Lecture notes on the Liouville quantum gravity metric

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Lecture notes for a mini course given at the Mini-school on Universality in Mathematical Physics, Lyon, Sept. 26-30, 2022.

Plan for the lectures:

- Definition of and motivation for Liouville quantum gravity: random fractal surfaces.
- Definition of the LQG metric.
- Techniques for proving things about the metric.
- Open problems.
- No prior background on LQG necessary to define and study the metric, or to understand the lectures.

Main references:

- Introductory articles on LQG [Gwy20b, She22].
- Book in progress on LQG [BP].
- Survey article on LQG metric [DDG21].
- Original papers on the construction of the LQG metric [DDDF20, GM20b, DFG$^+$20, GM20a, GM21b].
- I have a list of exercises on LQG and related topics which I can provide on request (probably too difficult if you have just followed this mini course).

1 Discrete motivation

- What is the most natural way of choosing a random curve in $\mathbb{R}^2$?
- Not obvious since the space of all curves is infinite-dimensional.
- Solution: consider random walk on $\mathbb{Z}^2$, take an appropriate limit, get Brownian motion.

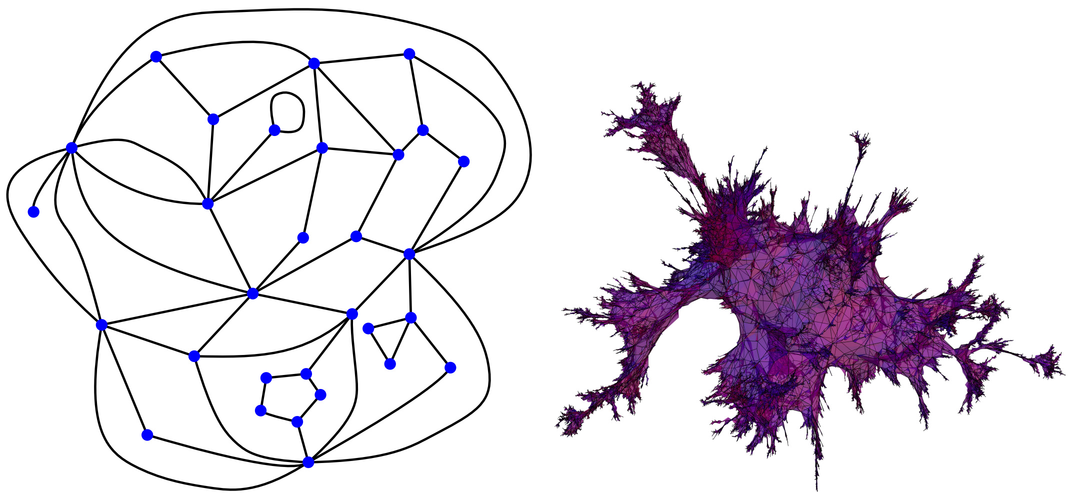

Figure 1: **Left.** One possible representation of a planar map. Other representations of the same planar map can be obtained by applying an orientation-preserving homeomorphism from $\mathbb{R}^2$ to $\mathbb{R}^2$. **Right.** Simulation of a large uniform triangulation which has been drawn in $\mathbb{R}^3$ in such a way that the embedding is, in a certain sense, as close as possible to preserving graph distances, made by J. Bettinelli. This can be viewed as a visual representation of the LQG metric space for $\gamma = \sqrt{8/3}$.

- What is the most natural way of choosing a random surface (2d Riemannian manifold)?
- Answer is provided by the theory of LQG.
- Could try discrete approximations: random planar maps.
- A planar map is a graph embedded in the plane, viewed modulo orientation-preserving homeomorphisms.
- Discrete surface: give each face the Riemannian metric of a polygon with unit side length, identify the polygons along the edges in a length-preserving way.
- Random planar maps.

  - Uniform: look at all possible planar maps with n edges, choose one so that each possibility is assigned equal probability. Can also consider triangulations, quadrangulations, etc.
  - Decorated: consider a pair (M, T), where M is a planar map with n edges and T is a spanning tree. Choose one uniformly at random. The marginal law of M is not uniform, but rather weighted by number of spanning trees. Similarly, can consider other decorations (Ising model, various special orientations, etc.).

- How to take a scaling limit of random planar maps?
- A planar map has a **measure** (counting measure) and a **metric** (graph distance).
- **Gromov-Hausdorff-Prokhorov topology**: natural topology on the space of compact metric measure spaces.
- **Embed** the map in some canonical way (Riemann uniformization, circle packing, Tutte embedding, etc.), then take a limit of measure and metric on the plane.
- Hence, we want our limiting objects to be defined as random metric measure spaces parametrized by (subsets of) the plane.
- Convergence only proven for **uniform** random planar maps (Le Gall, Miermont, Miller-Sheffield, Holden-Sun). Still open for weighted models.

2 Definition of LQG via the Gaussian free field

- We want to define a "random Riemannian metric" on the plane which describes the scaling limit of random planar maps.
- Isothermal coordinates: in local coordinates, metric takes the form

$$e^f(dx^2 + dy^2)$$

  for some function $f : \mathbb{C} \to \mathbb{R}$.

- Area measure $\mu_f = e^f \, dx \, dy$
- Length of a path P: $\mathrm{len}_f(P) = \int_0^1 e^{f(P(t))/2} |P'(t)| \, dt$.
- Distance: $D_f(z, w) = \inf_{P: z \to w} \mathrm{len}_f(P)$.
- Need to make a random choice of f.
- David-Distler-Kawai (DDK) ansatz: f should be a version of the planar Gaussian free field (GFF).

Definition 1. The **whole-plane Gaussian free field (GFF)** is the centered Gaussian random function with covariance function

$$\mathrm{Cov}(h(z), h(w)) = G_{\mathbb{C}}(z, w) := \log \frac{\max\{|z|, 1\} \max\{|w|, 1\}}{|z - w|}.$$

- Not defined pointwise, but makes sense as a generalized function (distribution).
- For "sufficiently nice" functions $\phi : \mathbb{C} \to \mathbb{R}$, the integral $\int_{\mathbb{C}} h(z)\phi(z)\, d^2 z$ is centered Gaussian with

$$\mathrm{Var} \int_{\mathbb{C}} h(z)\phi(z)\, d^2 z = \int_{\mathbb{C}\times\mathbb{C}} \phi(z)\phi(w)G(z, w)\, d^2 z\, d^2 w.$$

- Similarly, we can define **circle average** $C_r(z) =$ average of h over $\partial B_r(z)$.
- Scale / translation invariance modulo additive constant: for any $z \in \mathbb{C}$ and $r > 0$, $h(r \cdot +z) - C_r(z) \stackrel{d}{=} h$.

Definition 2. For $\gamma \in (0, 2]$, the γ**-Liouville quantum gravity surface** described by h is the random Riemannian manifold parametrized by U with Riemannian metric tensor

$$e^{\gamma h}(dx^2 + dy^2).$$

- Doesn't make literal sense.
- Expected to describe scaling limits of random planar maps.
  - $\gamma = \sqrt{8/3}$: uniform.
  - $\gamma = \sqrt{2}$: spanning tree decorated.
  - $\gamma = \sqrt{3}$: critical Ising decorated.
  - $\gamma = 2$: discrete GFF decorated.
- **Remark:** strictly speaking, the scaling limits of random planar maps are described not by $e^{\gamma h}\,(dx^2+dy^2)$ but by $e^{\gamma \tilde{h}}\,(dx^2+dy^2)$, where $\tilde{h}$ is a special variant of the Gaussian free field corresponding to the so-called **quantum sphere** [DKRV16, DMS21]. However, h and $\widetilde{h}$ have the same local behavior, in the sense of local absolute continuity. So, most geometric features of LQG are the same with h instead of $\widetilde{h}$.

3 Motivation

String theory (Polyakov, 1980's):

- A string is a path in $\mathbb{R}^n$ which evolves in time.
- Two parameters: parametrization of string + time.
- Traces out a "surface" in $\mathbb{R}^n$ (allowed to have self-intersections).
- To analyze this, Polyakov wanted to develop a notion of "sums over surfaces" analogous to Feynman path integral (which can be thought of as a "sum over paths").
- Need a notion of "random surfaces weighted by the number of possible embeddings into $\mathbb{R}^n$".
- Polyakov argued that this should correspond to LQG with

$$n = 25 - 6\left(\frac{2}{\gamma} + \frac{\gamma}{2}\right)^2.$$

- n is typically called the "matter central charge" and is denoted by $\mathbf{c}_{\mathrm{M}}$.
- For $\gamma \in (0, 2]$, we have $\mathbf{c}_{\mathrm{M}} \in (-\infty, 1]$: "embedding into space of dimension ≤ 1".
- To get embedding into $\mathbb{R}^n$ for $n \geq 2$, need a complex value of γ with $|\gamma| = 2$.
- We are just starting to understand LQG in this case (more on this later).

Conformal field theory:

- LQG is closely related to Liouville conformal field theory.
- Simplest example of a CFT with non-rational spectrum.
- Studied rigorously by Kupiainen-Rhodes-Vargas, et. al. [Var17,KRV20,DKRV16].
- Source of exact formulas for objects related to LQG.

Relationships to other random objects:

- Schramm-Loewner evolution: quantum zipper [She16], mating of trees [DMS21]; see [GHS19] for a survey of this work and its applications.
- Random planar maps (rigorously): convergence for $\gamma = \sqrt{8/3}$, to be discussed later; other values of γ via "peanosphere convergence".

- Random fractals: dimension / exponent computations via the KPZ formula [KPZ88, DS11, RV11, GHM20]. For example, this was used by Duplantier-Kwon [DK88] to predict the Brownian intersection exponents before they were rigorously computed by Lawler-Schramm-Werner [LSW01a, LSW01b].

- Random matrix theory: LQG measure describes limit of characteristic polynomial for various random matrix models (see, e.g., [Web15, BF22]).

- Random permutations: limits of various random permutations can be expressed in terms of LQG, e.g. [Bor21, BHSY22, BGS22].

4 LQG area measure

- We want to define area measure and metric associated with LQG.

- Let $\{h_\varepsilon\}_{\varepsilon>0}$ be a family of continuous functions which approximate h, define objects with h_ε instead of h, take a limit as $\varepsilon \to 0$.

- For concreteness, let $p_t(z) := \frac{1}{2\pi t} e^{-|z|^2/2t}$ and define

$$h^*_\varepsilon(z) := (h * p_{\varepsilon^2/2})(z) = \int_{\mathbb{C}} h(w) p_{\varepsilon^2/2}(z-w)\, d^2w.$$

- $\operatorname{Var} h^*_\varepsilon(z) \sim \log \varepsilon^{-1}$.

- $h^*_\varepsilon \to h$ as $\varepsilon \to 0$.

Theorem 3 (Kahane [Kah85], Duplantier-Sheffield [DS11], et. al.)**.** *The random measures $\varepsilon^{\gamma^2/2} e^{\gamma h^*_\varepsilon(z)}\, d^2 z$ a.s. converge weakly to a limiting measure μ_h, called the **LQG area measure**.*

- $\mu_h(\text{open}) > 0$, $\mu_h(\text{point}) = 0$, mutually singular with respect to Lebesgue measure.

- Should be scaling limit of counting measure on embedded random planar maps.

- **LQG coordinate change:** [DS11] suppose $\phi : V \to U$ is a conformal map. Then $\phi_* \mu_{\widetilde{h}} = \mu_h$, where

$$\widetilde{h} = h \circ \phi + Q \log |\phi'|, \quad Q = \frac{2}{\gamma} + \frac{\gamma}{2}.$$

- “Two different parametrizations of the same LQG surface”.

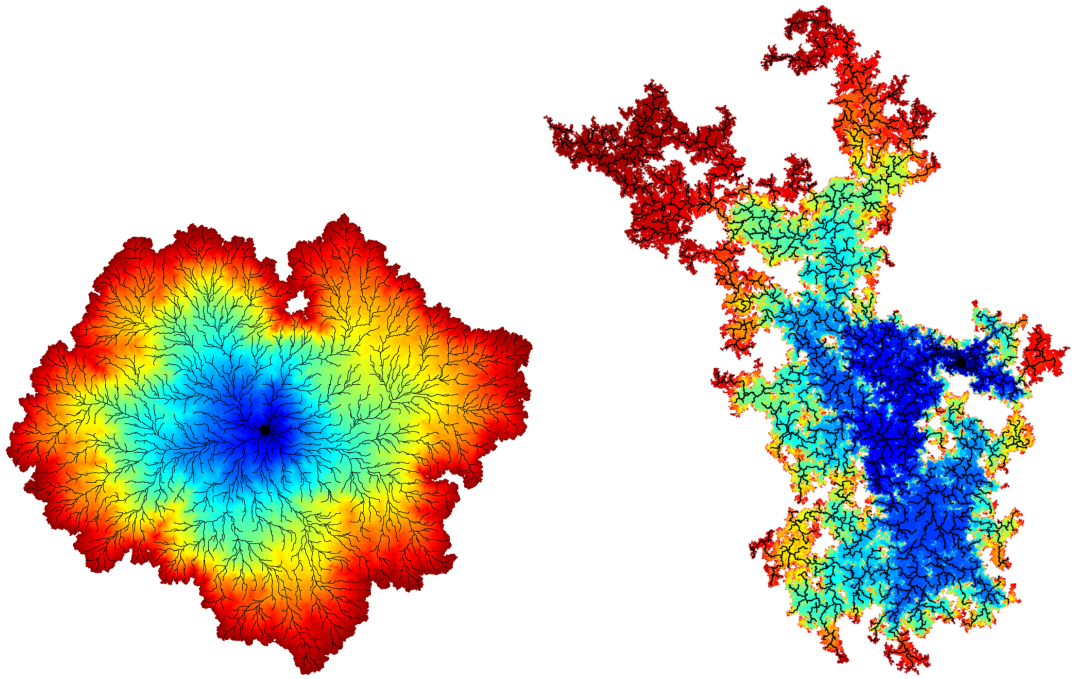

Figure 2: **Left.** Simulation of an LQG metric ball for $\gamma = 1.75$. Colors indicate the distance to the center point and the black curves are geodesics from the center point to other points in the ball. **Right.** Simulation of a supercritical LQG metric ball for $\xi = 2$. Both simulations were made by A. Bou-Rabee.

5 LQG metric

- We want to use a similar procedure to construct the LQG metric.
- Let $\xi > 0$ to be chosen later (depending on γ).
- For $\varepsilon > 0$, let
$$D_h^\varepsilon(z, w) = \inf_{P: z \to w} \int_0^1 e^{\xi h_\varepsilon^*(P(t))} |P'(t)| \, dt,$$
where the infimum is over piecewise C^1 paths from z to w.
- We want to take a limit of D_h^ε as $\varepsilon \to 0$ to get the LQG metric.
- What should ξ be?
- Scaling areas by C $\Leftrightarrow$ adding $\frac{1}{\gamma} \log C$ to h $\Leftrightarrow$ scaling distances by $C^{\xi/\gamma}$.
- γ/ξ should be the "dimension" of LQG.
- $\exists\, d_\gamma > 2$ such that for random planar maps in the γ-LQG universality class, $\# B_r(\text{typical vertex}) \approx r^{d_\gamma}$ when r is large [DZZ19, DG18].

- d_γ is the "dimension" of the random planar map.
- Not known explicitly except that $d_{\sqrt{8/3}} = 4$ (comes from results for uniform random planar maps).
- Watabiki [Wat93] prediction:

$$d_\gamma^{\mathrm{Wat}} = 1 + \frac{\gamma^2}{4} + \frac{1}{4}\sqrt{(4+\gamma^2)^2 + 16\gamma^2},$$

  disproven in [DG19], but from numerical simulations is "close" to the actual value of d_γ [AB14, BB19].
- Alternative guess due to Ding-Gwynne [DG18]:

$$d_\gamma^{\mathrm{DG}} = 2 + \frac{\gamma^2}{2} + \frac{\gamma}{\sqrt{6}}.$$

  Not disproven rigorously, but believed to be false (see, e.g., [DGS21]).
- We want $\gamma/\xi = d_\gamma$, i.e.,

$$\xi = \frac{\gamma}{d_\gamma}.$$

- A posteriori, can show that d_γ is the Hausdorff dimension of the LQG metric space [GP19b].
- Note: relationship between ξ and γ is not known explicitly.
- How to scale D_h^ε to get a non-trivial limit?
- For $\varepsilon > 0$, let

$$\mathfrak{a}_\varepsilon = \mathfrak{a}_\varepsilon(\xi) = \text{median of } D_h^\varepsilon\text{-distance across } [0,1]^2.$$

Proposition 4 (Ding-Gwynne [DG18])**.** *For $\xi = \gamma/d_\gamma$, we have $\mathfrak{a}_\varepsilon = \varepsilon^{1-\xi Q + o(1)}$ as $\varepsilon \to 0$, where $Q = 2/\gamma + \gamma/2$.*

Theorem 5 (Ding-Dubédat-Dunlap-Falconet [DDDF20])**.** *The random metrics $\{\mathfrak{a}_\varepsilon^{-1} D_h^\varepsilon\}_{\varepsilon>0}$ are tight with respect to the topology of uniform convergence on compact subsets of $\mathbb{C}\times\mathbb{C}$. Every subsequential limit is a random metric on $\mathbb{C}$ (not a pseudometric) which induces the same topology as the Euclidean metric.*

Theorem 6 (Gwynne-Miller [GM21b])**.** *The subsequential limit is uniquely characterized by a list of axioms, and one has $\mathfrak{a}_\varepsilon^{-1} D_h^\varepsilon \to D_h$ in probability as $\varepsilon \to 0$.*

- The limiting object is defined to be the **Liouville quantum gravity metric**.

- Convergence is *much* harder than for the measure since the minimizing path depends on ε.
- Proofs of tightness and uniqueness are quite involved, but use *only basic properties of the GFF*: nothing about LQG measure, relationship to SLE, relationship to random planar maps, exact formulas, special LQG surfaces, etc.
- Euclidean topology, but very different geometry.
- Hausdorff dimension $d_\gamma > 2$ [GP19b].
- $\exists$ LQG geodesic (length-minimizing path) between any two points (take limit of D_h^ε-geodesic).
- Confluence of geodesics [GM20a].
- Metric ball boundary is fractal, infinitely many connected components [Gwy20a, GPS22].
- **LQG coordinate change:** [GM21a] Let $\phi : U \to V$ be a conformal map. Then a.s.

$$D_{\widetilde{h}}(z,w) = D_{h|_V}(\phi(z),\phi(w)), \quad \forall z,w \in U \quad \text{where} \quad \widetilde{h} = h\circ\phi + Q\log|\phi'|, \quad Q = \frac{2}{\gamma} + \frac{\gamma}{2}.$$

- Same coordinate change rule as for μ_h.

6 Miller-Sheffield construction and convergence of uniform random planar maps

- Miller-Sheffield [MS20, MS21a, MS21b]: Earlier construction of the LQG metric for $\gamma = \sqrt{8/3}$.
- Use a process called **quantum Loewner evolution** to build a candidate for LQG metric balls.
- Show that there is a unique metric with these metric balls.
- Relies on special symmetries for $\gamma = \sqrt{8/3}$, does not generalize to other values of γ.

Theorem 7 (Le Gall [Le 13], Miermont [Mie13])**.** *Let M_n be a uniform quadrangulation with n edges, μ^n = counting measure on vertices, D^n = graph distance. Then $(M_n, n^{-1/4}D^n, n^{-1}\mu^n)$ converges in law to a random metric measure space called the **Brownian map**, w.r.t. the Gromov-Hausdorff-Prokhorov topology.*

- Also works for other uniform-type random planar maps, e.g., triangulations, unconstrained face degree [BJM14].

Theorem 8 (Miller-Sheffield [MS21a]). *For a special variant of the GFF called the* ***quantum sphere****, the $\sqrt{8/3}$-LQG metric measure space constructed via quantum Loewner evolution is isometric to the Brownian map.*

Theorem 9 (Gwynne-Miller [GM21b]). *The Miller-Sheffield $\sqrt{8/3}$-LQG metric coincides the the limit of $\mathfrak{a}_\varepsilon^{-1} D_h^\varepsilon$ for $\gamma = \sqrt{8/3}$.*

- Hence, uniform random planar maps converge to $\sqrt{8/3}$-LQG in the Gromov-Hausdorff-Prokhorov topology.
- Building on this, Holden and Sun showed that one also has convergence under the so-called **Cardy embedding** [HS19].
- We don't know how to see that $\gamma = \sqrt{8/3}$ is special directly from the properties of $\mathfrak{a}_\varepsilon^{-1} D_h^\varepsilon$. Connection to uniform random planar maps has to go through Miller-Sheffield construction.
- Random planar map convergence is still conjectural for $\gamma \neq \sqrt{8/3}$.

7 The supercritical case

- The quantity $\xi = \gamma/d_\gamma$ is increasing in γ [DG18].
- So, $\gamma/d_\gamma \leq 2/d_2 \approx 0.41$.
- What happens when $\xi > 2/d_2$?
- Can still define the approximating metrics D_h^ε and the normalizing factors $\mathfrak{a}_\varepsilon$.

Theorem 10 (Ding-Gwynne [DG20, DG21c]). *For all $\xi > 0$, the random metrics $\mathfrak{a}_\varepsilon^{-1} D_h^\varepsilon$ converge in probability with respect to the topology on lower semicontinuous functions $\mathbb{C} \times \mathbb{C} \to \mathbb{R} \cup \{\infty\}$ (weaker than local uniform topology).*

- We say that $z \in \mathbb{C}$ is a **singular point** if
$$D_h(z, w) = \infty, \ \forall w \neq z.$$
- For each fixed $z \in \mathbb{C}$, a.s. z is not a singular point (singular points have zero Lebesgue measure).
- A.s., for any two non-singular points z, w, we have $D_h(z, w) < \infty$ (typical points lie at finite distance).

- For $\xi > 2/d_2$, the set of singular points is uncountable and Euclidean dense.
- Non-Euclidean topology.
- Metric balls have positive Lebesgue measure but empty Euclidean interior.
- For $\alpha > 0$, an **thick point** of h is a point z such that $\limsup_{\varepsilon\to 0} h_\varepsilon(z)/\log \varepsilon^{-1} \geq \alpha$.
- Singular points are (almost) the same as Q-thick points [Pfe21].
- In the critical case $\gamma = 2$, $\xi = 2/d_2$, there are no singular points and the metric induces the Euclidean topology [DG21b].
- For $\xi > 2/d_2$, the metric D_h satisfies the LQG coordinate change rule with $Q \in (0, 2)$.
- If we choose $\gamma = \gamma(\xi)$ so that $Q = 2/\gamma + \gamma/2$, then $\gamma \in \mathbb{C}$ with $|\gamma| = 2$.
- Central charge $\mathbf{c}_{\mathrm{M}} = 25 - 6Q^2 \in (1, 25)$. Note that positive integer values of $\mathbf{c}_{\mathrm{M}}$ are important in Polyakov's "evolving strings" story.
- $\mathbf{c}_{\mathrm{M}} \in (1, 25)$ is much more mysterious than $\mathbf{c}_{\mathrm{M}} \leq 1$ (i.e., $\gamma \in (0, 2]$), lots of open questions about this phase of LQG. See Figure 3.

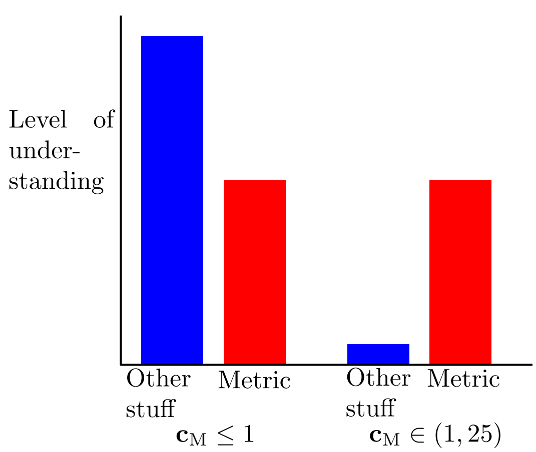

Figure 3: Visual representation of our level of understanding of different aspects of LQG. "Other stuff" includes the LQG measure, connections to random planar maps, connections to SLE, connections to conformal field theory, etc.

8 Axiomatic definition

- Suppose we are given a random metric on the plane, coupled with the GFF. What properties would it need to satisfy for us to say that it is the LQG metric?

- Let us formalize the problem. Let $h \mapsto D_h$ be a measurable function

$$\{\text{generalized functions on } \mathbb{C}\} \to \{\text{metrics on } \mathbb{C}\}.$$

- We require that whenever h is a GFF or a GFF plus a (possibly random) continuous function, the following is true.

  1. **Euclidean topology.** Same topology as Euclidean metric.
  2. **Length metric.** $D_h(z,w)$ is the infimum of the D_h-lengths of paths from z to w.
  3. **Locality.** For $U \subset \mathbb{C}$, define the **internal metric** by

     $$D_h(z,w;U) = \inf\{D_h\text{-length of } P : P \text{ is a path in } U \text{ from } z \text{ to } w\}.$$

     Then $D_h(z,w;U)$ is a measurable function of $h|_U$.
  4. **Weyl scaling.** Almost surely, for each continuous function $f : \mathbb{C} \to \mathbb{R}$,

     $$D_{h+f}(z,w) = \inf_{P:z\to w} \int_0^{D_h(z,w)} e^{\xi f(P(t))}\, dt,$$

     where the infimum is over paths parametrized by D_h-length.
  5. **LQG coordinate change.** Let $a \in \mathbb{C} \setminus \{0\}$, $b \in \mathbb{C}$. Almost surely,

     $$D_{h(a\cdot+b)+Q\log|a|}\left(\frac{z-b}{a}, \frac{w-b}{a}\right) = D_h(z,w), \quad \forall z,w \in \mathbb{C}.$$

- At first glance, there seems to be a two-parameter family (ξ and Q), but one can show that in fact ξ and Q must be related by $\xi = \gamma/d_\gamma$, $Q = 2/\gamma + \gamma/2$ (rough comparison to D_h^ε).

Theorem 11 (Gwynne-Miller [GM21b])**.** *Let D and $\widetilde{D}$ be two metrics satisfying the above axioms. There is a deterministic constant $C > 0$ such that a.s. $D_h = \widetilde{D}_h$ whenever h is a GFF or a GFF plus a continuous function.*

- Does this imply the uniqueness of the subsequential limit of $\mathfrak{a}_\varepsilon^{-1} D_h^\varepsilon$?

- Euclidean topology proven by DDDF, length metric, locality, Weyl scaling easy to check.

- LQG coordinate change is a problem since if we scale space by C, we replace D_h^ε by $D_h^{C\varepsilon}$.
- This might give us a different subsequence.
- To get around this, we prove a stronger characterization theorem with LQG coordinate change replaced by **tightness across scales**. Roughly speaking, this condition says that we can get up-to-constants comparisons between $D_{h(a\cdot)+Q\log|a|}(z/a, w/a)$ and $D_h(z,w)$ with high probability.
- Most of the proofs are only slightly harder when we replace LQG coordinate change by tightness across scales.
- Once tightness is proven, every proof about the LQG metric uses only the axioms (we don't need to go back to the definition of $\mathfrak{a}_\varepsilon^{-1} D_h^\varepsilon$).
- Existence of the metric can be taken as a black box.

9 Adding a bump function

- The following Cameron-Martin type lemma is one of the most useful tools for studying the LQG metric (see, e.g., [BP, Proposition 1.29]).

Lemma 12. *Let h be the whole-plane GFF and let $f : \mathbb{C} \to \mathbb{R}$ be a compactly supported function whose Dirichlet energy $(f,f)_\nabla = \int_{\mathbb{C}} |\nabla f(z)|^2 \, d^2 z$ is finite. Then the laws of $h+f$ and h are mutually absolutely continuous, and the Radon-Nikodym derivative of the law of $h+f$ with respect to the law of h is*

$$\exp\big((h,f)_\nabla - (f,f)_\nabla^2\big).$$

- If we want to show that D_h does something with positive probability, we just need to find a suitable bump function f such that D_{h+f} has the desired behavior with positive probability.

Lemma 13. *Fix $z, w \in \mathbb{C}$ and let $U \subset \mathbb{C}$ be a deterministic open set which contains a path from z to w. With positive probability, every D_h-geodesic from z to w is contained in U.*

Proof. Choose a deterministic smooth bump function f which is supported on a compact subset of U and which is equal to 1 on a neighborhood of a path in U from z to w. By Weyl scaling, if C is large then with high probability there is a path from z to w which is contained $\operatorname{supp} f$ and whose D_{h-Cf}-length is much smaller than the D_{h-Cf}-distance from $\operatorname{supp} f$ to ∂U. Thus every D_{h-Cf}-geodesic from z to w is contained in U. By absolute continuity, it holds with positive probability that every D_h-geodesic from z to w is contained in U. $\square$

10 Independence of the GFF across disjoint concentric annuli

- One of the most important tools for studying the LQG metric is the following lemma.

Lemma 14. *Let h be a whole-plane GFF. For $k \in \mathbb{N}$, let E_k be an event which is determined by the restriction of h to the annulus $B_{2^{-k}}(0) \setminus B_{2^{-k-1}}(0)$, viewed modulo additive constant.*

1. *For each $p \in (0,1)$, there exists $q = q(p) \in (0,1)$ such that if $\mathbb{P}[E_k] \geq p$ for each k, then for each $K \in \mathbb{N}$,*

$$\mathbb{P}[E_k \text{ occurs for at least one } k \in \{1, \dots, K\}] \geq 1 - q^K. \tag{10.1}$$

2. *For each $q \in (0,1)$, there exists $p = p(q) \in (0,1)$ such that if $\mathbb{P}[E_k] \geq p$ for each k, then for each $K \in \mathbb{N}$,* (10.1) *holds.*

- Show that $h|_{B_{2^{-k}}(0) \setminus B_{2^{-k-1}}(0)}$ are approximately independent, apply concentration for binomial(q, K) distribution.

- Idea originally due to Miller-Qian [MQ20], formulated precisely by Gwynne-Miller [GM20b].

- Various improvements are possible.

  - Replace $B_{2^{-k}}(0) \setminus B_{2^{-k-1}}(0)$ by disjoint concentric annuli with uniformly bounded aspect ratios.
  - If q is close enough to 1, then E_k has to occur for "most" $k \in \{1, \dots, K\}$.

Lemma 15. *For each $\gamma \in (0,2)$, there exists $\alpha = \alpha(\gamma) > 0$ and $c = c(\gamma) > 0$ such that the following is true. For each $z \in \mathbb{C}$ and each $\varepsilon > 0$, the probability that there is a D_h-geodesic between two points in $\mathbb{C} \setminus B_{\varepsilon^{1/2}}(z)$ which enters $B_\varepsilon(z)$ is at most $c\varepsilon^\alpha$.*

Roughly speaking, Lemma 15 says that "most" points in $\mathbb{C}$ are not hit by D_h-geodesics except at their endpoints. Lemma 15 immediately implies that the Hausdorff dimension of the union of all of the LQG geodesics w.r.t. the Euclidean metric is strictly less than 2. See [GP19b] for an explicit upper bound for the Hausdorff dimension of a single LQG geodesic. Similar (but more complicated) ideas to the ones in the proof of Lemma 15 are used in the proof of confluence of geodesics in [GM20a, DG21a].

Definition 16. For a Euclidean annulus $A \subset \mathbb{C}$, we define $D_h(\text{across } A)$ to be the D_h-distance between the inner and outer boundaries of A. We define $D_h(\text{around } A)$ to be the infimum of the D_h-lengths of paths in A which separate the inner and outer boundaries of A.

Both $D_h(\text{across } A)$ and $D_h(\text{around } A)$ are determined by the internal metric of D_h on A, so by locality these quantities are a.s. determined by $h|_A$.

For $z \in \mathbb{C}$ and $r > 0$, let

$$E_r(z) := \{D_h(\text{around } B_{3r}(z) \setminus B_{2r}(z)) < D_h(\text{across } B_{2r}(z) \setminus B_r(z))\}. \tag{10.2}$$

As noted above, $E_r(z)$ is a.s. determined by $h|_{B_{3r}(z) \setminus B_r(z)}$. In fact, adding a constant to h results in scaling D_h-distances by a constant (Weyl scaling), so adding a constant to h does not affect whether $E_r(z)$ occurs. Hence $E_r(z)$ is a.s. determined by $(h - h_{4r}(z))|_{B_{3r}(z) \setminus B_r(z)}$.

Lemma 17. *There exists $\alpha = \alpha(\gamma) > 0$ and $c = c(\gamma) > 0$ such that for each $z \in \mathbb{C}$ and each $\varepsilon > 0$,*

$$\mathbb{P}\left[\exists r \in \left[\varepsilon, \frac{1}{4}\varepsilon^{1/2}\right] \textit{ such that } E_r(z) \textit{ occurs}\right] \geq 1 - c\varepsilon^{\alpha}.$$

Proof. By the scale and translation invariance of the law of h, modulo additive constant, $\mathbb{P}[E_r(z)]$ does not depend on z or r. Using a "subtracting a bump function" argument, one can show that $p := \mathbb{P}[E_1(0)] > 0$. Hence $\mathbb{P}[E_r(z)] = p$ for each $z \in \mathbb{C}$ and $r > 0$. We now apply Lemma 14 with $K \asymp \log \varepsilon^{-1}$ to get

$$\mathbb{P}\big[\exists r \in [\varepsilon, \varepsilon^{1/2}] \text{ such that } E_r(z) \text{ occurs}\big] \geq 1 - q^{\log \varepsilon^{-1}}$$

for $q = q(p) \in (0,1)$. This last quantity is at least $1 - c\varepsilon^{\alpha}$ for an appropriate $c, \alpha > 0$. $\square$

Proof of Lemma 15. By Lemma 17, it suffices to show that if there is an $r \in [\varepsilon, \frac{1}{4}\varepsilon^{1/2}]$ such that $E_r(z)$ occurs, then no D_h-geodesic between two points in $\mathbb{C} \setminus B_{\varepsilon^{1/2}}(z)$ can enter $B_\varepsilon(z)$. Indeed, assume that $E_r(z)$ occurs, let $u, v \in \mathbb{C} \setminus B_{\varepsilon^{1/2}}(z)$, and let P be a path from u to v which hits $B_r(z) \supset B_\varepsilon(z)$. We will show that P is not a D_h-geodesic. By the definition (10.2) of $E_r(z)$, there is a path π in $B_{3r}(z) \setminus B_{2r}(z)$ which disconnects the inner and outer boundaries of this annulus and has D_h-length strictly less than $D_h(\text{across } B_{2r}(z) \setminus B_r(z))$. Let σ (resp. τ) be the first (resp. last) time that P hits π. Since P hits $B_r(z)$ and $u, v \notin B_{3r}(z)$, the path P crosses between the inner and outer boundaries of $B_{2r}(z) \setminus B_r(z)$ between times σ and τ. Hence

$$\big(D_h\text{-length of } P|_{[\sigma,\tau]}\big) \geq D_h(\text{across } B_{2r}(z) \setminus B_r(z)). \tag{10.3}$$

But, since $P(\tau), P(\sigma) \in \pi$,

$$\begin{aligned} D_h(P(\sigma), P(\tau)) \leq (D_h\text{-length of } \pi) &< D_h(\text{across } B_{2r}(z) \setminus B_r(z)) \\ &\leq \big(D_h\text{-length of } P|_{[\sigma,\tau]}\big). \end{aligned} \tag{10.4}$$

This implies that P is not a D_h-geodesic since it is not the D_h-shortest path from $P(\sigma)$ to $P(\tau)$. $\square$

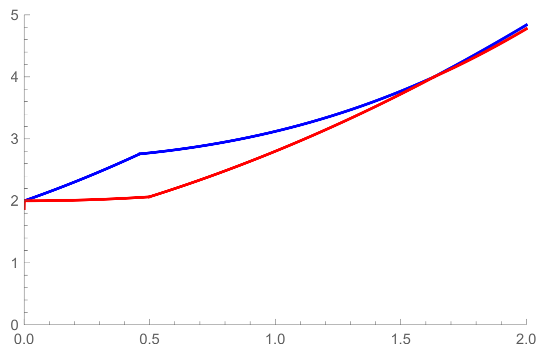

Figure 4: Graph of best known upper and lower bounds for d_γ as a function of γ. The graphs meet only at $(\sqrt{8/3}, 4)$.

11 Open problems

Gromov-Hausdorff limit of non-uniform random planar maps.

- Can compare distances up to polylog errors using mating of trees techniques, see [GHS20]. No up-to-constants comparison, though.

Compute d_γ.

- Reasonably good bounds by the following method: let $\lambda(\xi) = 1 - \xi Q_\xi$. Get differential inequality for $\lambda(\xi)$ from extremely crude estimates for D_h^ε. Plug in known values $\lambda(0) = 0$, $\lambda(1/\sqrt{6}) = 1/6$ (equivalent to $d_{\sqrt{8/3}} = 4$) [DG18, GP19a, Ang19]. See Figure 4, left.
- Lots of other quantities can be expressed in terms of d_γ.
  - Distance exponents for random planar maps [DG18].
  - Optimal Hölder exponents for LQG metric w.r.t. Euclidean metric [DFG+20].
  - Dimension of metric ball boundary [Gwy20a].

What can be said about LQG geodesics?

- Euclidean dimension (even in terms of d_γ)? Proven to be strictly bigger than 1 in [FG22].
- Any description of their laws?
- Miller-Qian: not SLE [MQ20].

Understand non-metric features of LQG in the supercritical case $\mathbf{c}_{\mathrm{M}} \in (1, 25)$.

- Analog of LQG measure (complex GMC)?

- Analog of SLE (Loewner evolution driven by complex Brownian motion)?
- Planar map connection (local limit of high genus maps)?
- Any duality for different values of $\mathbf{c}_{\mathrm{M}}$?
- Any nice interaction with n independent GFFs when $\mathbf{c}_{\mathrm{M}} = n$ is an integer?

See [GHPR20, DG21c] for more discussion.

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