BALANCE LAWS

In Bondi gauge, Einstein's equations reduce to a set of algebraic constraints in addition to a countable infinite set of local flux-balance equations on future null infinity:

$$N_{ab} = \partial_u C_{ab}$$

$$n=0 \quad : \qquad \frac{1}{4} D_b D_c N^{bc} = -\mathcal{F}(u) + \partial_u m,$$

$$n=1 \quad : \qquad -\frac{u}{2} D_c D_{\langle a} D_{b \rangle} N^{bc} = -\mathcal{F}_a(u) + \partial_u \mathcal{N}_a,$$

$$n=2 \quad : \qquad \frac{u^2}{12} \mathrm{STF}_{ab}[D_a D_c D_{\langle b} D_{d \rangle} N^{cd}] = -\underset{(2)}{\mathcal{F}_{ab}}(u) + \partial_u \underset{(2)}{\mathcal{E}_{ab}},$$

$$n \geq 3 \quad : \qquad \frac{(-u)^n}{6\, n!} \mathcal{D}_{n-3} \cdots \mathcal{D}_0 \, \mathrm{STF}_{ab}[D_a D_c D_{\langle b} D_{d \rangle} N^{cd}] = -\underset{(n)}{\mathcal{F}_{ab}}(u) + \partial_u \underset{(n)}{\mathcal{E}_{ab}} \,.$$

[Grant-Nichols, 2109.03832]
also [Freidel-Pranzetti-Raclariu, 2112.15573]

- LHS ~ LINEAR IN THE NEWS TENSOR, SOMETIMES REFERRED TO AS SOFT / MEMORY TERM
- RHS ~ FLUXES + TIME DERIVATIVE OF ``IMPROVED" BONDI FIELDS:

FLUXES VANISHES WHEN THE NEWS VANISHES

$$\mathcal{F} \equiv -\frac{1}{8} N_{ab} N^{ab}$$

``IMPROVED" BONDI FIELDS HAVE CONTRIBUTIONS PROPTO TO POWERS OF U

$$\mathcal{N}_a \equiv N_a - \frac{1}{4} C_{ab} D_c C^{bc} - \frac{1}{16} \partial_a (C_{bc} C^{bc}) - u D^b m_{ab}$$

BMS CHARGES AND HIGHER SPIN CHARGES

Let $\hat{n}_L = \text{STF}[n_{i_1 \ldots i_l}]$ be the symmetric and trace-free product of l unit directional vectors n_i

From the Bondi mass aspect ($n = 0$) and improved angular momentum aspect ($n = 1$):

$$\boxed{\mathcal{P}_L = \oint_S m\, \hat{n}_L, \qquad -\mathcal{J}_L = \frac{1}{2}\oint_S \epsilon^{ab}\partial_b \hat{n}_L \mathcal{N}_a, \qquad \mathcal{K}_L = \frac{1}{2}\oint_S \partial^a \hat{n}_L \mathcal{N}_a.}$$

[Compère, RO, Seraj, 1912.03164]

The 10 Poincaré charges are recovered for $l = \{0,1\}$. BMS charges are defined for $l \geq 2$.

From the Bondi improved sub-leading E's ($n \geq 2$):

$$\boxed{\mathcal{Q}^+_{n,L}(u) \equiv \oint_S \underset{(n)}{\mathcal{E}^{ab}}\, D_a D_b \hat{n}_L, \qquad \mathcal{Q}^-_{n,L}(u) \equiv \oint_S \underset{(n)}{\mathcal{E}^{ab}}\, \epsilon_{ac} D_b D^c \hat{n}_L.}$$

[Compère, RO, Seraj, *to appear*]

TWO QUALITATIVE DIFFERENT SETS OF LOCAL FLUX-BALANCE LAWS

Recall

$$n=0 \quad : \quad \frac{1}{4} D_b D_c N^{bc} = -\mathcal{F}(u) + \partial_u m,$$

$$n=1 \quad : \quad -\frac{u}{2} D_c D_{\langle a} D_{b\rangle} N^{bc} = -\mathcal{F}_a(u) + \partial_u \mathcal{N}_a,$$

$$n=2 \quad : \quad \frac{u^2}{12} \mathrm{STF}_{ab}[D_a D_c D_{\langle b} D_{d\rangle} N^{cd}] = -\underset{(2)}{\mathcal{F}_{ab}}(u) + \partial_u \underset{(2)}{\mathcal{E}_{ab}},$$

$$n \geq 3 \quad : \quad \frac{(-u)^n}{6\, n!} \mathcal{D}_{n-3} \cdots \mathcal{D}_0\, \mathrm{STF}_{ab}[D_a D_c D_{\langle b} D_{d\rangle} N^{cd}] = -\underset{(n)}{\mathcal{F}_{ab}}(u) + \partial_u \underset{(n)}{\mathcal{E}_{ab}}\,.$$

FACT 1:
LHS REMINDS THE MELLIN TRANSFORM:

$$\mathcal{M}_n[f] = \int_0^{+\infty} u^n\, f(u)$$

FACT 2:

$$\mathcal{D}_n \equiv -\frac{n+2}{2(n+1)(n+4)} (\Delta + n^2 + 5n + 2)$$

ANNIHILATES THE FIRST $l = n+2$ HARMONIC MODES

$$\partial_u \mathcal{Q}^+_{n,L}(u) = \oint_S \underset{(n)}{\mathcal{F}^{ab}}\, D_a D_b \hat{n}_L + \frac{(-u)^n}{6\ n!} \oint_S \hat{n}_L D^{\langle b} D^{a\rangle} \mathcal{D}_{n-3} \cdots \mathcal{D}_0 D_a D_c D_{\langle b} D_{d\rangle} N^{cd},$$

MEMORY-LESS FLUX-BALANCE LAWS

$n = 0 : \ \ell = 0$ ENERGY LOSS FORMULA AND $\ell = 1$ MOMENTUM LOSS FORMULA
$n = 1 : \ \ell = 1$ ANGULAR AND CENTER-OF-MASS LOSS FORMUALAE

$n = 2 : \varnothing$
$n \geq 3 : 2 \leq \ell \leq n-1$

MEMORY-FULL FLUX-BALANCE LAWS

$n = 0, \ell \geq 2$, DISPLACEMENT MEMORY EFFECT
$n = 1, \ell \geq 2$, SPIN AND CENTER-OF-MASS MEMORY EFFECTS

$n \geq 2, \ell \geq n$, SUBLEADING MEMORY EFFECTS

FACT3: IN NON-RADIATIVE REGIONS, THE HIGHER SPIN CHARGES ARE CONSERVED

HIGHER SPIN CHARGES - EXPLICIT EXPRESSIONS IN LINEARISED THEORY (1/2)

To compute $\mathcal{Q}^{+}_{n,L}(u) \equiv \oint_S \underset{(n)}{\mathcal{E}^{ab}}\, D_a D_b \hat{n}_L$, one needs access to the sub-leading E_{ab} in g_{ab}

Explicit expressions in linear theory of the Bondi fields are in [Blanchet et al., 2010:10000]

$$g_{uu} = -1 - G(\Delta + 2)\dot{f} + 2G\left(\frac{m}{r} + \sum_{n=2}^{+\infty} \frac{1}{r^n} \underset{(n)}{K}\right) + \mathcal{O}(G^2)\,,$$

$$g_{ua} = G\left(\frac{1}{2} D_b C^b_{\ a} + \frac{2}{3}\frac{N_a}{r} + e^i_a \sum_{n=2}^{+\infty} \frac{1}{r^n} \underset{(n)}{P^i}\right] + \mathcal{O}(G^2)\,,$$

$$g_{ab} = r^2\left[\gamma_{ab} + 2GD_{\langle a}Y_{b\rangle} + G\left(\frac{C_{ab}}{r} + e^i_{\langle a}e^j_{b\rangle} \sum_{n=2}^{+\infty} \frac{1}{r^n} \underset{(n)}{E^{ij}}\right)\right] + \mathcal{O}(G^2)\,.$$

"MONSIER DE DONDER MEETS THE SIR BONDI"

MPM (IN HARMONIC GAUGE) —> RADIATIVE/BONDI GAUGE

- **START FROM LINEARISED METRIC IN HARMONIC GAUGE**
- **IMPOSE RADIATIVE GAUGE CONDITIONS**
- **SOLVE THEM UP TO BMS TRANSFORMATIONS**
- **READ OFF THE RADIATIVE FIELDS IN TERMS OF MOMENTS**

In particular:

$$\underset{(n)}{E_{ab}} = 4 e^i_{\langle a} e^j_{b\rangle} \frac{n-1}{n+1} \sum_{\ell \geq n} \frac{1}{\ell!} \frac{(\ell+n)!}{2^n n! (\ell-n)!} n_{L-2} \left[M^{(\ell-n)}_{ijL-2} + \frac{2\ell}{\ell+1} \epsilon_{ipq} n_p S^{(\ell-n)}_{jqL-2}\right] + \mathcal{O}(G),$$

HIGHER SPIN CHARGES – EXPLICIT EXPRESSIONS IN LINEARISED THEORY (2/2)

In linear theory:

$$\mathcal{Q}^{+}_{n,L}(u) \equiv \oint_S \overset{(n)}{\mathcal{E}^{ab}}\, D_a D_b \hat{n}_L = \begin{cases} \displaystyle\sum_{p=n-l-1}^{n-3} q_{n,\ell,p}\, u^{p+1} M_L^{(\ell-n+p+1)} + b_{n,\ell}\, u^{n-1}\left(1-\frac{u}{n}\partial_u\right) M_L^{(\ell-1)} + \mathcal{O}(G) & 2 \le \ell \le n-1 \\ a_{n,\ell}\, M_L^{(\ell-n)} + \displaystyle\sum_{p=0}^{n-3} q_{n,\ell,p}\, u^{p+1} M_L^{(\ell-n+p+1)} + b_{n,\ell}\, u^{n-1}\left(1-\frac{u}{n}\partial_u\right) M_L^{(\ell-1)} + \mathcal{O}(G) & \ell \ge n \end{cases}$$

$\mathcal{Q}^{-}_{n,L}(u)$ same expression with $M_L \to \dfrac{2l}{l+1} S_L$

Assuming no-radiation: $$0 = N_{ab} = \dot{C}_{ab} \propto \sum_{\ell=2}^{+\infty} \frac{n_{L-2}}{\ell!}\left[M^{(\ell+1)}_{klL-2} - \frac{2\ell}{\ell+1}\varepsilon_{kpq} n_p S^{(\ell+1)}_{lqL-2}\right] + \mathcal{O}(G)$$

NON-RADIATIVE REGIONS: $$M_L^{(l+1)} = 0 = S_L^{(l+1)} \Longrightarrow M_L(u) = \sum_{k=0}^{l} M_{L,k}\, u^k \quad \text{and} \quad S_L(u) = \sum_{k=0}^{l} S_{L,k}\, u^k$$

THE HIGHER SPIN CHARGES FOR NON-RADIATIVE REGIONS : $$\begin{cases} 0 & 2 \le \ell \le n-1 \\ a_{n,\ell}\,(\ell - n)!\, M_{L,\ell-n} + \mathcal{O}(G) & \ell \ge n \end{cases}$$

HIGHER SPIN CHARGES - PHYSICAL INTERPRETATION

NON-RADIATIVE REGIONS: $M_L(u) = \sum_{k=0}^{l} M_{L,k}\, u^k \Longrightarrow \mathcal{Q}^+_{n,L} = a_{n,\ell}\,(\ell-n)!\, M_{L,\ell-n} + \mathcal{O}(G), \quad \ell \geq n \geq 2$

BMS charges		non-stationary features			Geroch-Hansen

$$M_L(u) = M_{L,\ell}\, u^\ell + M_{L,\ell-1}\, u^{\ell-1} + M_{L,\ell-2}\, u^{\ell-2} + \ldots + M_{L,1}\, u + M_{L,0}$$

$$\{g_{uu}, r g_{ra}, r^2 g_{ab}\} \propto \frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3} + \ldots + \frac{1}{r^\ell} + \frac{1}{r^{\ell+1}}$$

CHARACTERISATION OF NON-RADIATIVE REGIONS

=

BMS CHARGES ($k = \ell, k = \ell - 1$)

+

HIGHER SPIN CHARGES ($0 \leq k \leq \ell - 2$)

(Linear) displacement memory

$$C_{ab} = 4 e^i_{\langle a} e^j_{b\rangle} \sum_{\ell=2}^{+\infty} n_{L-2} \left[M_{ijL-2,\ell} - \cancelto{0}{\frac{2\ell}{\ell+1} \varepsilon_{ipq} n_p S_{jqL-2,\ell}} \right]$$

$k \rightarrow$; $L \downarrow$

$M_{\emptyset,0} \sim \mathcal{E}$			
$M_{i,0} \sim \mathcal{K}_i$	$M_{i,1} \sim \mathcal{P}_i$		
$M_{ij,0} \sim \mathcal{Q}^+_{2,ij}$	$M_{ij,1} \sim \mathcal{K}_{ij}$	$M_{ij,2} \sim \mathcal{P}_{ij}$	
$M_{ijk,0} \sim \mathcal{Q}^+_{3,ijk}$	$M_{ijk,1} \sim \mathcal{Q}^+_{2,ijk}$	$M_{ijk,2} \sim \mathcal{K}_{ijk}$	$M_{ijk,3} \sim \mathcal{P}_{ijk}$

NEW PERSPECTIVES

Higher spin charges defined here $\mathcal{Q}^{\pm}_{n,L}$ are (proportional to) gravitational multipole moments

The $\mathcal{Q}^{\pm}_{n,L}$ are proportional to the (real part of the) $Lw_{1+\infty}$ charges proposed in [Freidel-Pranzetti-Raclariu, 2112.15573]

$$\partial_u Q_{a_1\cdots a_s} = D_{\langle a_1} Q_{a_2\cdots a_s\rangle} + \frac{s+1}{2} C_{\langle a_1 a_2} Q_{a_3\ldots a_s\rangle}.$$

$$\{Q_s(\tau), Q_{s'}(\tau')\}^{\text{lin}} = (s'+1) Q_{s+s'-1}(\tau' D \tau) - (s+1) Q_{s+s'-1}(\tau D \tau'),$$

TO DERIVE: ALGEBRA OF MULTIPOLE MOMENTS

$$\left[T^{(\Delta,-s)}, T'^{(\Delta',-s')}\right]_*^{a_1\cdots a_{s+s'-1}} = \frac{n_s n_{s'}}{n_{s'+s-1}} \frac{(s+s'-1)!}{s!s'!} s T^{b(a_1\cdots a_{s-1}}\mathcal{D}_b T'^{a_s\cdots a_{s+s'-1})} - s' T'^{b(a_1\cdots a_{s'-1}}\mathcal{D}_b T^{a_{s'}\cdots a_{s+s'-1})}$$

[Compère, RO, Seraj, *to appear*]

THANK YOU FOR YOUR ATTENTION!

QUESTIONS?