Critical phenomena for dually weighted graphs and spanning forests via matrix models

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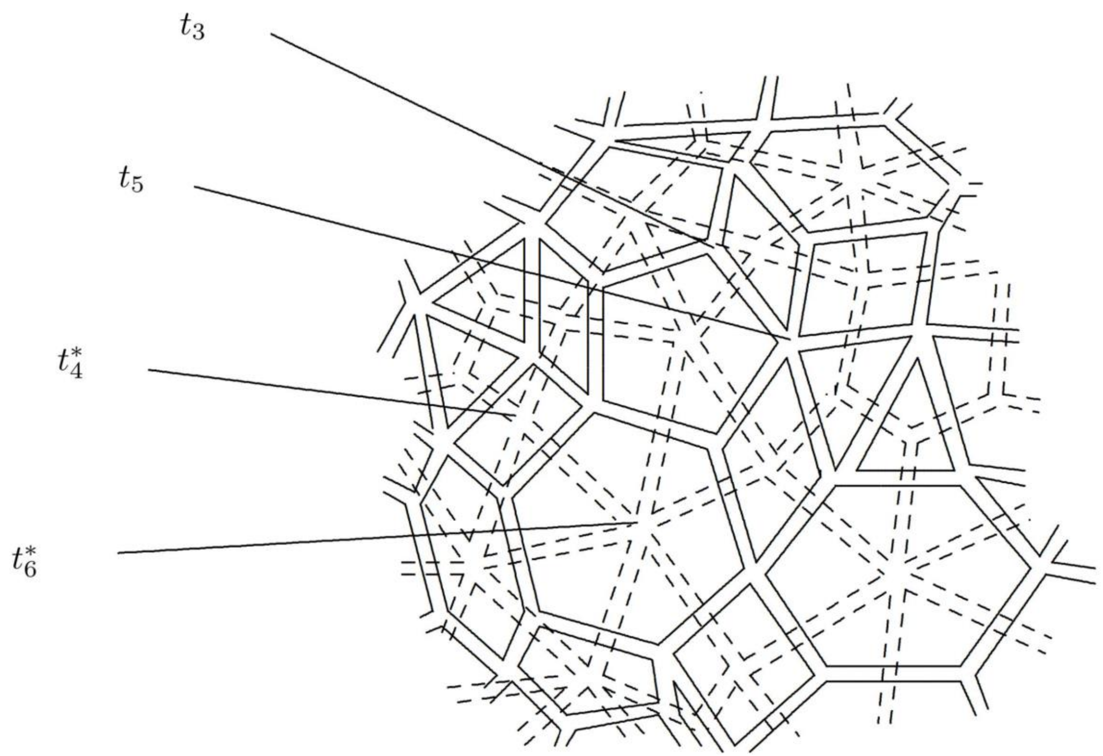
based on

[JHEP 01 (2022) 190, arXiv:2110.10104] with V. Kazakov

[to appear] with A. Gorsky, V. Kazakov, V. Mishnyakov

PART 1

Dually weighted graphs



Dually weighted graphs (DWG) partition function

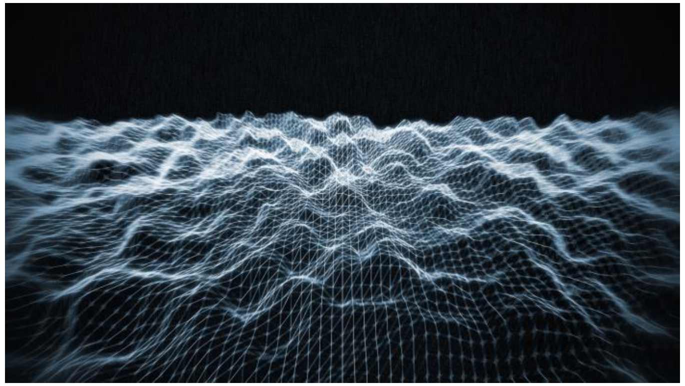
$$Z_N(t,t^*) = \sum_G N^{2-2g_G} \prod_{v_q^*, v_q \in G} t_q{}^{\# v_q}\, t_q^*{}^{\# v_q^*}$$

- counting of graphs, polyhedra, tilings

Related works in mathematics [Kazarian, Zograf, Zorich, …]

- 2d quantum gravity

Dual weights control
local curvature (R^2 coupling)

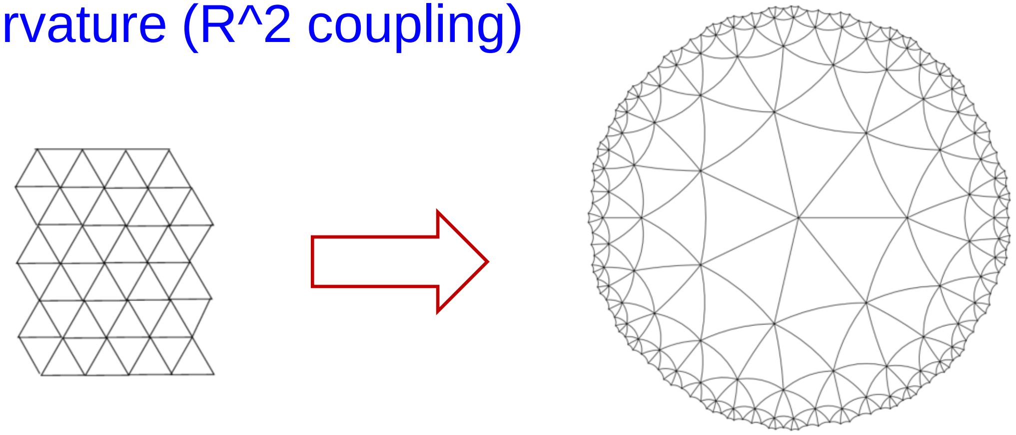


continuum limit gives
sum over surfaces

Our focus:
close to flat space

Kazakov '85 David '85
Kazakov, Kostov, Migdal '85

Kazakov, Staudacher, Wynter '96



Lattice model of AdS_2 ?

http://www.geom.at/fade2d/html/

DWG matrix model

Das, Dhar, Sentgupta, Wadya '90
Kazakov, Staudacher, Wynter '95

Matrix models provide a useful description for 2d gravity

Matrix integral $\longrightarrow$ sum over graphs $\longrightarrow$ sum over 2d surfaces

large N, continuum limit

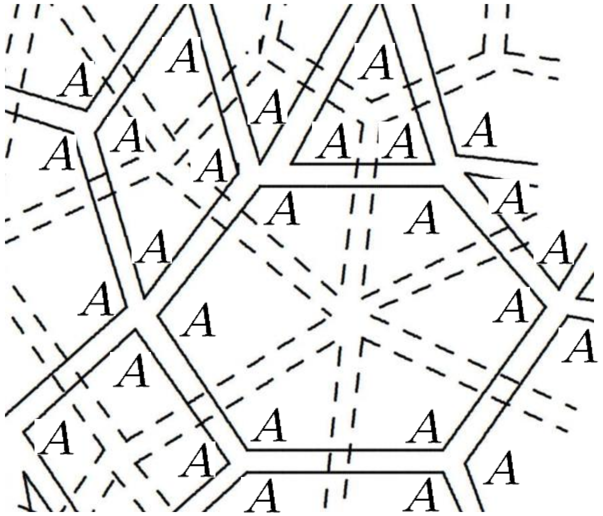
We study a refined version of usual 1-matrix model

$$Z_N(t,t^*) = \int \mathcal{D}M \exp N \,\mathrm{tr}\left(-\frac{1}{2}M^2 + \sum_{q=1}^{Q+1}\frac{1}{q}t_q(AM)^q\right)$$

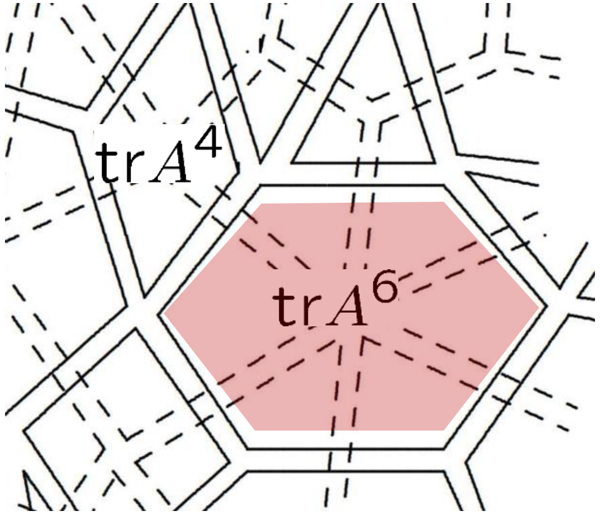
n-vertex $\to t_n$

n-face $\to {t_n}^*$

Dually weighted graphs (DWG)



$$t_n^* = \frac{1}{N}\mathrm{tr}A^n$$



Dual couplings $\longrightarrow$ better control the geometry of graphs / surfaces

DWG matrix model

Das, Dhar, Sentgupta, Wadya ’90
Kazakov, Staudacher, Wynter ‘95

$$Z_N(t,t^*) = \int \mathcal{D}M \exp N\ \mathrm{tr}\left(-\frac{1}{2}M^2 + \sum_{q=1}^{Q+1} \frac{1}{q} t_q (AM)^q\right)$$

Goal: compute observables like $\langle \mathrm{Tr} M^n \rangle$ and $\langle \mathrm{Tr}(AM)^n \rangle$, explore continuum limit

Recent motivation – Jackiw-Teitelboim (JT) gravity

Saad, Shenker, Stanford ‘19
Witten, Mirzakhani

Gravity theory in AdS_2, fascinating dualities with matrix models as well as AdS_2/CFT_1

Speculative hope – study it using DWG

Solving the matrix model

Kazakov, Staudacher, Wynter ’95, ‘96

DWG Matrix Model and sum over Young tableaux

Di Francesco, Itzykson '93
Kazakov, Staudacher, Wynter '95

$$Z_N(t,t^*) = \int (d^{N^2}M)\ \exp\left(-\frac{1}{2}\mathrm{tr}M^2 + \sum_{q=1}^{Q+1}\frac{1}{q}t_q\,\mathrm{tr}(AM)^q\right)$$

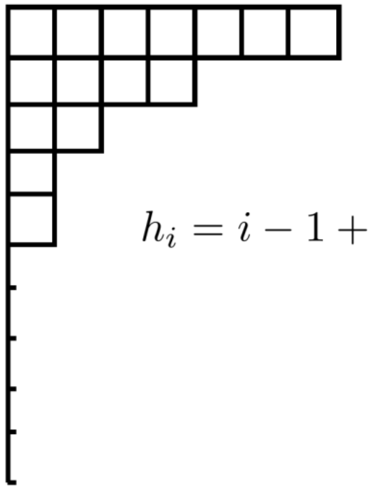
Cannot reduce to eigenvalues like usual MM, instead use characters

$$\exp\mathrm{tr}\left(\sum_{q=1}^{Q+1}\frac{1}{q}t_q(AM)^q\right) = \sum_{\{h\}}\chi_{\{h\}}[t]\ \chi_{\{h\}}(MA) \overset{\Omega\, int}{\Longrightarrow} \sum_{\{h\}}\frac{1}{\mathrm{dim}_h}\chi_{\{h\}}[t]\ \chi_{\{h\}}[t^*]\chi_{\{h\}}(M) \qquad t_q^* = \frac{1}{N}\mathrm{tr}A^q$$

sum over Young tableaux

angular integration $\quad MA \to M\Omega^\dagger A\Omega\,, \qquad \Omega \in SU(N)$

$i = \mathrm{N}$



i $\qquad h_i = i - 1 + \#\ \text{boxes in row}\ i$

$i = 1$

Schur character: $$\chi_{\{h\}}[t] = \det_{1\le j,k\le N} P_{h_j - j + 1}[t]\,,$$
$$\sum_{n=1}^{\infty} z^n P_n[t] = \exp\left(\sum_{m=1}^{\infty} z^m t_m\right)$$

What remains is a Gaussian integral of character

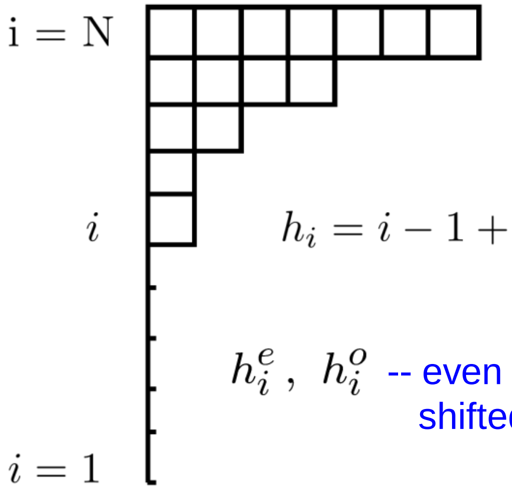
DWG Matrix Model and sum over Young tableaux

Di Francesco, Itzykson '93
Kazakov, Staudacher, Wynter '95

see also Kostov, Staudacher, Wynter 97

Result for partition function:

$$Z_N(t, t^*) = c(N) \sum_{\{h^e, h^o\}} \frac{\prod_i (h_i^e - 1)!!\, h_i^o!!}{\prod_{i,j} (h_i^e - h_j^o)} \chi_{\{h\}}[t]\, \chi_{\{h\}}[t^*]$$



$h_i = i - 1 + \#$ boxes in row i

h_i^e , h_i^o -- even and odd
shifted highest weights

Large Young tableaux dominate at large N

Saddle point gives a single Young tableau

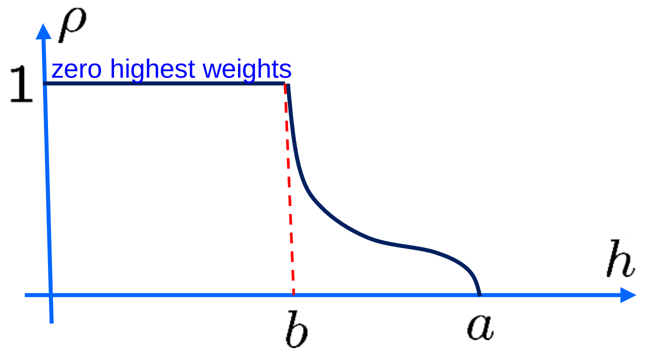
$$h_i \sim N$$

$$\chi_{\{h\}} \sim e^{N^2(\dots)}$$

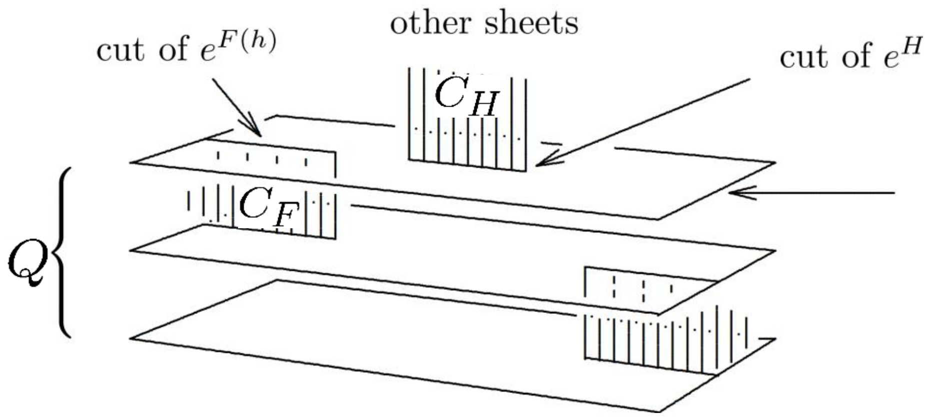
Algebraic curve of DWG

At large N we parameterize the dominating Young tableau by some continuous shape

$$H(h) = \frac{1}{N}\sum_{k=1}^{N}\frac{1}{h - h_k/N} \qquad H(h) = \int_0^a \frac{\rho(s)ds}{h-s}$$



It can be found from a nontrivial Riemann-Hilbert problem



$$he^{-H(h)} = \frac{(-1)^{Q-1}}{t_Q}\prod_{q=1}^{Q} g_q(h)$$

Product over Q sheets of Riemann surface of $g(h)$

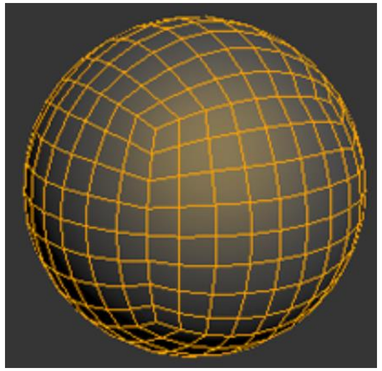
$$h = \sum_{q=1}^{Q+1} t_q\, g^q + \sum_{k=0}^{\infty}\langle\frac{\mathrm{tr}}{N}(MA)^k\rangle\, g^{-k}$$

This gives (in principle) solution of the model

Special case: quadrangulations

Disc Quadrangulations

Particular case: counting quadrangulations of sphere and disc



Here we impose :

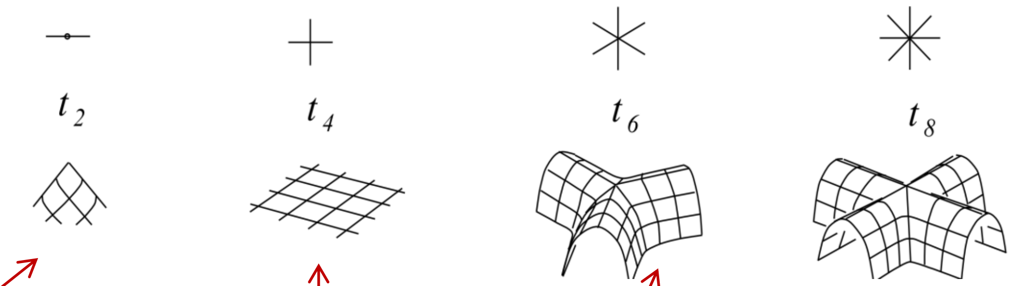
$$t_q^* = \lambda\, \delta_{q,4}$$

Special choice of weights makes the model tractable:

$$t_2 = \lambda\, \beta^2, \qquad t_{2n} = \lambda$$

Kazakov, Staudacher, Wynter '96

Build surface from:



....

positive curvature flat negative curvature

Overall curvature must be balanced (sphere topology)

Graph combinatorics gives

$$Z \propto \sum_G \lambda^F \beta^{2(\#v_2-4)}$$

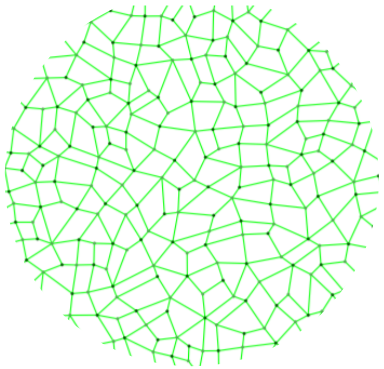
λ controls area $\quad$ β controls curvature

Z was computed by [Kazakov, Staudacher, Wynter 96] Corresponds to sphere topology

We will compute two resolvents, i.e. disc partition functions

Type 1 $\mathcal{W}(g) = \langle \frac{\mathrm{tr}}{N} \frac{1}{g - MA} \rangle$

Type 2 $W(g) = \langle \frac{\mathrm{tr}}{N} \frac{1}{g - M} \rangle$



Disc topology, two types of boundary

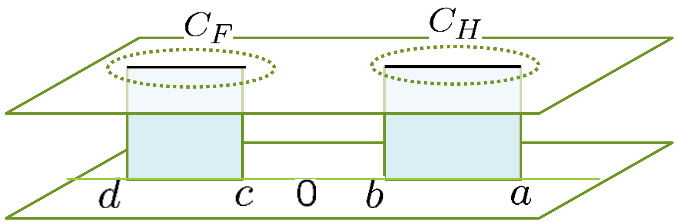
$$W(g) = \sum_{n \geq 0} \frac{1}{g^{n+1}} \frac{1}{N} \langle \mathrm{tr} M^n \rangle$$

n = length of boundary

Saddle point equation

We have a 2-cut problem

$$h - 1 = \frac{\lambda}{\beta^2}\left((\beta^2-1)g^2 + \frac{g^2}{1-\beta^2 g^2}\right) + \mathcal{O}(1/g)$$



Elliptic Riemann surface of $2F + H + \log h$

Saddle point equation: $2F + \cancel{H} + \log h = 0\,, \qquad h \in [b,a]$

From algebraic curve: $2\cancel{F} + H + \ln h = \log[\lambda(\beta^2-1)]\,, \qquad h \in [d,c]$

Solved by elliptic integrals

$$H(h)+2F(h) = \log\frac{(h-c)^2}{h(h-b)} + r(h)\left[\oint_{C_H}\frac{dh}{2\pi i}\frac{\log\frac{(s-b)}{(s-c)^2}}{(h-s)r(s)} + \oint_{C_F}\frac{dh}{2\pi i}\frac{\log\frac{\lambda(\beta^2-1)(s-b)}{(s-c)^2}}{(h-s)r(s)}\right]$$

$$r(x) = \sqrt{(a-x)(x-b)(x-c)(x-d)}$$

Solution of Riemann-Hilbert problem

We get density in terms of theta-functions Kazakov, Staudacher, Wynter ‘96

$$\rho(h) = \frac{u}{K} - \frac{i}{\pi}\ \ln\left[\frac{\theta_4(\frac{\pi}{2K}(u-iv),q)}{\theta_4(\frac{\pi}{2K}(u+iv),q)}\right]$$

$$u = \mathrm{sn}^{-1}\sqrt{\frac{(a-h)(b-d)}{(h-d)(a-b)}}$$

Then we find H(h) etc

$$H(h) = \int_b^a \frac{\rho(s)ds}{h-s}$$

(already not a standard special function)

Parameters are fixed from consistency requirements (e.g. asymptotics)

$$\mathrm{sn}^2(v,m') = \sqrt{\frac{a-c}{a-d}} \qquad m = \frac{(a-b)(c-d)}{(a-c)(b-d)}$$

$$v = -K' - \frac{K}{\pi}\log(\lambda(1-\beta^2)) \qquad \ldots$$

Asymptotics of Large Area & Boundary

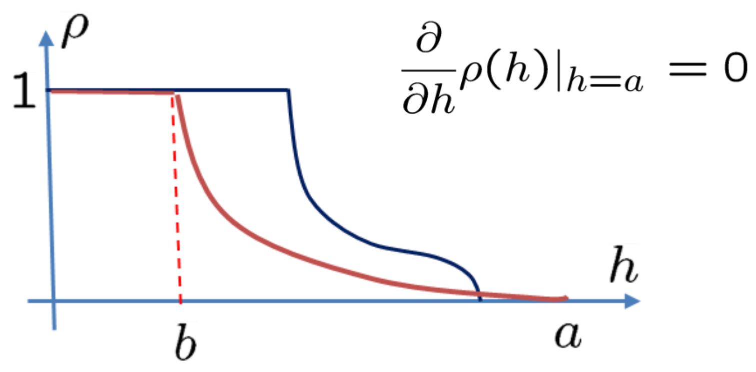
Search for universal regimes
when the details of discretization do not matter

Quadrangulations become continuous 2d geometries

Asymptotics of Large Area & Boundary

Kazakov, Staudacher, Wynter '96

$$Z \propto \sum_G \lambda^F \beta^{2(\#v_2 - 4)}$$

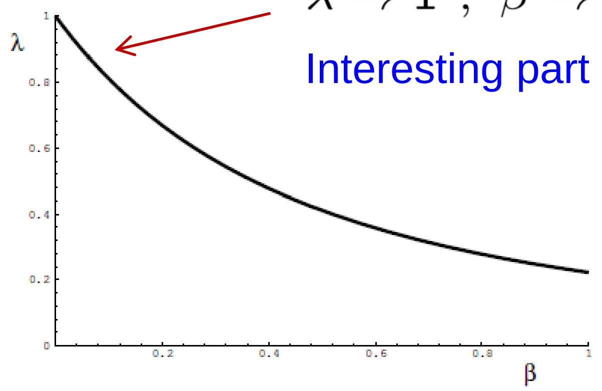


Critical regime: density has zero slope at endpoint

Gives $\lambda_c(\beta)$

$\lambda \to 1$, $\beta \to 0$

Interesting part (the rest gives pure gravity)



elliptic modulus

$$\lambda \simeq q = e^{-\pi\tau} \qquad \tau \to 0$$

Parametrise via $x = \frac{\sqrt{2}(1-\lambda)}{\pi\beta}$

$$V = \frac{1}{\sqrt{2}}(x - \sqrt{x^2 - 1})$$

finite ratio

Three key regimes:

- All orders in τ (exp precision)
- Flattening: 1st order in τ, $V =$ finite
  Small curvature, various n-vertices
- Almost flat: 1st order in τ with $V \to 0$
  Small curvature, only 2- and 4- vertices

Solution in exp. approximation

Kazakov, FLM

$\mathcal{O}(e^{-\frac{\pi}{\tau}})\,,\qquad \tau\to 0$ We need to compute $H(h)=\int_b^a\frac{\rho(s)ds}{h-s}$

Density simplifies $\rho=-\frac{i}{\pi}\log\left[-\frac{\theta_1\left(\frac{\pi}{2K}(u-i(v+K')),q\right)}{\theta_1\left(\frac{\pi}{2K}(u+i(v+K'),q\right)}\right]\simeq\frac{4\tau}{\pi^2}u(v+\pi/2)-\frac{i}{\pi}\log\left(\frac{\cosh\left(u-iv\right)}{\cosh\left(u+iv\right)}\right)$

We get integrals of the kind $\int_0^\infty dw\left(\frac{\sinh w}{\cosh w+\sinh u}+\dots\right)\log\frac{\cosh(w+iv)}{\cosh(w+iv)}$ $\longrightarrow$ Lots of polylogs

Long calculation gives result in parametric form $h\;=\;b+e^{\tau(4v^2/\pi-\pi)}\frac{\sin^2(2v)}{\sin^2(v)-\sin^2(r)}\,,\quad g^4=\frac{1}{h-b}\,e^{-\frac{i}{\pi}F(r)}$

where

$$\begin{aligned}F\;=\;&-\frac{i}{\pi}\tau(2v+\pi)\left(-2r^2-2\pi r+2v^2+\pi^2\right)-\left(2(r-v)(r-3v)-2\pi r+\frac{5\pi^2}{6}\right)\\&+i(2v-2r-\pi)\log\left(1-e^{2i(r-v)}\right)+i(2(r+v)+\pi)\log\left(1-e^{2i(r+v)}\right)-\frac{i}{6\pi}2\pi\log\left(1+e^{2iv}\right)\\&+\mathrm{Li}_2\left(e^{-2i(r-v)}\right)+\mathrm{Li}_2\left(e^{2i(r+v)}\right)+2\,\mathrm{Li}_2\left(1-e^{2iv}\right)+2\,\mathrm{Li}_2\left(1+e^{2iv}\right)\end{aligned}$$

[thanks to Omer Gurdogan
for help with polylogs]

Parameters from couplings:

$$\tau=-\frac{\log[\lambda(1-\beta^2)]}{2v+\pi}\qquad v=V\beta$$

$$\lambda\simeq(\lambda(1-\beta^2))^{1-\frac{2v}{\pi}}\left[\left(1-\log[\lambda(1-\beta^2)]\left(\frac{2v}{\pi}+1\right)\right)\sin(2v)+\pi(\frac{2v}{\pi}+1)\cos(2v)\right]$$

Solution in exp. approximation

$$\begin{aligned} h &= b + e^{\tau(4v^2/\pi - \pi)} \frac{\sin^2(2v)}{\sin^2(v) - \sin^2(r)} \\ g^4 &= \frac{1}{h-b}\, e^{-\frac{i}{\pi} F(r)} \end{aligned}$$

$$h = \sum_{q=1}^{Q+1} t_q\, g^q + \sum_{k=0}^{\infty} \langle \frac{\mathrm{tr}}{N} (MA)^k \rangle\, g^{-k}$$

A rich phase structure to be explored

Describes large 2d surfaces with strong fluctuations of curvature

We can compare $\langle \mathrm{Tr} M^2 \rangle$ with exact result

[Kazakov, Staudacher, Wynter 96]

$$\begin{aligned} \langle \mathrm{Tr} M^2 \rangle = \frac{\xi^2}{4\lambda} \Big[& \frac{K k'^2 \mathrm{sn}^2}{8\pi\ \mathrm{cn}\ \mathrm{dn}} (1 - k'^2 \mathrm{sn}^4) - \frac{k'^2 \mathrm{sn}^2}{4} - \frac{1}{2\pi} (K\, \Xi - \frac{\pi}{2})(\mathrm{dn}^2 + k'^2 \mathrm{cn}^2) \\ & + \frac{3\, \Upsilon\, \Phi}{8\pi\ \mathrm{sn}^2} - \frac{\Phi^2}{2\pi^2 \mathrm{sn}^2} - \frac{K\, \Upsilon}{4\pi\ \mathrm{sn}} (1 - \mathrm{dn}^2) + \frac{\Phi\, \Xi}{2\pi\ \mathrm{sn}} + \frac{\Omega\, \Phi}{\pi\ \mathrm{sn}} - \frac{\Omega^2}{2} \Big] \end{aligned}$$

Expand our solution to 5th order in tau, find pefect match!

$$\langle \mathrm{Tr} M^2 \rangle = 1 - \frac{8\beta^5 V^3 (6V^2 - 5)}{15\pi} + \dots$$

‘Flattening’ regime – type 1 resolvent

To get “flattening” limit we expand to leading order in τ

$$\beta \sim v \sim (1-\lambda) \sim \tau \to 0$$

Curvature is suppressed, Young tableau almost empty

Resolvent = disc partition function:

$$\mathcal{W}(g) = \frac{1}{g} + \frac{1}{g^3}\langle\frac{\text{tr}}{N}(AM)^2\rangle + \frac{1}{g^5}\langle\frac{\text{tr}}{N}(AM)^4\rangle + \frac{1}{g^7}\langle\frac{\text{tr}}{N}(AM)^6\rangle + \dots$$

We get

Kazakov, FLM

$$\frac{\mathcal{W}}{\beta^3} = \frac{\left(\frac{4V^3}{\pi}+\frac{2V}{\pi}\right)\arcsin(y)^2}{y^2} + \left(\frac{4V^3+2V}{y^2} + \frac{16V^3\sqrt{1-y^2}}{\pi y^3}\right)\arcsin(y)$$
$$+ \frac{8V^3\left(\sqrt{1-y^2}-1\right)}{y^3} + \frac{4V^3}{3\pi} - \frac{16V^3}{\pi y^2} - \frac{2V}{y} - \frac{2V}{\pi}$$

where $y = \dfrac{2\beta V}{g^2}$

Interpolates between pure 2d QG and almost flat case

2d QG from flattening solution

Kazakov, FLM

In 2d QG limit we have small bulk and boundary cosmological constants $\lambda_c \simeq 1 - \frac{\pi\beta}{\sqrt{2}}$

1st resolvent: $\Lambda = 2\sqrt{2}\frac{\lambda_c - \lambda}{\pi\beta} \to 0\,, \qquad \eta = 1 - \frac{\sqrt{2}\beta}{g^2} \to 0$ with $z = \frac{\sqrt{\Lambda}}{\eta} = \text{fixed}$

We get the standard disc partition function for pure gravity

$$\frac{\mathcal{W}}{\beta^3} = \underbrace{\frac{-28 - 18\pi + 9\pi^2}{3\sqrt{2}\pi} + \frac{\sqrt{2}\left(-12 - 7\pi + 3\pi^2\right)\eta}{\pi}}_{\text{non-universal terms}} - \frac{8}{3}\underbrace{\eta^{3/2}(z-2)\sqrt{z+1}}_{\text{universal part}} + O\left(\eta^2\right)$$

Matches known prediction! Kazakov '88

2nd resolvent: $\frac{1}{\beta}\left[W - \frac{L}{2D}\right] = \text{regular terms} + \zeta^{3/2}(z-2)\sqrt{z+1}\,\frac{4\sqrt{2}D^3}{3\left(2D^2-1\right)}$ Get the same universal part

So universal part does not depend on type of boundary

Almost flat regime

Only 2- and 4-vertices

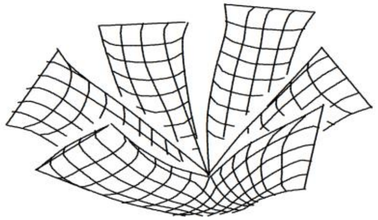
$$\beta \to 0\ ,\ V \to 0$$

1st resolvent: $$\frac{\mathcal{W}}{\beta^3} = \frac{2V}{\pi y^2}\left[(\arcsin y)^2 - y^2 + \pi(\arcsin y - y)\right]$$

Kazakov, Staudacher, Wynter '96

$$y = \frac{t_2}{2\pi\tau\, g}$$

Describes trees



2nd resolvent: $$W(P) = \frac{2\xi}{T} + \tau\frac{T\xi}{\pi(T^2 - 4\xi^2)}\left(T^2 + \frac{8\pi^2}{T^2}\xi^2 - \frac{\arcsin^2\xi}{\xi^2}\right)$$

Kazakov, FLM

$$\xi = \frac{T\,(P - \sqrt{P^2 - 4})}{4}$$

$$T = \frac{t_2}{\tau} \to \text{fixed}$$

Near criticality we find again a tree-like behavior :

$$W(g) - \frac{g - \sqrt{g^2 - 4}}{2} \simeq t\frac{\pi^2 - 4}{4\pi} + t\sqrt{2}\sqrt{1 - \xi} + \dots$$

$$T \sim g \to \infty$$

$$\xi = \frac{T}{2g} = \frac{t}{2g\tau} \to \text{fixed}$$

Interpretation: $\beta \neq 0$ controls local curvature fluctuations

Produces R^2 term in QG action

$$S_{\rm QG} = \int_{\mathcal{M}} \sqrt{g}\, (\tilde{\Lambda} + \frac{\log N}{2\pi} R + \frac{1}{\tilde{\beta}} R^2) \quad + \text{boundary terms}$$

- Curvature strongly fluctuates at distance $\tilde{\Lambda}^{1/2} \gg \tilde{\beta}$ $\longrightarrow$ QG regime
- Curvature is suppressed at distance $\tilde{\Lambda}^{1/2} \ll \tilde{\beta}$ $\longrightarrow$ "almost flat" regime

Our solution interpolates between the two regimes

We see no phase transition, in agreement with expectations from [Kazakov, Staudacher, Wynter '96]

Part 1 – summary & prospects

We count the number of dually weighted graphs, topology of sphere, disc (& more?)

Compute 2 disc partition functions, observe universality in 2d QG regime

Future:

- Explore our exponential solution and its phase structure
- General spectral curve of DWG, integrability/tau function
- Jackiw-Teitelboim gravity? May need more involved observables / develop character methods

PART 2

Critical phenomena in the forest

Forests on random graphs

Key properties of a graph are encoded in its Laplacian matrix $\Delta = D - A$

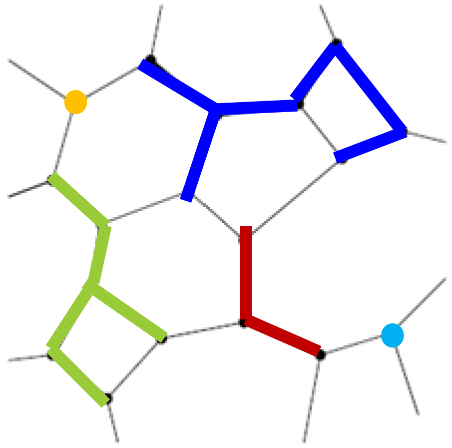
vertex degrees

adjacency matrix

Matrix tree theorem: $\det'\Delta = \#$ of spanning trees

Generalization: matrix forest theorem [Chelnokov, Kelmanis] [David, Duplantier]

$$\det(\Delta + M^2) = \sum_{F=(F_1 \ldots F_l) \in G} M^{2l} \prod_{i=1}^{l} |F_i|$$



We study the partition function:

$$Z = \sum_{\text{graphs } G} \lambda^{|G|} \det(\Delta + M^2) = \sum_{F=(F_1 \ldots F_l) \in G} \lambda^{|G|} M^{2l} \prod_{i=1}^{l} |F_i|$$

Sum over random regular graphs (RRG) with triple vertices

We count weighted rooted spanning forests

Related work:
[Caracciolo, Jacobsen, Saleur, Sokal 04]
[Caracciolo, Sportiello 09]
[Bondesan, Caracciolo, Sportiello 16]

$$Z = \sum_{\text{graphs } G} \lambda^{|G|} \det(\Delta + M^2)$$

Many reformulations!

- Counting rooted forests
- Massive spinless fermions interacting with 2d gravity
- Hermitian 1-matrix model with nontrivial potential
- Related to O(n) / loop model [Kostov, Staudacher]

Parisi-Sourlas reformulation via fermions

$$Z = \sum_G N^{2-2g} \lambda^{|G|} \int \prod_{i \in G} d^2\theta_i \prod_{<ij> \in G} e^{-(\bar\theta_i - \bar\theta_j)(\theta_i - \theta_j)} \prod_{i \in G} e^{-\frac{q}{2} M^2 \bar\theta_i \theta_i}$$

Massive spinless fermions on random graphs

To get sum over graphs we introduce NxN supermatrix $\Phi(\theta) = \phi + \bar\theta\psi + \theta\bar\psi + \bar\theta\,\theta\epsilon$

$$Z = \int d^{N^2}\phi\, d^{2N^2}\psi\, d^{2N^2}\epsilon\; e^{NS(\Phi)} \qquad S(\Phi) = \mathrm{tr} \int d^2\theta \left(-\frac{1}{2}\Phi^2(\theta) - \frac{1}{2}\partial_\theta\Phi(\theta)\partial_{\bar\theta}\Phi(\theta) + \frac{\lambda}{3} e^{-\frac{3}{2}M^2\bar\theta\theta} [\Phi(\theta)]^3 \right)$$

Feynman graphs give our partition function!

Notice that for unrooted trees one finds massive fermions + 4-fermion interaction
Instead we have rooted trees and no interaction

[Bondesan, Caracciolo, Sportiello 16]

The matrix model

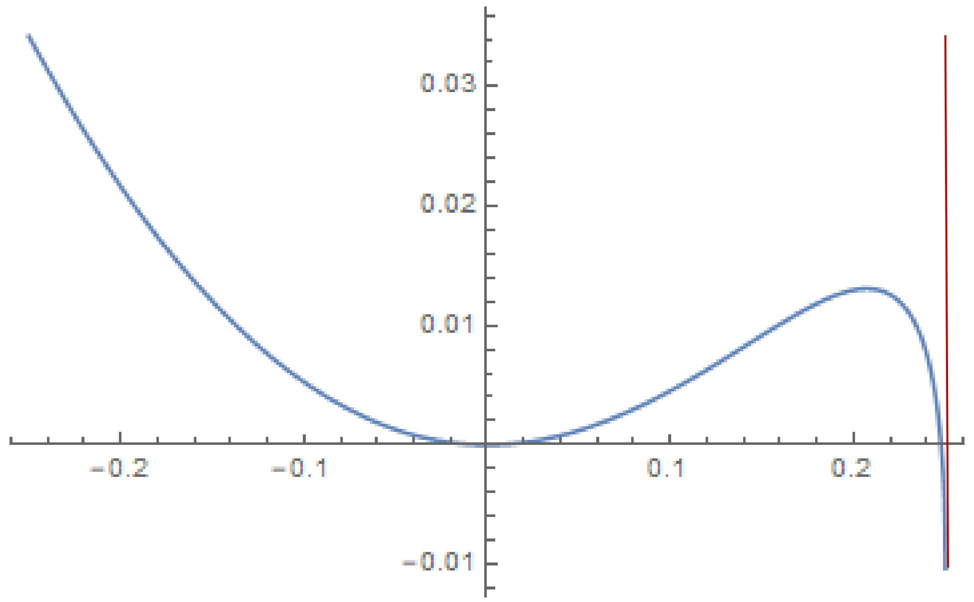
Next we integrate out the fermions
and get an Hermitian 1-matrix model

$$Z = \int d^{N^2} X e^{N S_X}$$

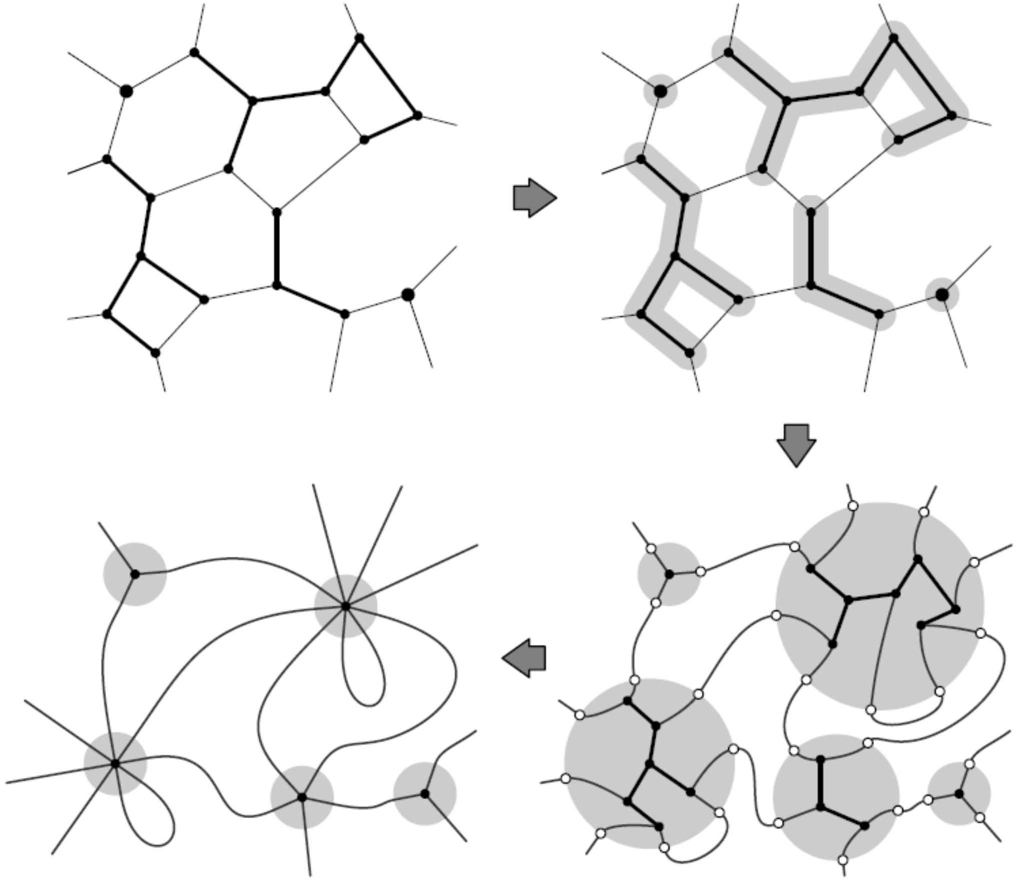
$$S_X = \mathrm{tr}\left[-\frac{1}{2}X^2 + \frac{\lambda M^2}{2}\left(\frac{1-\sqrt{1-4\lambda X}}{2\lambda}\right)^3\right]$$

generating function for trees
i.e. Catalan numbers

Potential:
two extrema + wall



Matrix model – combinatoric derivation



picture from
[Bondesan, Cracciolo, Sportiello 16]

Another way to derive the same matrix model

Extension of [Bondesan, Caracciolo, Sportiello 16]
who found the potential for unrooted trees:

$$\tilde{V}(z) = \frac{1}{12z^2}\left(-6z^2 + (1-4z)^{3/2} + 6z - 1\right)$$
$$= \sum_{n=1}^{\infty} \tilde{c}_n z^{n+2}, \qquad \text{where } \tilde{c}_n = \frac{(2n)!}{n!(n+2)!}.$$

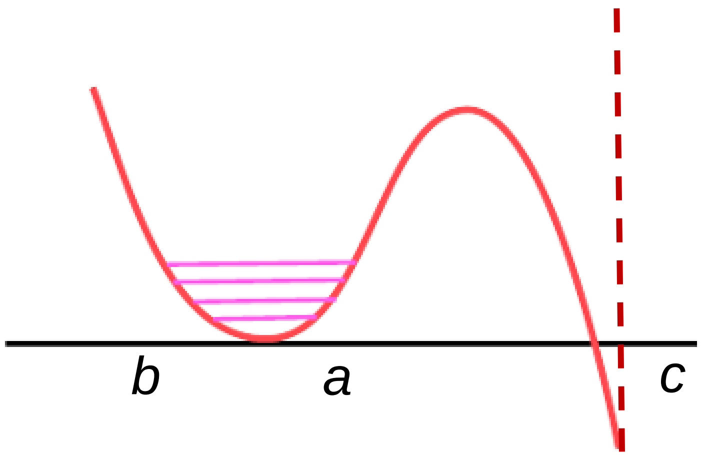
For our rooted trees we find:

$$V = \frac{\lambda}{3}\left(\frac{1-\sqrt{1-4\lambda X}}{2\lambda}\right)^3 = \lambda\partial_\lambda\left(X^2\tilde{V}(\lambda X)\right)$$

i.e. the same potential as from fermion realization !

Solution of the model

One-cut solution: $$G(x)=\int_b^a \frac{\rho(y)dy}{x-y}$$ Solved in elliptic functions



$$G(x)=\frac{9}{4\pi}\sqrt{c}M^2\sqrt{x-a}\sqrt{x-b}\left(\frac{2(x-2c)\Pi\left(\frac{a-b}{x-b}\middle| m\right)}{\sqrt{c-b}(x-b)}-\frac{2K(m)}{\sqrt{c-b}}\right)$$
$$+\frac{1}{2}\left(-\sqrt{x-a}\sqrt{x-b}+9cM^2+x\right)$$

$$c=\frac{1}{4\lambda} \qquad B^2=c-b\,,\ m=\frac{a-b}{c-b}$$

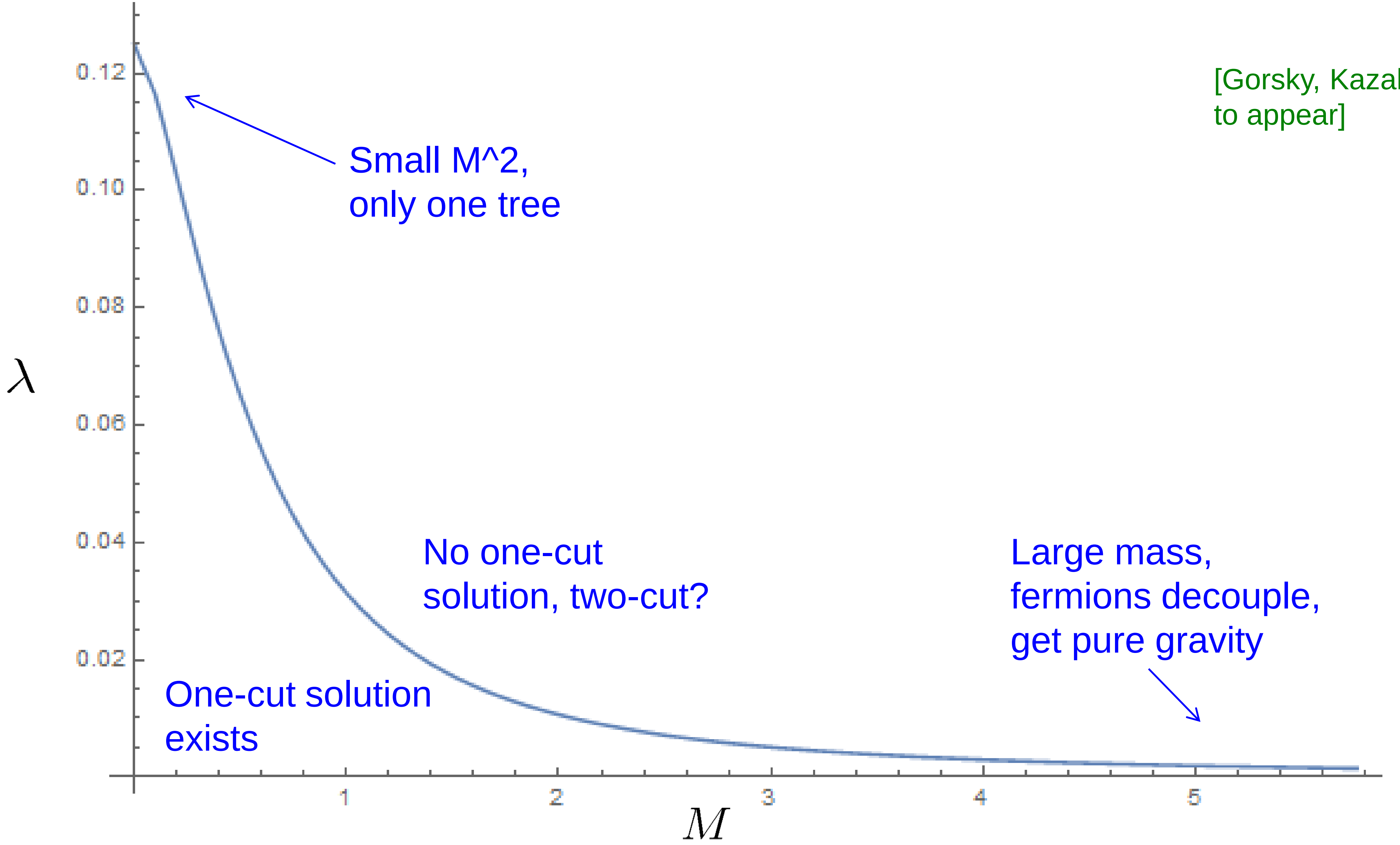
Consistency with asymptotics fixes endpoints

$$\pi B^3(m-2)-18B^2\sqrt{c}M^2E+2\pi Bc\left(9M^2+1\right)-18c^{3/2}M^2K=0$$

$$3\pi B^4m^2-36B^3\sqrt{c}M^2((m-2)E-2(m-1)K)+108Bc^{3/2}M^2((m-2)K+2E)-48\pi=0$$

Merging of solutions $\longrightarrow$ critical curve $$9\sqrt{c}M^2E(m)\left(c-B^2(m-1)\right)+(m-1)\left(\pi B^3m+9\sqrt{c}M^2\left(B^2+c\right)K(m)\right)=0$$

Critical curve



[Gorsky, Kazakov, FLM, Mishnyakov to appear]

Limiting regimes

$M \to \infty :$ Heavy fermions decouple $\qquad \lambda_{eff} = M^2 \lambda$

We reproduce pure gravity (c=0) prediction $(\lambda_{eff})^2_{crit} = \frac{1}{12\sqrt{3}}$

[Brezin, Itzyckson, Parisi, Zuber 78]
[Kazakov, Migdal, Kostov 85]

$M \to 0 :$ Only 1 tree in the forest $\qquad$ Resolvent becomes Gaussian $\quad G(x) \sim \sqrt{x^2 - 4}$

$$\det{}'(\Delta) \sim \frac{d}{dM^2} \det(\Delta + M^2)\Big|_{M=0} \sim \frac{d}{dM^2} Z\Big|_{M=0} = \langle \phi^3(x) \rangle \sim \int \sqrt{x^2 - 4}(1 - \sqrt{1 - 4\lambda x})^3$$

When branch points collide we reproduce the (c=-2) behavior $\lambda_{crit} = 1/8$

[David; Kazakov, Kostov, Migdal]

Our model interpolates between these regimes !

Interesting double scaling limits to be explored

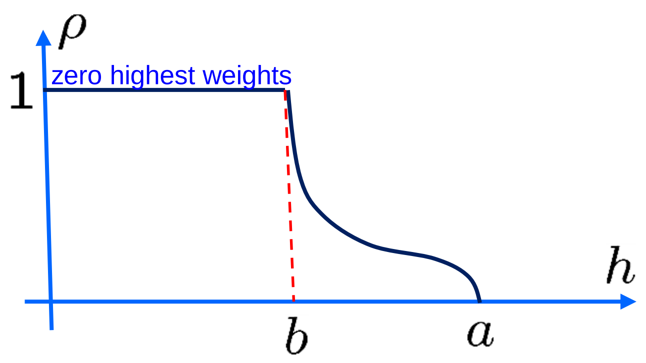
Future

- Links with Kesten – McKay resolvent
- Nonperturbative effects, i.e. instantons
- Multicut solutions
- Generalization to Dirac (spinful) fermions
- RRG as model for Hilbert space, ergodicity and many-body localization (MBL)

Characters at large N

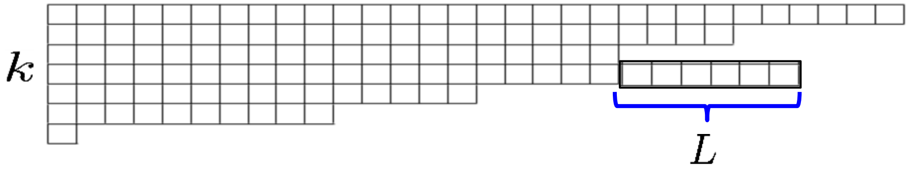
Vershik '77
Douglas, Kazakov '93
Kazakov, Wynter '94
Kazakov, Staudacher, Wynter '95, '96

$$H(h) = \frac{1}{N}\sum_{k=1}^{N}\frac{1}{h - h_k/N} \qquad H(h) = \int_0^a \frac{\rho(s)ds}{h-s}$$



We use the identity for Schur polynomials

$$\boxed{t_L = \sum_{k=1}^{N}\frac{\chi_{\{h+L\delta_k\}}[t]}{\chi_{\{h\}}[t]}} = \sum_{k=1}^{N}\frac{\mathrm{dim}(h+L\delta_k)}{\mathrm{dim}(h)}e^{\log\frac{\chi_{\{h+L\delta_k\}}}{\mathrm{dim}(h+L\delta_k)}-\log\frac{\chi_{\{h\}}}{\mathrm{dim}(h)}} = \sum_{k=1}^{N}\prod_{j(\neq k)}\left(1+\frac{L}{h_j-h_k}\right)e^{L\,F(h_k)}$$



where $\quad F(h_k) = \partial_{h_k}\log\frac{\chi_{\{h\}}}{\mathrm{dim}(h)}$

So we get $\; t_L = \oint_{C_H}\frac{dh}{2\pi i}[g(h)]^{-L} = \oint\frac{dg}{2\pi i}g^{-L-1}h(g)$

$$g(h) = e^{-H(h)-F(h)}$$

Similarly $\; \langle \mathrm{tr}(MA)^L\rangle = \oint_{C_H}\frac{dg}{2\pi i}\,g^{L-1}h(g)$

Some motivation

- Matrix models offer another way to understand nonperturbative physics, complementary to AdS/CFT integrability

- Potential links to JT gravity in AdS and its dual matrix model/topological recursion

  Saad, Shenker, Stanford '19

- New integrability structures to be uncovered

'Flattening' regime – type 2 resolvent

$$W(g) = \frac{1}{g} + \frac{1}{g^3}\langle \frac{\mathrm{tr}}{N} M^2 \rangle + \frac{1}{g^5}\langle \frac{\mathrm{tr}}{N} M^4 \rangle + \frac{1}{g^7}\langle \frac{\mathrm{tr}}{N} M^6 \rangle + \dots$$

$$\beta \sim v \sim (1-\lambda) \sim \tau \to 0$$

We get from inversion formula

$$\frac{W(g) - \frac{L}{2D}}{\beta} =$$

$$= \frac{16 D^3}{L^2(4D^2 - L^2)} \left[L^2 \frac{\sin^{-1}(\xi)^2 - \xi^2}{4\pi\xi} + \frac{\xi^3 + 12\sqrt{1-\xi^2}\sin^{-1}(\xi) - 12\xi + 3\xi\sin^{-1}(\xi)^2}{6\pi} \right]$$

Kazakov, FLM

$$\xi = \frac{1}{\sqrt{2}}\left(x - \sqrt{x^2-1}\right) L\,, \qquad L = D\left(g - \sqrt{g^2-4}\right)$$

sums up boundary "trees"

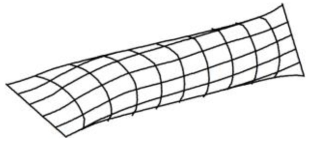
Highly similar to previous resolvent though not exactly equal

Related by a simple but formal replacement (meaning to be understood)

Asymptotics of large area & boundary

$$Z \propto \sum_G \lambda^F \beta^{2(\# v_2 - 4)}$$

Simplest case : only 4 2-vertices -- Dedekind function

$\sim \sum$  $= \frac{1}{8}\frac{\partial^2}{\partial q^2} \log \left[\prod_{n=1}^{\infty} (1 - q^{2n}) \right]$

Large area: elliptic modulus = 1 $\lambda = q \rightarrow 1$

What happens when we add other vertices?