Supersymmetric black holes and modularity

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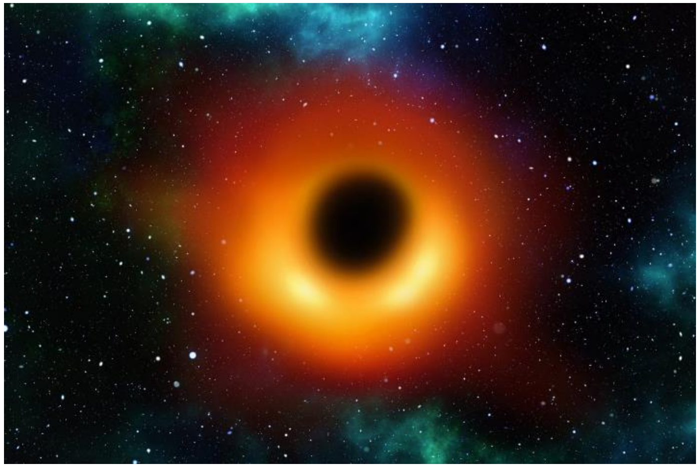
Théorie, Univers et Gravitation
October, 2022

Black hole entropy



Black holes are known to be *thermodynamical* objects! [Bekenstein '72, Hawking '74]

They possess *entropy*, temperature and emit radiation.

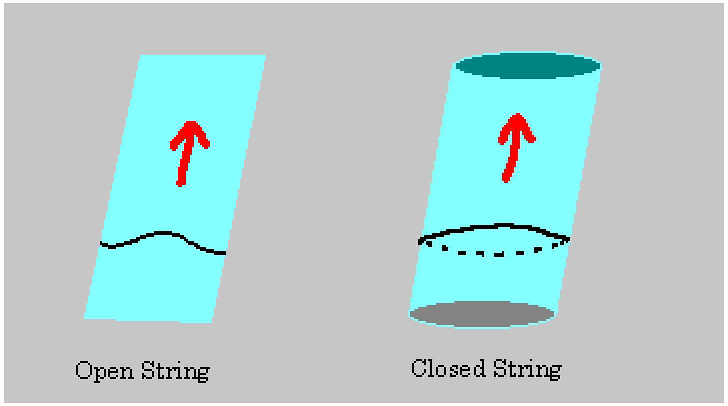
$$\boxed{S_{\mathrm{BH}} = \frac{A_{\mathrm{hor}}}{4\ell_{\mathrm{Pl}}^2}} \Rightarrow N = \log S_{\mathrm{BH}}$$

number of black hole states

What are the microstates counted by black hole entropy?

Black holes have no hairs, only few parameters… → challenge for ***quantum gravity!***

String theory solution

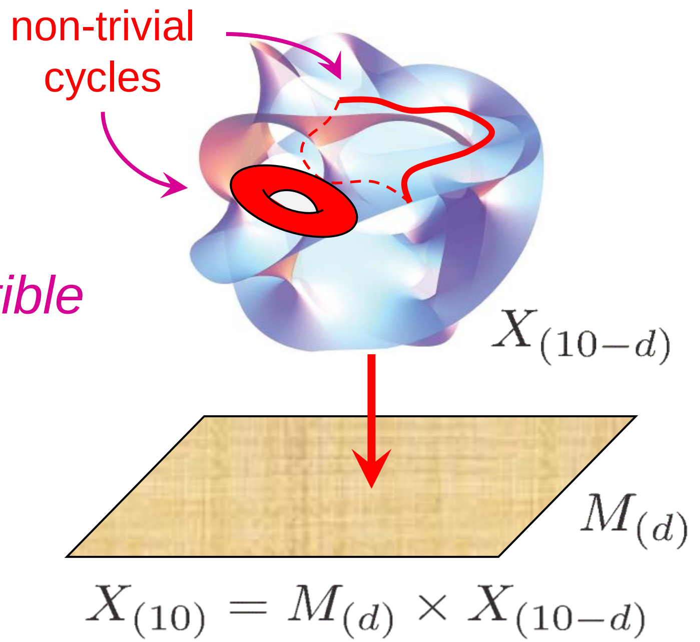


- String theory — a theory of one-dimensional extended objects (*strings*)
- It lives in *10-dimensional* spacetime
- It possess *spacetime SUSY*
- Besides strings, it includes other extended objects — *D-branes*
- Its different formulations are all related by *dualities* and are considered as different vacua of the same theory (*M-theory*), which also appears as a theory of *M2* and *M5-branes* in 11 dimensions

Realistic vacua are supposed to arise upon *compactification* to *d(=4)* dimensions

Black holes in string theory are *solitonic* objects constructed from D-branes wrapping *non-contractible cycles* on the internal manifold

BH microstates — D-brane configurations



BPS black holes in N=8 SUGRA

In some simple situations one can count BH entropy *exactly*!

Simple $\longrightarrow$ sufficiently many SUSY $\longrightarrow$ sufficiently simple internal manifold

The simplest case $X_{(6)} = T^6$ — 6d torus

Effective theory in 4d — N=8 SUGRA

This theory has black hole solutions preserving a part of SUSY — *BPS black holes*

⅛ BPS black holes are characterized by one integer valued charge

It is possible to find a generating function of their degeneracies
[Moore,Maldacena,Strominger '99, Pioline '05, Shih,Strominger,Yin '05]

$$\frac{\theta_3(2\tau)}{\eta^6(4\tau)} = \sum_{n\geq -1} \Omega(n) q^n = q^{-1} + 2 + 8q^3 + 12q^4 + 39q^7 + \cdots \qquad q = e^{2\pi i\tau}$$

theta function

Dedekind eta function

modular functions

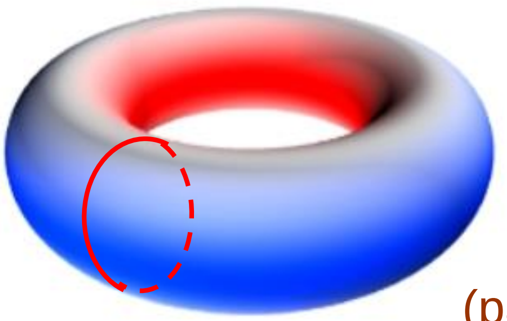
$$\tau \mapsto \frac{a\tau + b}{c\tau + d} \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$$

For large charge, it reproduces the area law

$$\Omega(n) \sim e^{\pi\sqrt{n}} \sim e^{A_{hor}(n)/4}$$

Modular symmetry governs BH entropy

Modular symmetry: torus

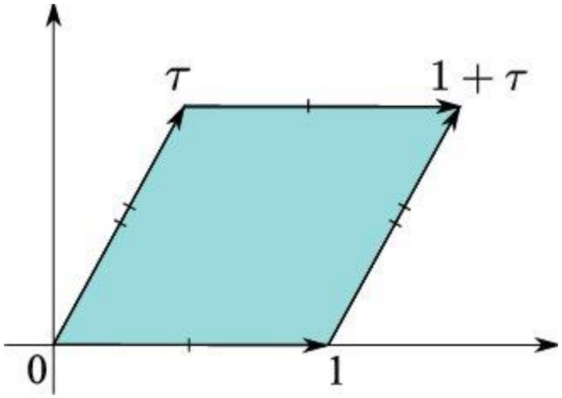


$$T^2 = \mathbb{C}/\{z \sim z + n + m\tau\}$$

$$n, m \in \mathbb{Z} \qquad \mathrm{Im}\tau > 0$$

τ — *torus modulus*

(parametrizes its *complex structure*)



Infinitesimal transformations of τ change the torus,
but global can leave it invariant:

$$\tau \mapsto \frac{a\tau + b}{c\tau + d} \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) \quad \Rightarrow \quad \begin{array}{c} am + cn = m' \\ bm + db = n' \end{array}$$

All well-defined quantities on torus must be *modular invariant*!

$$h\left(\frac{a\tau + b}{c\tau + d}\right) = h(\tau)$$

Modular forms

$$h_\mu\left(\frac{a\tau+b}{c\tau+d}\right) = \sum_\nu \rho_{\mu\nu}(g)\,(c\tau+d)^w\, h_\nu(\tau)$$

w — *modular weight*

multiplier system
(phase factor)

Asymptotic properties of Fourier coefficients:

$$q=e^{2\pi \mathrm{i}\tau} \qquad h_\mu(\tau) = \sum_{n\geq n_{\min}} \Omega_\mu(n)\, q^n$$

$$n_{\min}>0 \;\Rightarrow\; \Omega_\mu(n)\sim n^{w/2}$$
$$n_{\min}=0 \;\Rightarrow\; \Omega_\mu(n)\sim n^{w-1}$$
$$n_{\min}<0 \;\Rightarrow\; \Omega_\mu(n)\sim e^{C\sqrt{n}}$$

relevant for black holes

In fact, there is an *exact* formula for Fourier coefficients with positive *n* in terms of ones with negative *n* (*polar coefficients*)

Rademacher expansion:

$$\Omega_\mu(n) = 2\pi \sum_\nu \sum_{n'<0} \Omega_\nu(n') \sum_{c=1}^{\infty} \frac{1}{c}\, K_\mu^\nu(n,n') \left(\frac{|n'|}{n}\right)^{(1-w)/2} I_{1-w}\left(\frac{4\pi}{c}\sqrt{-nn'}\right),$$

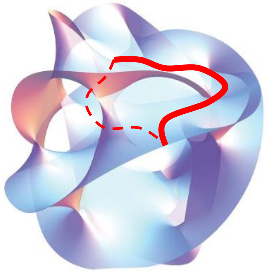
Kloosterman sum

Bessel function

$$K_\mu^\nu(x,y) = \mathrm{i}^{-w} \sum_{\substack{-c\leq d<0\\ (c,d)=1}} \left(\rho^{-1}\right)_\mu^\nu\, e^{2\pi\mathrm{i}\left(\frac{a}{c}\,y+\frac{d}{c}\,x\right)}$$

D4-D2-D0 black holes in Type IIA/CY

More complicated setup: Type IIA string theory on a Calabi-Yau threefold



Effective theory in 4d — N=2 SUGRA

Important class of ½ BPS black holes is constructed as bound states of D4, D2 and D0-branes characterized by electro-magnetic charge

$$\gamma = (0, p^a, q_a, q_0) \qquad a = 1, \dots, b_2(CY)$$

no D6

label 4- and 2-dim cycles wrapped by D4 and D2-branes

BPS index $\Omega(\gamma)$ — black hole degeneracy = generalized Donaldson-Thomas invariant of CY

Natural generating function $h^{\mathrm{DT}}_{p^a,q_a}(\tau) = \sum_{q_0 > q_{0,\min}} \Omega(\gamma)\, e^{2\pi \mathrm{i} q_0 \tau}$

But this function secretly depends on *CY moduli* and hence is *not* expected to have any nice properties

Wall-crossing

The reason: multi-centered black holes (bound states of black holes) are stable only in a region of the moduli space and decay in another

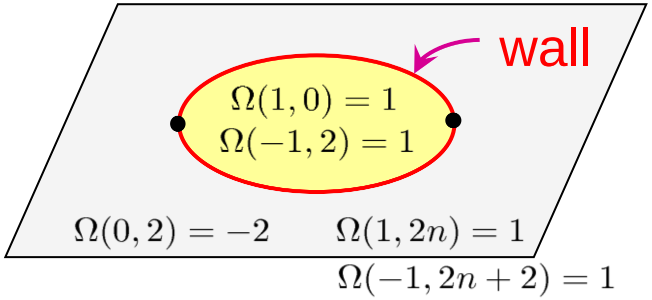
$\Omega(\gamma)$ *is only piecewise constant*

wall of marginal stability

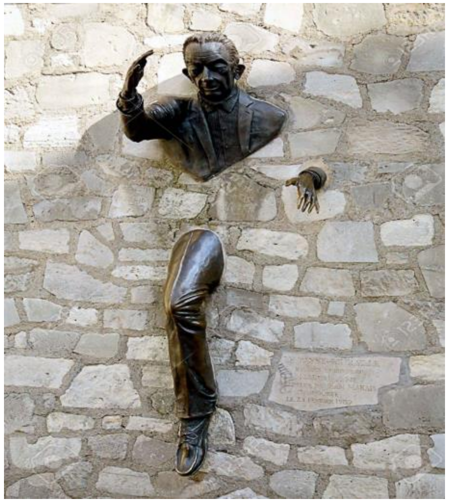
bound state $\gamma_1+\gamma_2$ does not exist	bound state $\gamma_1+\gamma_2$ exists

position of walls $\quad M_{\gamma_1}+M_{\gamma_2}=M_{\gamma_1+\gamma_2}$

Example: pure SU(2) $\mathcal{N}$=2 SUYM



Seiberg,Witten ’94



How to cross a wall?

Kontsevich-Soibelman formula

allows to evaluate $\Omega(\gamma)$ on one side of a wall from their values on the other side

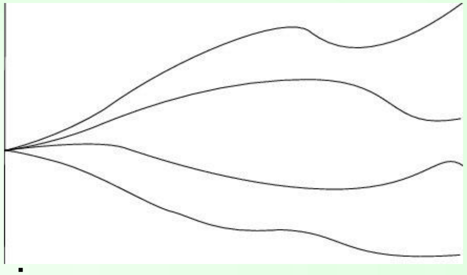
It is enough to find $\Omega(\gamma)$ in one chamber

Attractor chamber

There is one special chamber in the moduli space: *attractor chamber*

Attractor mechanism [Ferrara,Kallosh,Strominger '95]

CY moduli — scalar fields in 4d which depend on the radial coordinate in the black hole background



SUGRA equations fix their values at the horizon in terms of the black hole charges and *independently* of their values at infinity $z^a_\star(\gamma)$

In the attractor chamber all multi-centered black holes are unstable (up to "*scaling solutions*")

Maldacena,Strominger,Witten '97

$$\Omega^{\rm MSW}_\gamma = \Omega(\gamma, z^a_\star(\gamma)) = \Omega_p(\hat q_0)$$

$\hat q_0 \equiv q_0 - \frac12\, \kappa^{ab} q_a q_b$ — invariant charge bounded from above

generating function of MSW invariants

$$h_p(\tau) = \sum_{\hat q_0 \le \hat q_0^{\rm max}} \Omega_p(\hat q_0)\, e^{-2\pi \mathrm{i} \hat q_0 \tau}$$

Where is a torus?

Why do we expect $h_p(\tau)$ to be modular?

Type IIA / CY = M-theory / $S^1 \times$ CY

D4-brane / Γ_4 = M5-brane / $S^1 \times \Gamma_4$

In the limit of small Γ_4, the world-volume theory on M5-brane reduces to a *2d CFT* where states are counted by the *elliptic genus*

$$Z_p(\tau, y) = \mathrm{Tr}\left[(2J_3)^2(-1)^{2J_3}\, e^{2\pi \mathrm{i}\left(\left(L_0-\frac{c_L}{24}\right)\tau-\left(\bar L_0-\frac{c_R}{24}\right)\bar\tau+q_a y^a\right)}\right]$$
$$\sim h_p(\tau)\overline{\theta_p(\tau, y)}$$

This is a finite temperature partition function with $\beta = T^{-1} \sim \tau_2$ ➡ Time is also compactified on a circle and the CFT is considered on a torus

⬇

$h_p(\tau)$ — *is a modular form*

But this conclusion turns out to be naive and is true only for irreducible Γ_4

SA,Banerjee,Manschot,Pioline ‘16

Where is another torus?

Can one find a torus in spacetime?

modular group

$$\text{Type IIA}/_{CY \times S^1} = \text{M-theory}/_{CY \times \overbrace{S^1 \times S^1}^{T^2}}$$

Effective theory in 3d — SUSY non-linear sigma model with target space $\mathcal{M}$

After compactification on the circle

D4-brane $/_{\Gamma_4 \times S^1}$ — pointlike object (*instanton*)

Affects the metric on $\mathcal{M}$ by instanton corrections

Each instanton is weighted by the BPS index $\Omega(\gamma)$

$\mathcal{M}$ carries an isometric action of the modular group

Restriction on (the generating function of) BPS indices $\Omega(\gamma)$

The functions $h_p(\tau)$ are (higher depth) *mock* modular forms

Mock modular forms

First examples of *mock theta functions* appeared in the last letter of Ramanujan to Hardy 100 years ago, but remained mysterious objects until recently….

Zwegers. '02

Mock modular form — *a holomorphic* function which is "almost" modular with an anomaly controlled by another modular form (*shadow*)

$$h(\tau) \mapsto (c\tau+d)^w \left(h(\tau) - \int_{-d/c}^{-i\infty} \frac{\overline{g(\bar z)}}{(\tau - z)^w}\, dz \right)$$

modular form of weight $2-w$

and (*non-holomorphic*) modular *completion* given by $\widehat{h}(\tau,\bar\tau) = h(\tau) - \int_{\bar\tau}^{-i\infty} \frac{\overline{g(\bar z)}}{(\tau - z)^w}\, dz$

- [Dabholkar,Murthy,Zagier '12] : ¼ BPS b.h. in **Type II / K3 × T²** (*immortal dyons*)
- [SA,Banerjee,Manschot,Pioline '16] : D4-D2-D0 b.h. in **Type II / CY** with D4-brane wrapped on a reducible divisor $\Gamma = \Gamma_1 + \Gamma_2$
- [SA, Pioline '18] : higher depth mock modularity if $\Gamma = \sum_{i=1}^{n} \Gamma_i,\ n>2$

Applications

- Mock modularity and knowledge of the shadow determine uniquely the Fourier coefficients in terms of polar terms → Rademacher expansion

Explicit expressions for generating functions of black hole degeneracies

↕

Donaldson-Thomas invariants for compact CY

- Non-compact CY (zoom in around a singularity) → geometric engineering of SUSY gauge theories

SU(N) Vafa-Witten invariants of complex surfaces (count instantons in a topological theory)

- New results on indefinite theta series

Explicit construction of non-holomorphic completions for arbitrary signature of quadratic form

↓

Development of the theory of higher depth mock modular forms

Conclusions

- Modular symmetry is ubiquitous in string theory and is a powerful tool to obtain *exact* results

- It controls generating functions of degeneracies of BPS black holes appearing in string compactifications

- When there are multi-centered black holes, modularity is replaced by its *mock* version

- These results have numerous applications ranging from calculation of exact black hole degeneracies to pure mathematics

- Modular symmetry = S-duality

  It is a tool to understand string theory at the non-perturbative level

- Modularity in string compactifications with N=1 supersymmetry?

Thank you!