Subsonic accretion and dynamical friction for a black hole moving through a self-interacting scalar dark matter cloud

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In collaboration with Patrick Valageas & Philippe Brax

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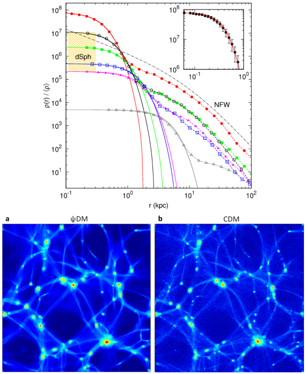
Self-interacting scalar dark matter ?



Upper panel *: Radial density profiles of haloes formed in the* Ψ*DM model compared to CDM*
Lower panel *: Comparison of large-scale structures formed by CDM and by* Ψ*DM*
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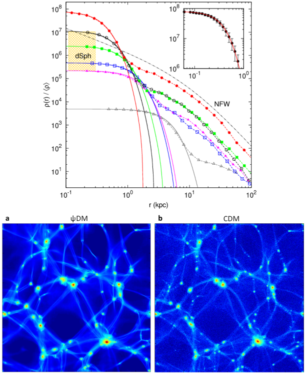

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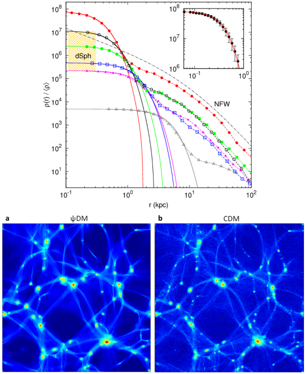

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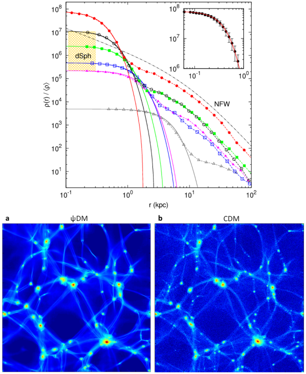

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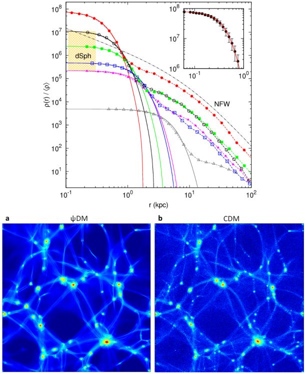

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- Masses : $10^{-22}eV < m < eV$
- Form stable equilibrium configurations, between ~~self-gravity and quantum pressure (Fuzzy DM)~~ self-gravity and pressure due to self-interactions → different behaviour at galactic scales
- Might help to solve some cosmological tensions (core-cusp problem, missing satellites, ...)
- Still recovers the successes of ΛCDM at large scale

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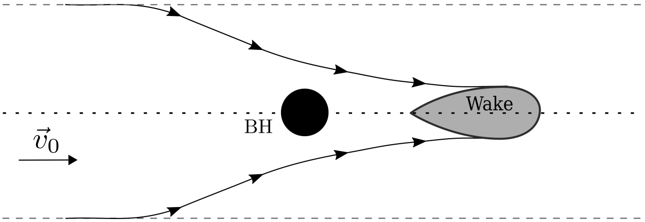

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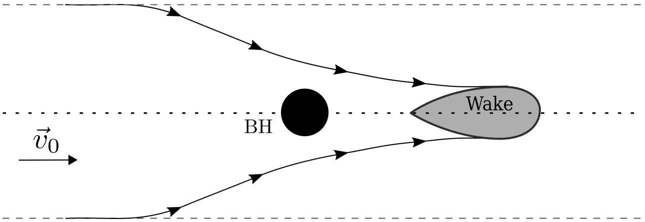

- Impact on : Gravitational waves emission dephasing in binary BHs
  [**Kocsis et al.** PhysRevD.84.024032 (2011), **Barausse et al.** PhysRevD.89.104059 (2014), **Cardoso & Maselli** AA 664 (2020), ...]

$$\Psi(f) = 2\pi f \left(t_c + \frac{r}{c}\right) - \Phi_c - \frac{\pi}{4} + \Psi_{GR}(f) + \Psi_{env}(f)$$

Action and field solution (large-m limit)

- SFDM Action : $S_\phi = \int d^4x\sqrt{-g}\left[-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right]$
- $V(\phi) = \frac{m^2}{2}\phi^2 + V_{\rm I}(\phi)\,, \qquad V_{\rm I}(\phi) = \frac{\lambda_4}{4}\phi^4$ (first term obey $\rho \propto a^{-3}$)

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- The Klein-Gordon equation [**Brax et al.** PhysRevD.101.023521 (2020)] :

$$\boxed{\frac{\partial^2\phi}{\partial t^2} - \sqrt{\frac{f}{h^3}}\vec{\nabla}\cdot\left(\sqrt{fh}\vec{\nabla}\phi\right) + f\frac{\partial V(\phi)}{\partial\phi} = 0}$$

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- If the radial and angular derivatives are discarded (local approximation), we can recognize the Duffing equation :

Amplitude — Phase related to the velocity

$$\boxed{\phi = \phi_0(r,\theta)cn[\omega(r,\theta)t - \mathbf{K}(k)\beta(r,\theta), k(r,\theta)]}$$

Angular frequency — Modulus (nonlinear oscillator)

A system of 3 equations

- At leading order in the large-mass limit, we obtain the system :

Relativistic Bernouilli equation : $(\nabla\beta)^2 = \frac{h}{f}\left(\frac{2\omega_0}{\pi}\right)^2 - \frac{hm^2}{(1-2k^2)\mathbf{K}^2}$

Deviation from harmonic oscillator : $\frac{\lambda_4\phi_0^2}{m^2} = \frac{2k^2}{1-2k^2}$

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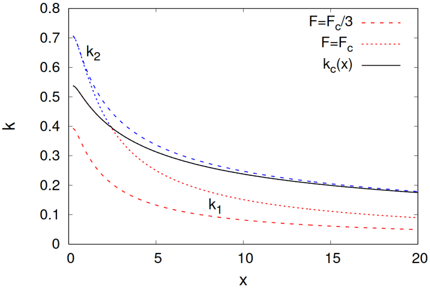
- From the conservation equation $\langle\nabla_\mu T_0^\mu\rangle = 0$ (where $\langle\ldots\rangle$ is the average over the oscillations, to ensure steady state) we obtain a result in the form of an effective continuity equation :

$$\nabla\cdot(\rho_{\rm eff}\nabla\beta) = 0\,, \qquad \rho_{\rm eff} = \sqrt{fh}\,\phi_0^2\omega\mathbf{K}\langle {\rm cn}'^2\rangle$$

Low and high velocity branches

- In the case of radial accretion, the effective continuity equation can be integrated at once, since only depending on radial derivatives. We get 2 solutions for k as for hydrodynamics infall (see [**Bondi** (1952), **Michel** (1972)])



Moduli k_1 and k_2 for a constant flux $F_c/3$ (dashed lines) and F_c (dotted lines). The critical modulus k_c (solid line) is equal to k_1 for $x < x_\star$ and to k_2 for $x > x_\star$, with $F = F_c$
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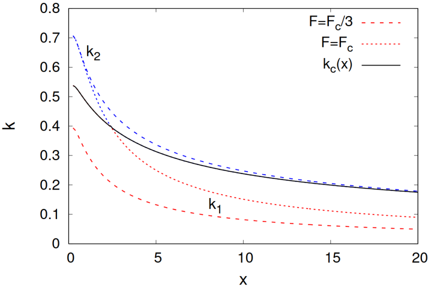

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- In our case, Same k near the BH + need to solve at large-radii along k_2

Low-k and subsonic regime

- Using the rescaled quantities $\hat{r} = \frac{r}{r_s}$ and $\hat{\beta} = \frac{\pi}{2mr_s}\beta$, we obtain :

$$(\hat{\nabla}\hat{\beta})^2 = \frac{3}{2}k_0^2 + v_0^2 + \frac{1}{\hat{r}} - \frac{3}{2}k^2 = \frac{3}{2}\left[k_+(\hat{r})^2 - k^2\right] \ ,$$

where we introduced $k_+(\hat{r})^2 = \boxed{k_0^2} + \boxed{\frac{2}{3}v_0^2} + \boxed{\frac{2}{3\hat{r}}}$

Enthalpy/Soliton density | Relative velocity term | BH contribution

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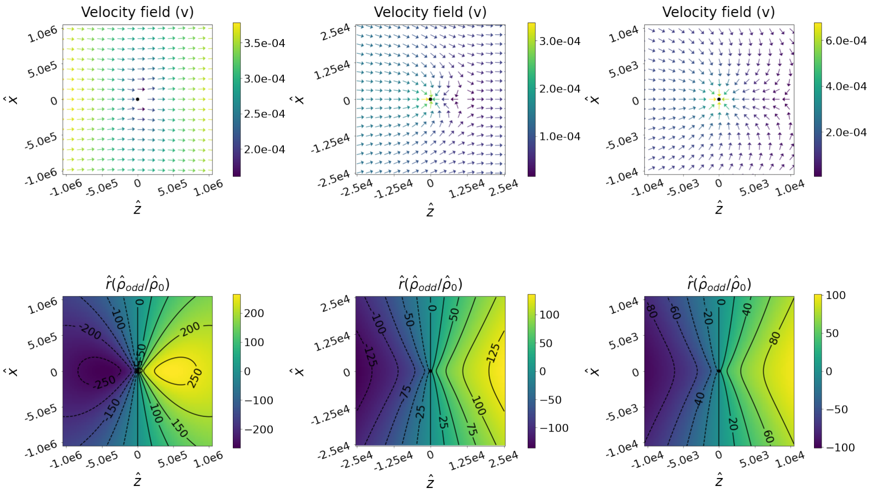
Enthalpy/Soliton density — Relative velocity term — BH contribution

- With this, we can re-express the conservation equation ($\rho_{\text{eff}} \propto k^2$) as :

$$\boxed{\hat{\nabla} \cdot \left[\left(k_+(x)^2 - \frac{2}{3}(\hat{\nabla}\hat{\beta})^2\right)\hat{\nabla}\hat{\beta}\right] = 0} \rightarrow \text{(subsonic regime)}\ \hat{\nabla}\cdot\left[\mathrm{k}_+(\mathrm{x})^2\hat{\nabla}\hat{\beta}\right] = \mathrm{S}$$

Velocity and density fields

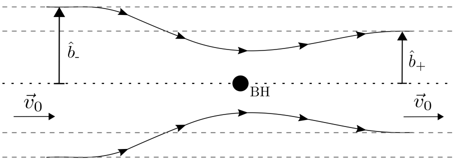


Flow (top panels) and odd-component of the density field $\hat{r}\,\hat{\rho}_{odd}/\hat{\rho}_0$ (bottom panels) for the scalar field cloud at different scales

Dynamical friction & mass accretion

- The mass accretion (mass conservation) :

$$\dot{\hat{M}}_{\rm BH} = -\int_{\hat{S}} d\vec{\hat{S}} \cdot \hat{\rho}\vec{v} = 2\pi \int_0^{\hat{b}_-} d\hat{b}\, \hat{b}\; \hat{\rho} v_z\big|_{\hat{z}_-} - 2\pi \int_0^{\hat{b}_+} d\hat{b}\, \hat{b}\; \hat{\rho} v_z\big|_{\hat{z}_+}$$



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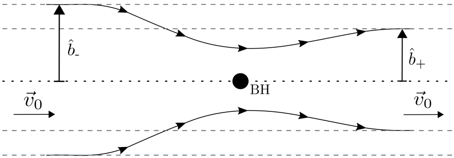

- Finding $\hat{b}_+$ from the streamlines, $\boxed{\dot{M}_{\rm BH} \sim \rho_0 r_s^2/c_s^2 \sim \rho_0 \mathcal{G}^2 M_{\rm BH}^2/c_s^2}$

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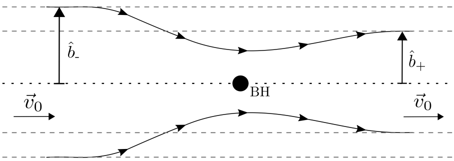

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- The dynamical friction (momentum conservation), exact analytical result :

$$F_z = \frac{dp_z}{dt} = -\int_{S_{\rm out}} d\vec{S}\cdot\rho\vec{v}v_z - \int_{S_{\rm out}} d\vec{S}\cdot P\vec{e}_z = \boxed{\dot{M}_{\rm BH}v_0}$$

Comparison with FDM and CDM

- SFDM : $v_0 < c_s, \quad F_z \sim \frac{\mathcal{G}^2 M_{\rm BH}^2 \rho_0 v_0}{c_s^2}$

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- So $\boxed{\dfrac{F_z}{F_{\rm free}} \sim \dfrac{c_s}{C} \ll 1 \ , \ \dfrac{F_z}{F_{\rm FDM}} \sim v_0 \ll 1 \ \text{ and } \ \dfrac{F_z}{F_{\text{perfect gas}}} \sim c_s \ll 1}$

Summary

In the large-mass regime + subsonic regime :

- The self-interaction modifies the accretion rate and the dynamical friction of a moving BH inside a soliton
- The system is closer to the case of a perfect gas than FDM
- Close to the BH, the self-interactions are able to significantly slow down the infall, thus the dynamical friction is smaller than for a perfect gas
- The obtained dynamical friction is also smaller than for FDM/Collisionless particles

Prospects

- In progress :
  - Calculation of the phase shift induced on GW emissions for BBHs
  - $\Phi(f) = \int_f^{f_c} \omega_{gw} df \propto f^{-13/3}$, an inverse PN contribution (-4 PN)
  - Compared to a fluid at rest : $\Phi(f) \propto f^{-16/3}$ (-5.5 PN)
    [**Barausse et al.** PhysRevD.89.104059 (2014), **Cardoso & Maselli** AA 664 (2020), ...]

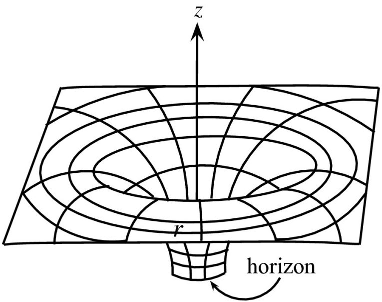
- For future work :
  - Generalize to supersonic flow, $v_0 > c_s = \frac{\sqrt{3}}{2} k_0$
  - Add spin to the BH (for more realistic results) $\rightarrow$ Kerr BH

Schwarzschild metric

- $ds^2 = -f(r)\,dt^2 + h(r)\,(dr^2 + r^2\,d\vec{\Omega}^2)$
- Isotropic metric functions :

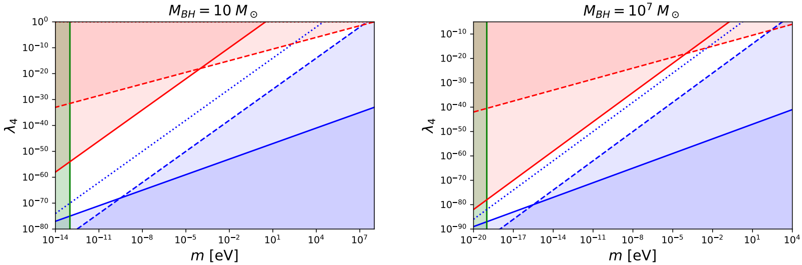
$$f(r) = \left(\frac{1 - r_s/(4r)}{1 + r_s/(4r)}\right)^2 ,$$
$$h(r) = (1 + r_s/(4r))^4$$

- In this coordinates, the BH horizon is located at $r = \dfrac{r_s}{4}$
- At large radii : $f = 1 + 2\Phi_\mathrm{N}$ and $h = 1 - 2\Phi_\mathrm{N}$



[**Manoukian** (2020)]

Parameter space (m, λ_4)



Domain in the parameter space (m, λ_4) *for a BH of mass* $10\, M_\odot$ *and* $10^7 M_\odot$

- Solid : Quantum pressure comes into play at cloud scale
- Dashed : The cloud radius is of the order of r_s
- Dotted : The Cloud radius does not extend above r_{sg} (self-gravity regime never reached)
- Solid : λ_4 becomes large enough to see deviations from CDM model at large scales
- Dashed : Obs. constraints on cross section [**Brax et al.** PhysRevD.100.023526 (2019)]
- Solid : Quantum pressure comes into play at r_s scale

Linear flow (1)

- At small radii (but far from BH), we are in low-velocity radial accretion regime and so $(\hat{\nabla}\hat{\beta})^2 \ll k_+^2$:

$$\hat{\nabla} \cdot \left[k_+(\hat{r})^2 \hat{\nabla}\hat{\beta}\right] = 0$$

- The spherical symmetry of k_+^2 implies that the angular part of the linear modes can be expanded over spherical harmonics

- We only need $Y_l^0(\theta, \phi)$ since we have axisymmetric solutions and so the Green function $G_\ell(\hat{r}, \theta) = G_\ell(\hat{r})\, P_\ell(\cos\theta)$ with :

$$\frac{d}{d\hat{r}}\left(\hat{r}^2 k_+^2 \frac{dG_\ell}{d\hat{r}}\right) - \ell(\ell+1)k_+^2 G_\ell = 0$$

Linear flow (2)

- The boundary condition at large radius, $\hat{r} \to \infty$: $\hat{\beta} = v_0 \hat{r} \cos\theta$
- The inner boundary condition, $\hat{r} = \hat{r}_{\mathrm{m}}$: $\dfrac{\partial \hat{\beta}}{\partial \hat{r}} \simeq v_r^{\mathrm{m}}, \quad \dfrac{\partial \hat{\beta}}{\partial \theta} \simeq 0$
- Thus, at linear level we only generate the monopole and the dipole :

$$\hat{\beta}^L = \hat{\beta}_0^L(\hat{r}) + \hat{\beta}_1^L(\hat{r}) \, \cos(\theta) \, ,$$

where :

$$\hat{\beta}_0^L(\hat{r}) = -v_r^{\mathrm{m}} \left(\gamma \hat{r}_{\mathrm{m}}^2 + \hat{r}_{\mathrm{m}}\right) \ln\left(\gamma + \frac{1}{\hat{r}}\right) \, ,$$

$$\hat{\beta}_1^L(\hat{r}) = \frac{v_0}{\gamma} \, (\gamma \hat{r})^{\sqrt{2}} \, \frac{\Gamma(-1+\sqrt{2})\,\Gamma(2+\sqrt{2})}{\sqrt{2}\,\Gamma(1+2\sqrt{2})} \times \, {}_2F_1(2+\sqrt{2}, -1+\sqrt{2}; 1+2\sqrt{2}; -\gamma \hat{r})$$

Beyond linear flow

- To go beyond linear flow, we can consider the second term in the conservation equation as a source term :

$$\nabla \cdot (k_+^2 \nabla \beta) = S, \quad S = \frac{2}{3} \nabla \cdot [(\nabla \beta)^2 \nabla \beta]$$

- In this case, using $\nabla \cdot (k_+^2 \nabla G) = \delta_D(\vec{r} - \vec{r}^{\,\prime})$ and developing G and β in spherical harmonics :

$$G(\vec{r}, \vec{r}^{\,\prime}) = \sum_{\ell,m} G_\ell(r, r') Y_\ell^m(\theta', \varphi')^* Y_\ell^m(\theta, \varphi) \,,$$

$$\beta(r, \theta) = \sum_\ell \beta_\ell(r) P_\ell(\cos\theta) \,,$$

we obtain :

$$\boxed{\beta_\ell = \beta_\ell^L + \int_{r_\mathrm{m}}^{\infty} dr' \; r'^2 G_\ell(r, r') S_\ell(r')}$$

Green function components

- By setting $\gamma = \frac{3k_0^2}{2} + v_0^2$ and $\frac{3}{2}k_+^2 = \frac{1}{\hat{r}} + \gamma$:

$$G_0^+(\hat{r}) = 1\,, \qquad G_0^-(\hat{r}) = \ln\left(\gamma + \frac{1}{\hat{r}}\right)\,,$$
$$G_\ell^+(\hat{r}) = \hat{r}^{a-\nu}\ {}_2F_1(a, 1-b; 1-b+a; -\gamma\hat{r})\,,$$
$$G_\ell^-(\hat{r}) = \hat{r}^{-\nu}\ {}_2F_1(a, b; c; -1/(\gamma\hat{r}))\,,$$

with :

$$\nu = \frac{1 + \sqrt{1 + 4\ell(\ell+1)}}{2}\,, \qquad a = \nu + \sqrt{\nu(\nu-1)}\,,$$
$$b = \nu - \sqrt{\nu(\nu-1)}\,, \qquad c = 2\nu$$

Green function

- Since β_ℓ^L already matches the boundary conditions, we require the Green function to become negligible at r_m and large radii, and $\frac{\partial G_0}{\partial r}(r_\mathrm{m}) = 0$ to recover the radial velocity :

$$
\begin{aligned}
r < r' : &\quad G_0(r,r') = -3G_0^-(r')/2 \\
r > r' : &\quad G_0(r,r') = -3G_0^-(r)/2 \\
r < r' : &\quad G_\ell(r,r') = A\left[G_\ell^-(r_\mathrm{m})G_\ell^+(r) - G_\ell^+(r_\mathrm{m})G_\ell^-(r)\right] , \\
r > r' : &\quad G_\ell(r,r') = B\, G_\ell^-(r) ,
\end{aligned}
$$

where :

$$
A = \frac{3G_\ell^-(r')}{2G_\ell^-(r_\mathrm{m})(r' + \gamma r'^2)[G_\ell^+(r')G_\ell^{-'}(r') - G_\ell^-(r')G_\ell^{+'}(r')]} ,
$$

$$
B = A\left[G_\ell^-(r_\mathrm{m})\frac{G_\ell^+(r')}{G_\ell^-(r')} - G_\ell^+(r_\mathrm{m})\right]
$$

Numerical simulation

- To obtain numerical results, one solution is to express S in spherical harmonics to :

$$S = \sum_{\ell} S_\ell P_\ell(\cos\theta)$$

- Then, we can solve :

$$\boxed{S_\ell = \int d\vec{\Omega} \frac{2}{3} \nabla \cdot [(\nabla\beta)^2 \nabla\beta] P_\ell(\cos\theta)}$$

- And finally, we can calculate the new value of β_ℓ
- This iteration go on until we reach convergence (if there is)