Self-similar solutions for Fuzzy Dark Matter

Published in Phys. Rev. D 105, 123528 (2022)
arXiv:2203.05995

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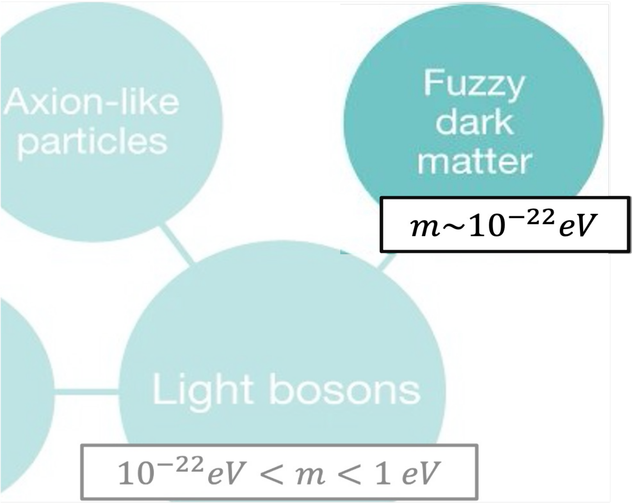
Théorie, Univers et Gravitation (TUG) Workshop, October 6, 2022

Introduction

Dark matter candidates

Dark matter candidates

Fuzzy dark matter (FDM): Field picture



Action:

$$S = \int d^4x \sqrt{-g} \left[-\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{m^2}{2} \phi^2 \right]$$

1. FIELD PICTURE

$$i \frac{\partial \psi}{\partial t} = -\frac{1}{2m} \nabla^2 \psi + m \Phi_{\rm N}\, \psi,$$

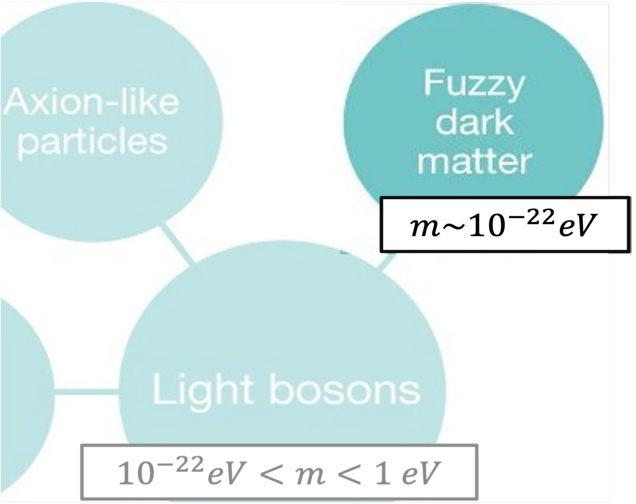
$$\nabla^2 \Phi_{\rm N} = 4\pi \mathcal{G}_{\rm N} \rho, \quad \rho = m \psi \psi^*$$

Schrödinger-Poisson (SP)

SP system scaling law

$$\{t, \vec{r}, \Phi_{\rm N}, \psi, \rho\} \to \{\lambda^{-2} t, \lambda^{-1} \vec{r}, \lambda^2 \Phi_{\rm N}, \lambda^2 \psi, \lambda^4 \rho\}\,.$$

FDM: Fluid picture



Action:

$$S = \int d^4x\sqrt{-g}\left[-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{m^2}{2}\phi^2\right]$$

1. FIELD PICTURE

$$i\frac{\partial\psi}{\partial t} = -\frac{1}{2m}\nabla^2\psi + m\Phi_{\mathrm{N}}\,\psi,$$

$$\nabla^2\Phi_{\mathrm{N}} = 4\pi\mathcal{G}_{\mathrm{N}}\rho, \quad \rho = m\psi\psi^*$$

Schrödinger-Poisson (SP)

2. HYDRODYNAMICAL PICTURE

$$\psi = \sqrt{\rho}\,e^{iS/\epsilon}, \quad \vec{v} = \nabla S,$$

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\vec{v}) = 0,$$

$$\frac{\partial\vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v} = -\nabla\left(\Phi_{\mathrm{N}} + \Phi_{\mathrm{Q}}\right)$$

$$\nabla^2\Phi_{\mathrm{N}} = 4\pi\rho.$$

Continuity, Euler and Poisson

SP system scaling law

$$\{t, \vec{r}, \Phi_{\mathrm{N}}, \psi, \rho\} \to \{\lambda^{-2}t, \lambda^{-1}\vec{r}, \lambda^2\Phi_{\mathrm{N}}, \lambda^2\psi, \lambda^4\rho\}.$$

Quantum pressure

$$\Phi_{\mathrm{Q}} = -\frac{\epsilon^2}{2}\frac{\nabla^2\sqrt{\rho}}{\sqrt{\rho}}.$$

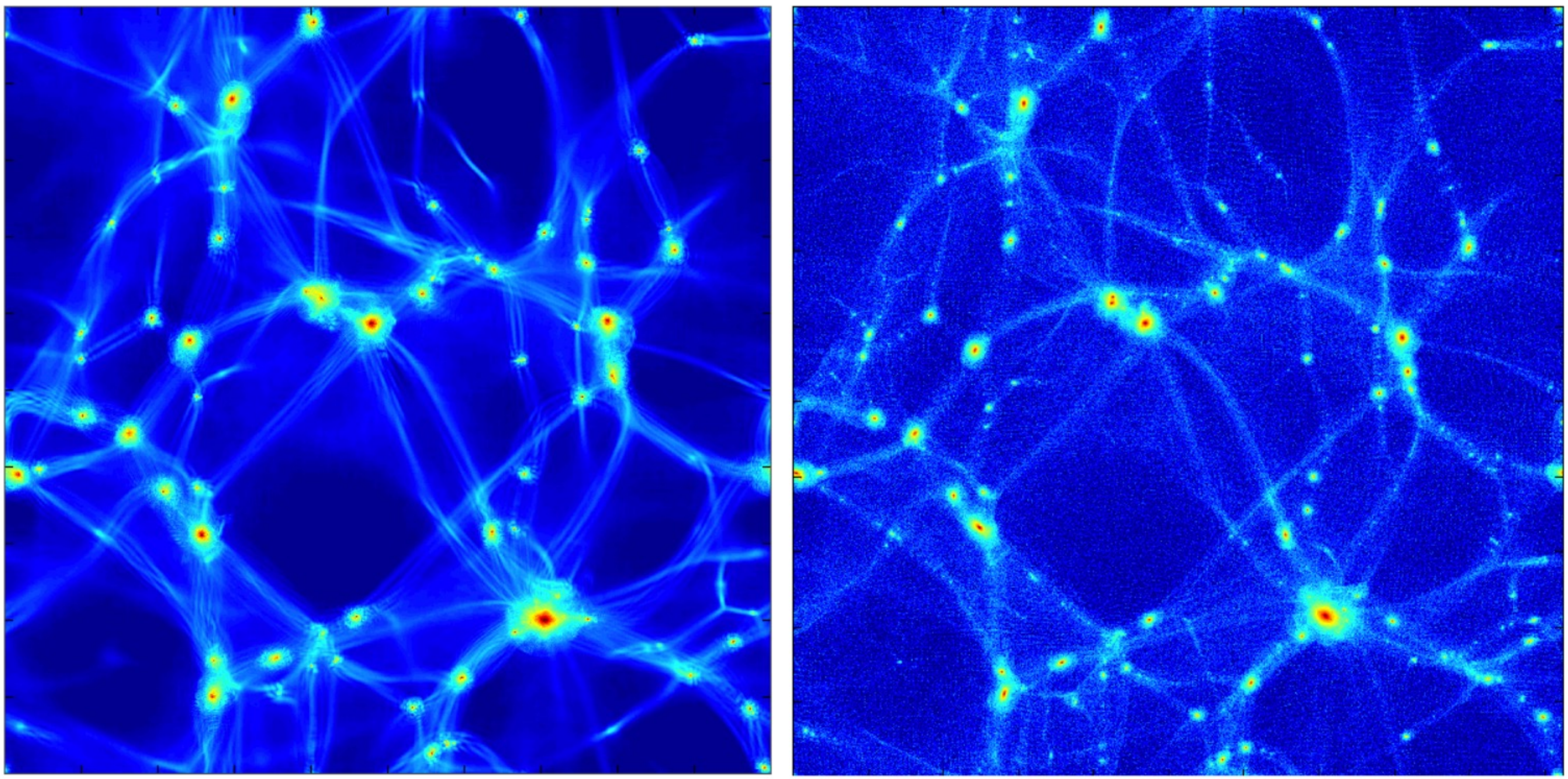
$$\epsilon = \frac{T}{mL^2} \sim \frac{\lambda_{DB}}{L}$$

$\lambda_{DB} \sim 0.5\ kpc$

FDM at large scales

Recover the success of CDM large scale distribution of filaments and voids.

a **Fuzzy Dark Matter (FDM)** b **Cold Dark Matter (CDM)**



Schive, Chiueh, and Broadhurst (2014)

FDM at small scales

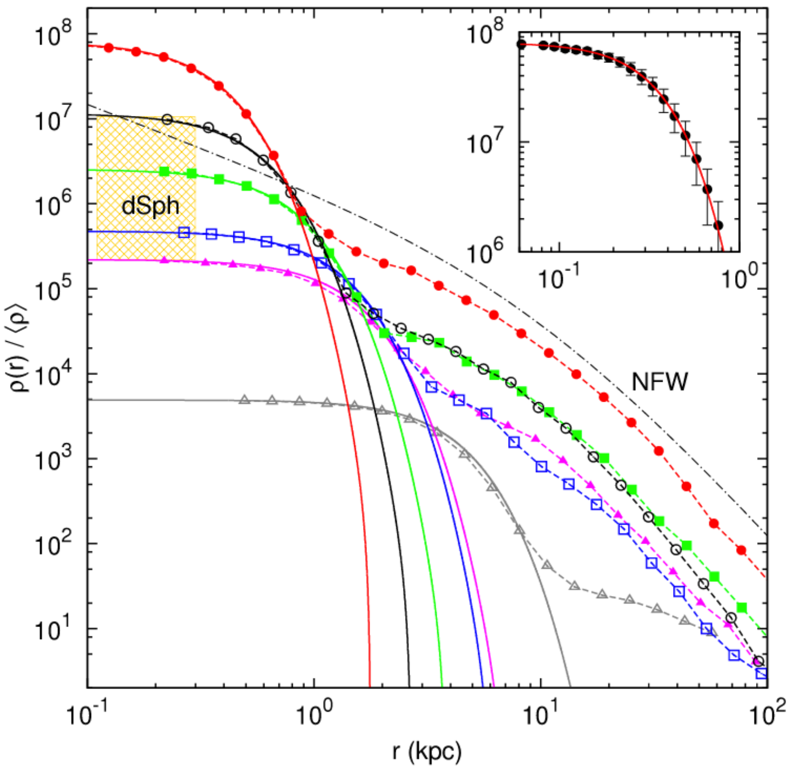
Hydrostatic equilibrium

$$\Phi_{\rm N} + \Phi_{\rm Q} = \alpha,$$

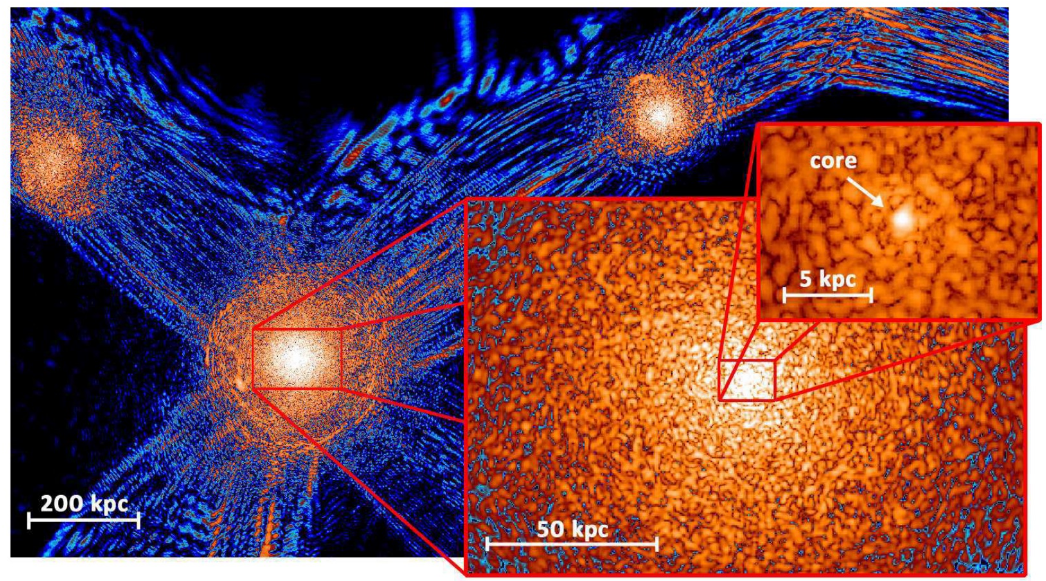
Soliton profile

$$\epsilon^2 \nabla^2 \psi_{\rm sol} = 2(\Phi_{\rm N} - \alpha)\psi_{\rm sol}.$$

$$\nabla^2 \Phi_{\rm N} = 4\pi \psi_{\rm sol}^2$$



Radial density profiles of haloes formed in the ψDM model



A slice of density field of ψDM simulation on various scales at z=0.1

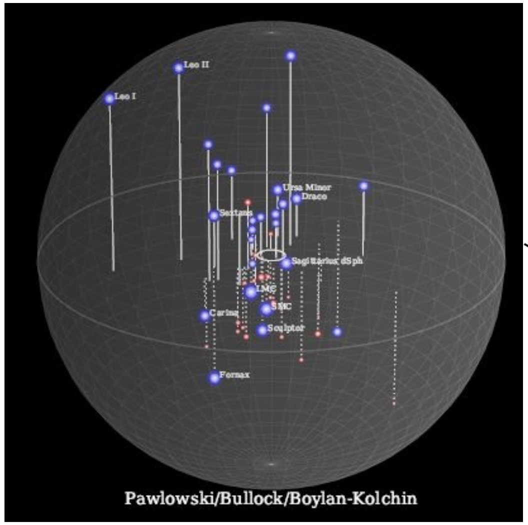
Schive, Chiueh, and Broadhurst (2014)

FDM Motivation 1) Explanation to CDM small-scales tensions

Missing satellite problem

Predicted ΛCDM substructure

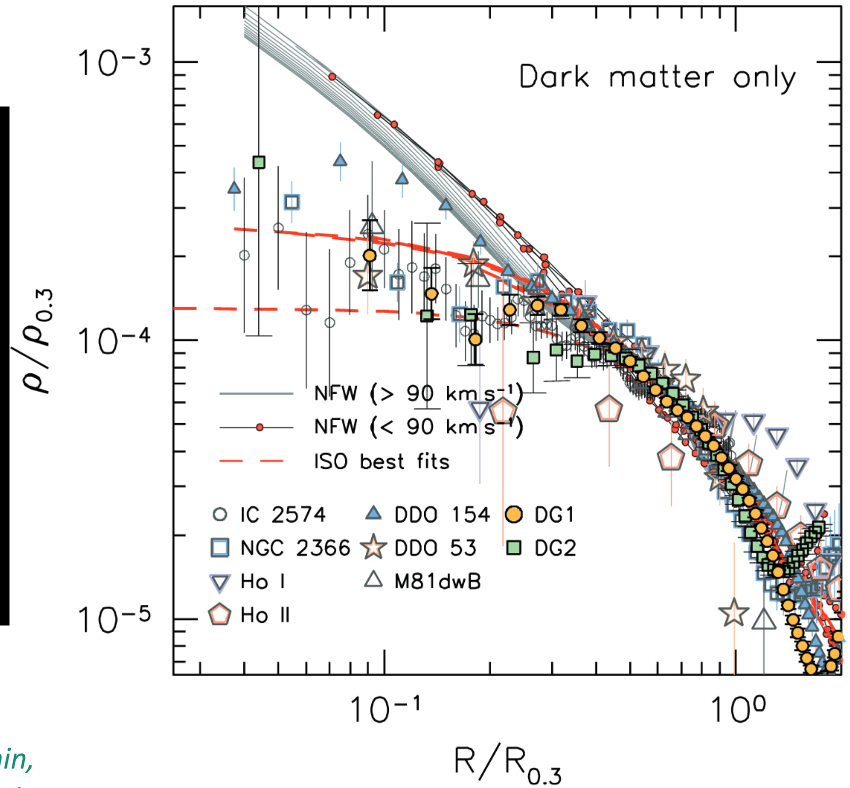
Simulation by V. Robles and T. Kelley and collaborators.



Known Milky Way satellites

James S. Bullock, M. Boylan-Kolchin, M. Pawlowski

Core/cusp problem



Density profiles observations and simulations

Antonino Popolo, Morgan Le Delliou (2017)

FDM Motivation 2) Alternative to CDM N-Body simulations

FDM comoving Vlasov equation

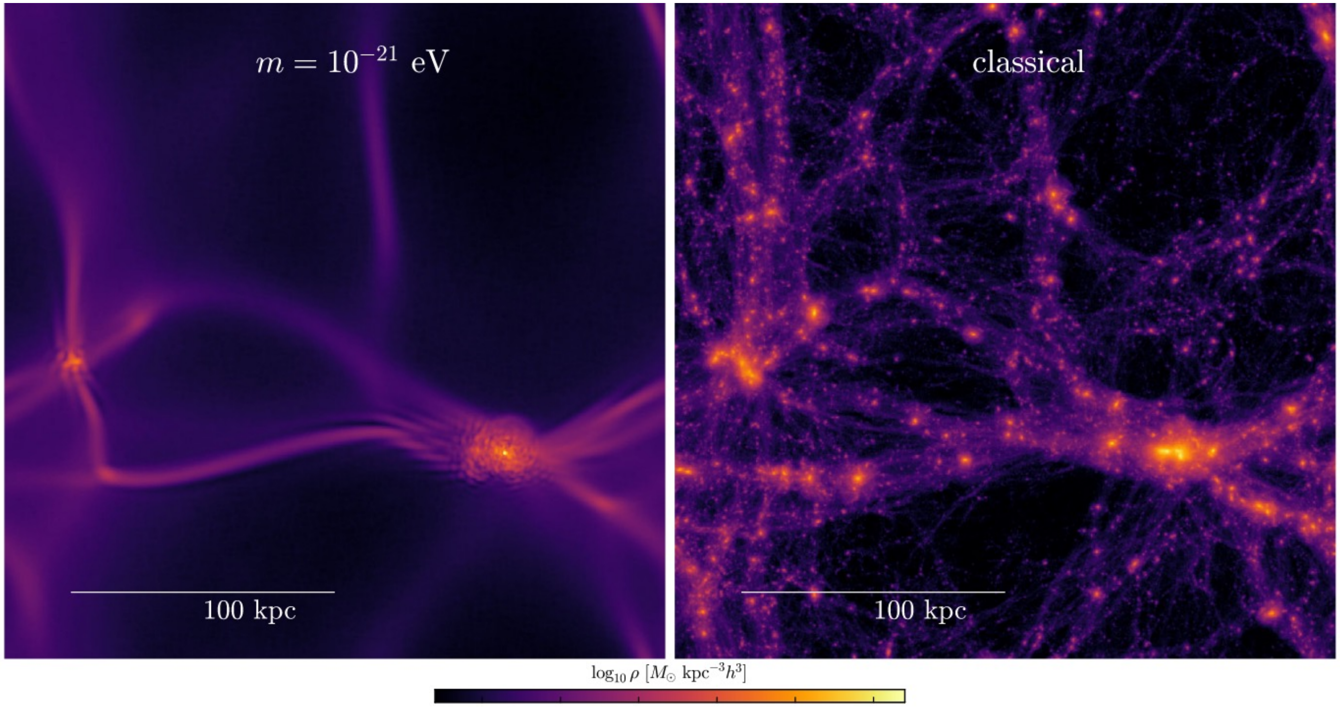
$$\frac{\partial f_{\mathrm{W}}}{\partial t} + \frac{\vec{p}}{a^2} \cdot \frac{\partial f_{\mathrm{W}}}{\partial \vec{x}} - \vec{\nabla}_x \varphi_{\mathrm{N}} \cdot \frac{\partial f_{\mathrm{W}}}{\partial \vec{p}} + \mathcal{O}(\epsilon) = 0.$$

CDM comoving Vlasov equation

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \frac{\boldsymbol{p}}{ma^2} \cdot \boldsymbol{\nabla}_x f - m(\boldsymbol{\nabla}_x \phi) \cdot \boldsymbol{\nabla}_p f = 0,$$

Kaiser (1993)

Link the Schrödinger wave function ψ to a function f in phase space.



Cosmological simulation at z=3, evolved either as CDM (VP eq) or as FDM (SP)

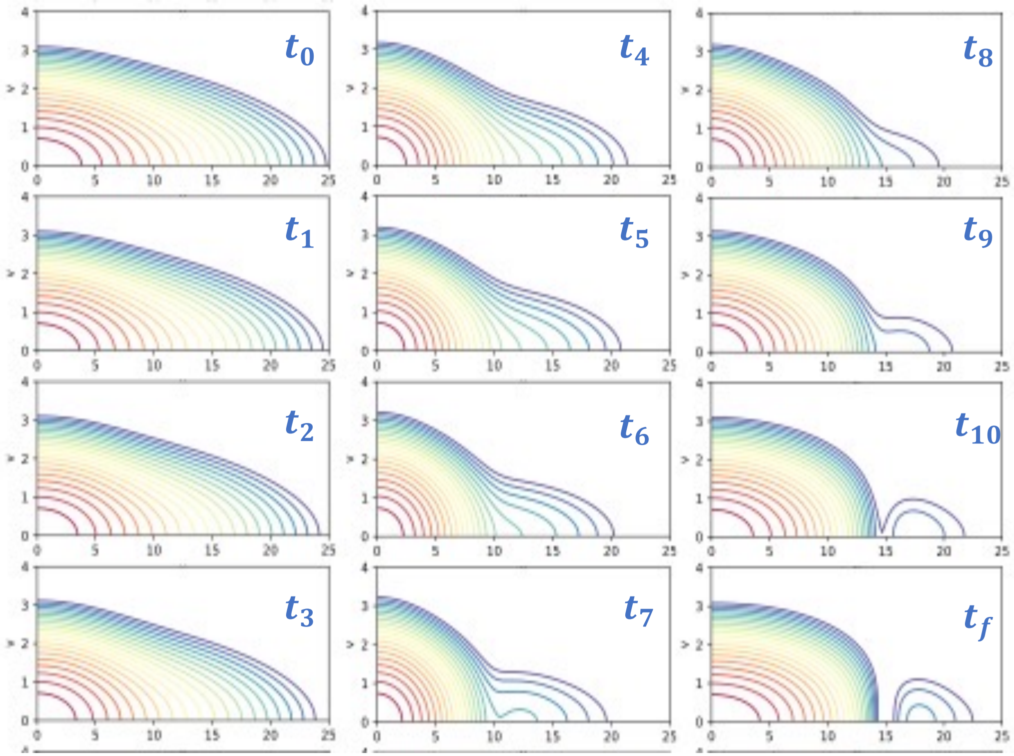
Philip Mocz and Lachlan Lancaste,Anastasia Fialkov and Fernando Becerra, Pierre-Henri Chavanis (2018).

Motivation of this work

1. **Go beyond the static solitons** by investigating **dynamical self-similar solutions.**
2. Understand physical processes: **gravitational cooling.**
3. Understand **comparison with self similar solutions for CDM.**

Dynamics: 3D Numerical simulations

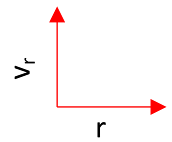
FDM out of equilibrium, ε=1



R.Galazo-Garcia

Husimi Phase-space distribution

Axis:



Gravitational Cooling

E.Seidel, W.M.Suen(1994)
F.S Guzmán, L.A. Ureña-López (2004)

Self-similar solutions for FDM

Cosmological Self-similar solutions

SELF-SIMILAR ANSTATZ

$$\rho = t^{-\alpha} f\left(\frac{r}{t^{\beta}}\right), \quad v = t^{-\delta} g\left(\frac{r}{t^{\beta}}\right), \quad \Phi_{\mathrm{N}} = t^{-\mu} h\left(\frac{r}{t^{\beta}}\right),$$

Continuity, Euler and Poisson

Cosmological Self-similar solutions

SELF-SIMILAR ANSTATZ

$$\rho = t^{-2} f\left(\frac{r}{\sqrt{t}}\right), \quad v = t^{-1/2} g\left(\frac{r}{\sqrt{t}}\right), \quad \Phi_{\rm N} = t^{-1} h\left(\frac{r}{\sqrt{t}}\right)$$

Continuity, Euler and Poisson

PERTURBATIONS AROUND THE EXPANDING COSMOLOGICAL BACKGROUND

$$\rho = \bar{\rho}(1+\delta), \quad \vec{v} = \bar{\vec{v}} + \vec{u}, \quad \Phi_{\rm N} = \bar{\Phi}_{\rm N} + \varphi_{\rm N},$$

Einstein de-Sitter Universe: $a \propto t^{2/3}$

Self-similar form

Cosmological Self-similar solutions

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Einstein de-Sitter Universe: $a \propto t^{2/3}$

Self-similar form ✓

COMOVING FLUID EQUATIONS

$$\frac{\partial \delta}{\partial t} + \frac{1}{a}\nabla_x \cdot [(1+\delta)\vec{u}] = 0,$$

$$\frac{\partial \vec{u}}{\partial t} + \frac{1}{a}(\vec{u}\cdot\nabla_x)\vec{u} + H\vec{u} = -\frac{1}{a}\nabla_x(\varphi_{\rm N} + \Phi_{\rm Q}),$$

$$\nabla_x^2 \varphi_{\rm N} = \frac{2}{3}\frac{\delta}{a},$$

Cosmological Self-similar solutions

SELF-SIMILAR ANSTATZ

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$$\nabla_x^2 \varphi_{\rm N} = \frac{2}{3}\frac{\delta}{a},$$

SPHERICAL SELF-SIMILAR SOLUTIONS

$$\delta(x,t) = \hat{\delta}(\eta), \quad u(x,t) = \epsilon^{1/2} t^{-1/2}\, \hat{u}(\eta),$$

$$\varphi_{\rm N}(x,t) = \epsilon t^{-1}\, \hat{\varphi}_{\rm N}(\eta), \quad \Phi_{\rm Q}(x,t) = \epsilon t^{-1}\, \hat{\Phi}_{\rm Q}(\eta),$$

$$\delta M(x,t) = \epsilon^{3/2} t^{-1/2}\, \delta\hat{M}(\eta),$$

SCALING VARIABLE

$$\eta = \frac{t^{1/6} x}{\epsilon^{1/2}} = \frac{r}{\sqrt{\epsilon t}}.$$

Linear regime: Fourier space

FOURIER SPACE

$$\ddot{\delta}_L + \frac{4}{3t}\dot{\delta}_L - \frac{2}{3t^2}\delta_L + \frac{\epsilon^2 k^4}{4t^{8/3}}\delta_L = 0.$$

FDM Growing and decaying modes: $D_\pm(k,t)$

$$D_+(k,t) = t^{-1/6} J_{-5/2}\left(\frac{3}{2}\epsilon k^2 t^{-1/3}\right),$$
$$D_-(k,t) = t^{-1/6} J_{5/2}\left(\frac{3}{2}\epsilon k^2 t^{-1/3}\right).$$

CDM Growing and decaying modes: $D_\pm(k,t)$

- Semi-classical limit, $\epsilon \to 0$, or on large scales $k \to 0$ → $D_+(k,t) \propto t^{2/3} \propto a$, $D_-(k,t) \propto t^{-1}$

- For $\epsilon \neq 0$, Φ_Q → Acoustic waves: $D_+(k,t) \sim \cos(3\epsilon k^2 t^{-1/3}/2)$ $D_-(k,t) \sim \sin(3\epsilon k^2 t^{-1/3}/2)$

To recover the **BKGD** density on large scales, we keep the decaying mode:

$$\delta_L(x,t) = 1 + \frac{\eta^4}{45} - \frac{8\eta^2}{9\pi}\, {}_2F_3\left(-\frac{1}{2}, 2; \frac{3}{2}, \frac{5}{4}, \frac{7}{4}; -\frac{\eta^4}{144}\right)$$

Linear regime: Real space

REAL SPACE

$$\delta_L^{(4)} + \frac{4}{\eta}\delta_L^{(3)} + \frac{\eta^2}{9}\delta_L'' + \frac{\eta}{3}\delta_L' - \frac{8}{3}\delta_L = 0,$$

FDM 4 independent linear modes

$$\delta_{L1} = 45 + \eta^4, \quad \delta_{L2} = \frac{1}{\eta}\,{}_2F_3\left(-\frac{5}{4}, \frac{5}{4}; \frac{1}{4}, \frac{1}{2}, \frac{3}{4}; -\frac{\eta^4}{144}\right),$$

$$\delta_{L3} = \eta\,{}_2F_3\left(-\frac{3}{4}, \frac{7}{4}; \frac{3}{4}, \frac{5}{4}, \frac{3}{2}; -\frac{\eta^4}{144}\right),$$

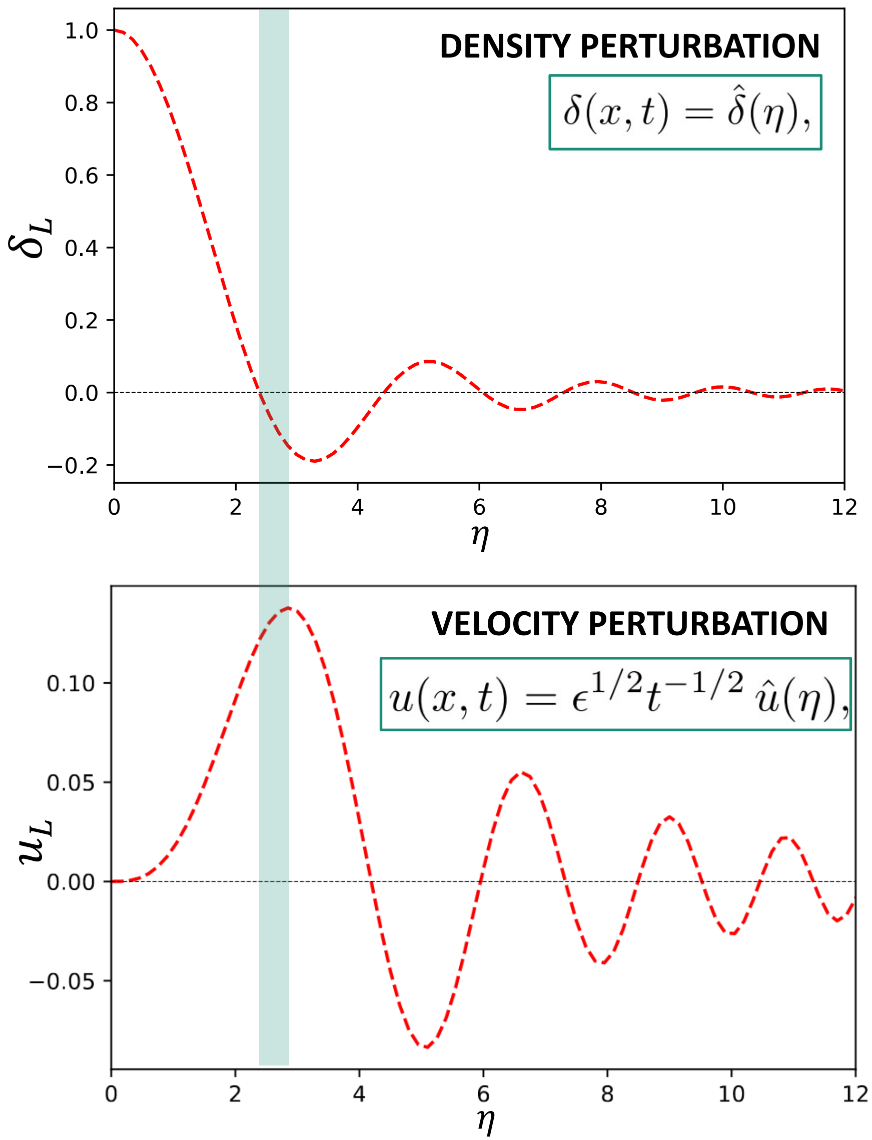
$$\delta_{L4} = \eta^2\,{}_2F_3\left(-\frac{1}{2}, 2; \frac{5}{4}, \frac{3}{2}, \frac{7}{4}; -\frac{\eta^4}{144}\right).$$

- Smooth solution at $\eta = 0$
- Satisfy the boundary conditions at infinity with $\delta(0) = 1$

$$\delta_L = -\frac{8}{9\pi}\left(\delta_{L4} - \frac{\pi}{40}\delta_{L1}\right)$$

✓ **Recover Fourier solution**

1) Linear regime, $\delta(0) = 1$

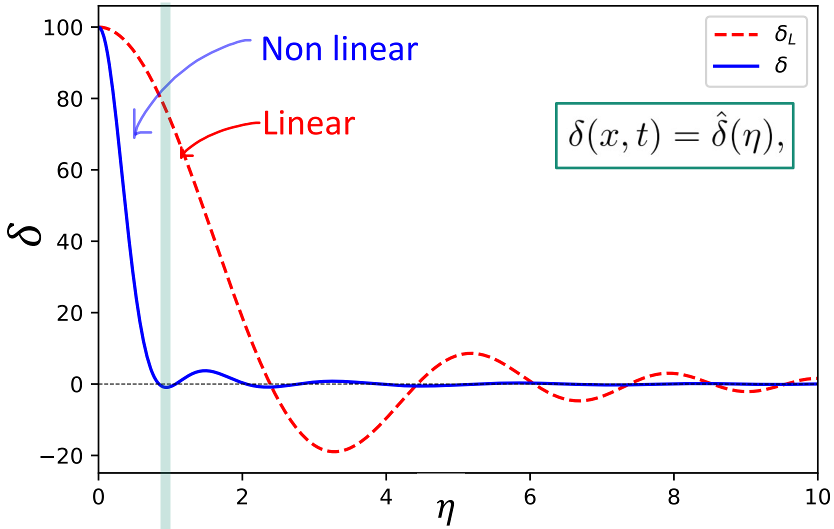


$$\delta_L(x,t) = 1 + \frac{\eta^4}{45} - \frac{8\eta^2}{9\pi}\, {}_2F_3\left(-\frac{1}{2}, 2; \frac{3}{2}, \frac{5}{4}, \frac{7}{4}; -\frac{\eta^4}{144}\right)$$

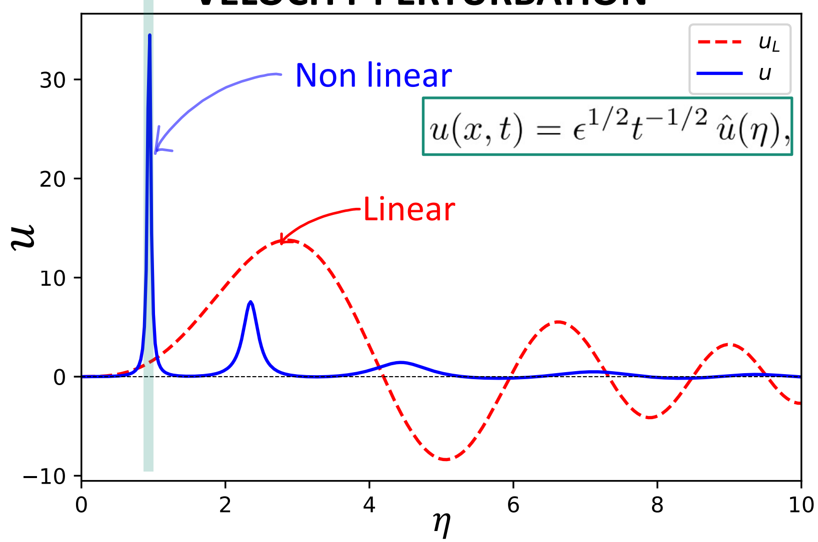
- **FDM: constant amplitude. (≠ CDM)**
- **FDM: grows in physical** coordinates but **shrinks in comoving** coordinates. **(≠ CDM)**
- **FDM** : Fields with **oscillations ,** Φ_Q. **(≠ CDM)**

2) Non-linear regime: Overdensity, $\delta(0) = 100$

DENSITY PERTURBATION



VELOCITY PERTURBATION

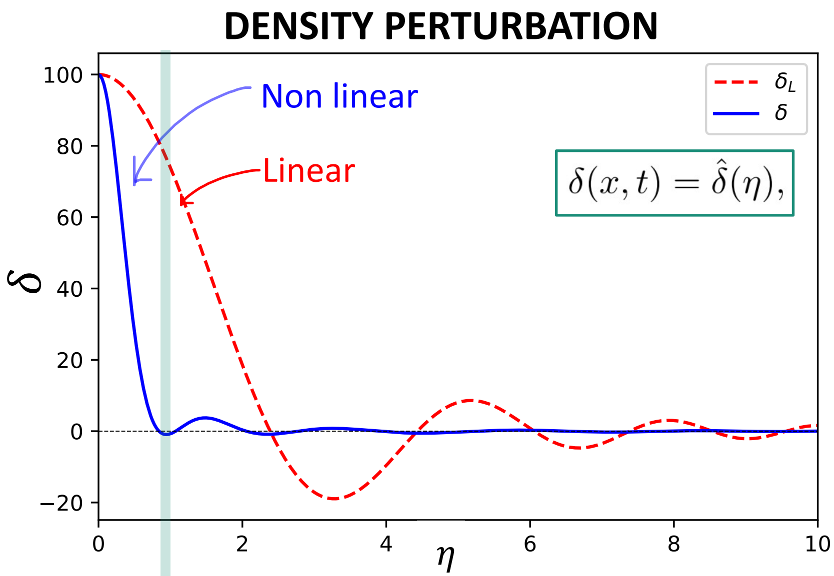


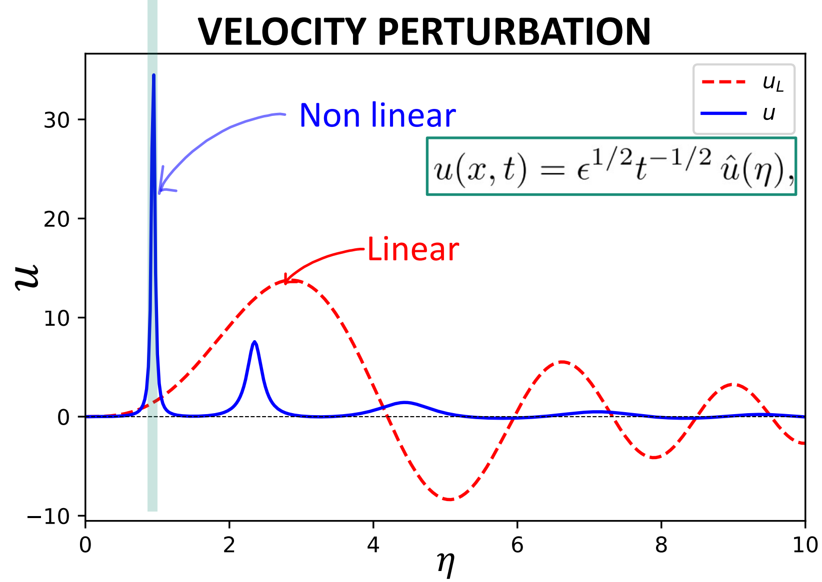
- **The nonlinear corrections make the peak narrower.**
- All the higher-order **peaks move closer to the center**.

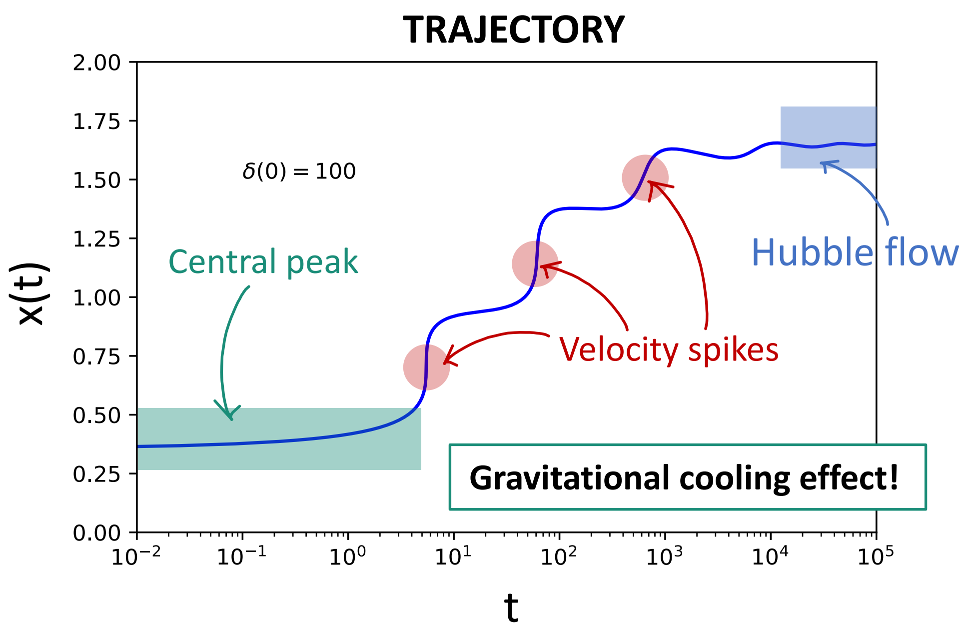
- **Oscillations of the velocity field grow & much sharper.**

- The **velocity** shows high and narrow **positive spikes at the density min.**

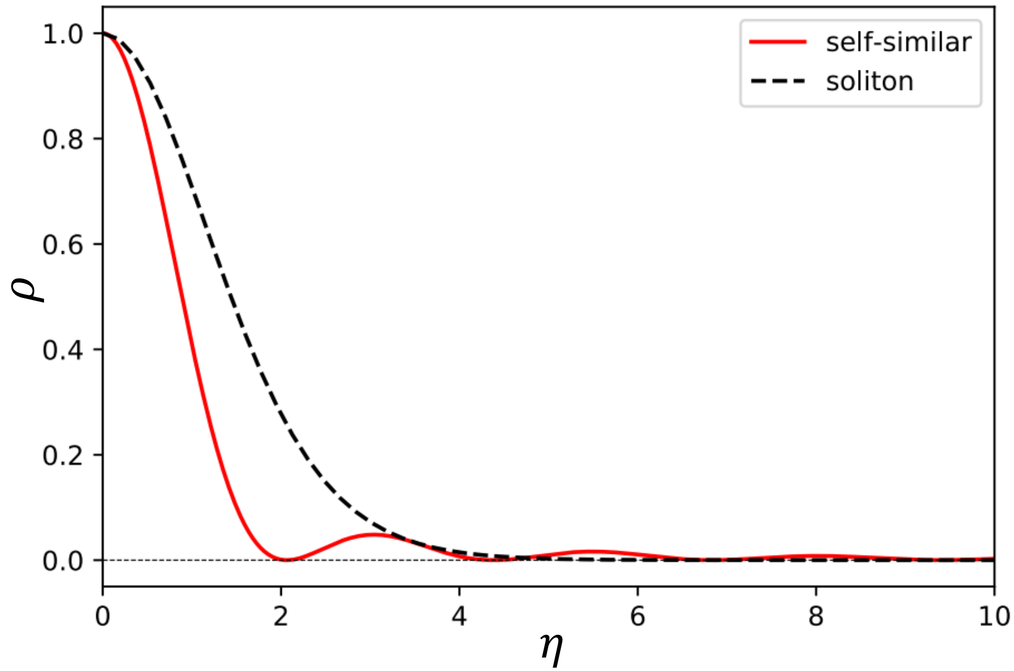
2) Non-linear regime: Overdensity, $\delta(0) = 100$







High-density asymptotic limit



- The **BKGD density becomes negligible** as compared with the central density.
- The inner profile converge to a limiting shape that obeys the scaling law:

$$\{\eta, \psi, \rho, M\} \to \{\lambda^{-1}\eta, \lambda^2\psi, \lambda^4\rho, \lambda M\}$$

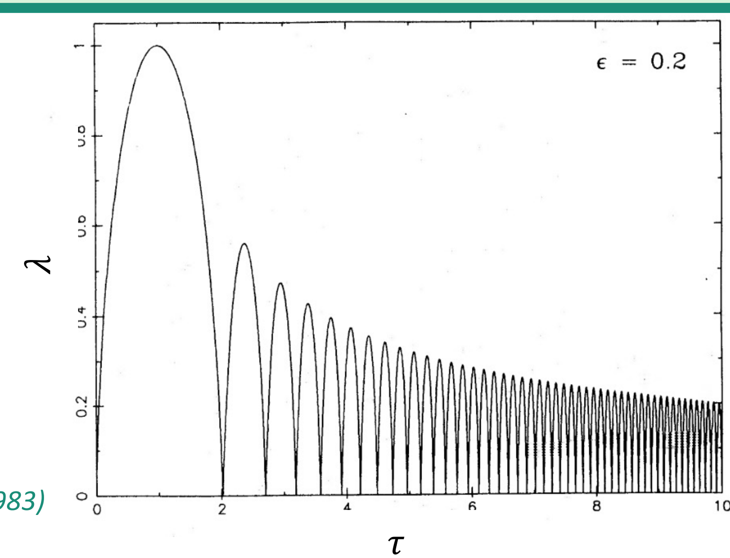
- The central peak of the **self-similar solution is narrower than the soliton peak**.
- **The shape of the central peak of the self-similar profile does not converge to the soliton equilibrium**. → **kinetic effects** (dominate near the boundary of the central peak).

Comparison with CDM self-similar solutions & Conclusions

Comparison: CDM vs FDM

CDM

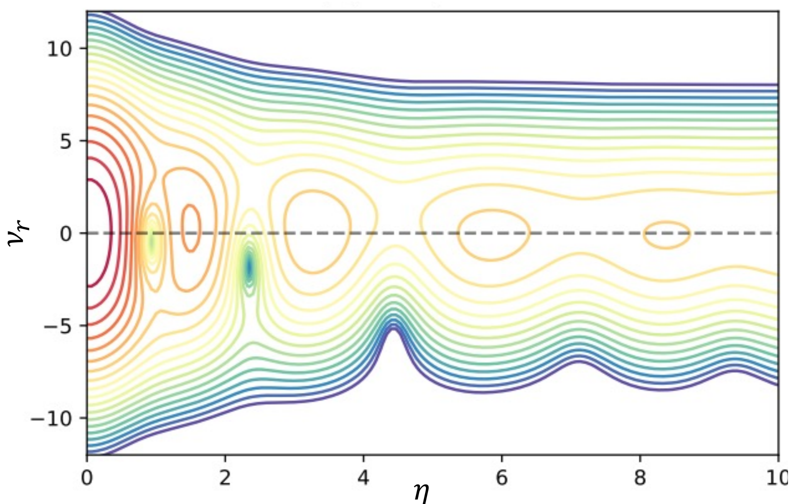
- **Transition from the linear regime to the non-linear regime.**
- **Gravitational collapse**→ Virial equilibrium in the inner nonlinear core.
- **The size grows** with time, in **physical and comoving** coordinates.

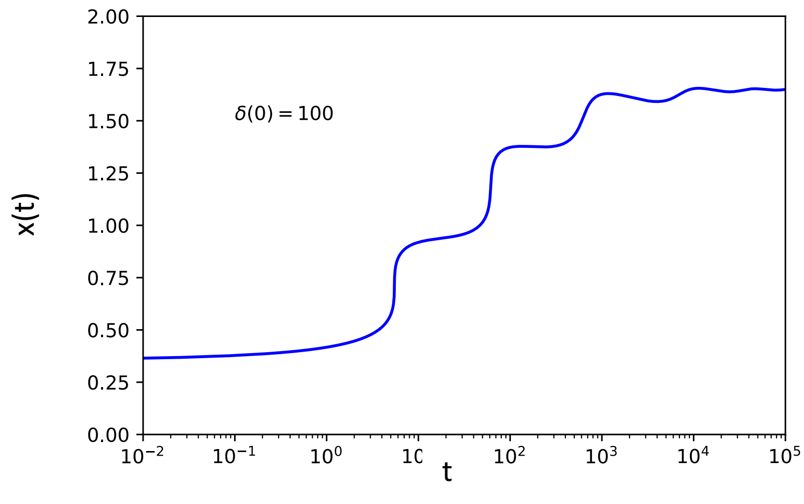


Bertschinger, E.(1985) Fillmore, J. A. and Goldreich, P. (1983)

FDM

- **No transition from the linear to the non-linear regime.**
- **Gravitational cooling**.
- **The size grows in physical** coordinates but **shrinks in comoving** coordinates.





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Thank you for your attention

Raquel Galazo García

TUG Workshop, October 6, 2022

This project has received funding from the European Union's Horizon 2020 research and innovation program under the Marie Skłodowska-Curie Grant Agreement No. 800945 – NUMERICS H2020-MSCA-COFUND-2017.

Back up: Introduction

Dynamics of Fuzzy Dark Matter

$m \sim 10^{-22}\ eV$ De Broglie wavelength $\sim 0.5 kpc$

Action:

$$S = \int d^4x\sqrt{-g}\left[-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{m^2}{2}\phi^2\right]$$

Non-Relativistic regime:

$$\phi = \frac{1}{\sqrt{2m}}\left(\psi \exp^{-imt} + \psi^* \exp^{imt}\right)$$

$|\ddot{\psi}| \ll m|\dot{\psi}|$ Factor-out the fast time oscillation of $\boldsymbol{\phi}$

SCHRÖDINGER-POISSON SYSTEM (SP)

$$i\frac{\partial\psi}{\partial t} = -\frac{1}{2m}\nabla^2\psi + m\Phi_{\rm N}\,\psi,$$

$$\nabla^2\Phi_{\rm N} = 4\pi\mathcal{G}_{\rm N}\rho, \quad \rho = m\psi\psi^*$$

HYDRODYNAMICAL PICTURE

$$\psi = \sqrt{\rho}\,e^{iS/\epsilon}, \quad \vec{v} = \nabla S,$$

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\vec{v}) = 0,$$

$$\frac{\partial\vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v} = -\nabla\left(\Phi_{\rm N} + \Phi_{\rm Q}\right)$$

$$\nabla^2\Phi_{\rm N} = 4\pi\rho.$$

SP system scaling law

$$\{t, \vec{r}, \Phi_{\rm N}, \psi, \rho\} \rightarrow \{\lambda^{-2}t, \lambda^{-1}\vec{r}, \lambda^2\Phi_{\rm N}, \lambda^2\psi, \lambda^4\rho\}\,.$$

Quantum pressure

$$\Phi_{\rm Q} = -\frac{\epsilon^2}{2}\frac{\nabla^2\sqrt{\rho}}{\sqrt{\rho}}.$$

$$\epsilon = \frac{T}{mL^2} \sim \frac{\lambda_{DB}}{L}$$

FDM Motivation 2) Alternative to CDM N-Body simulations

CDM: A classical collisionless fluid is governed by the **Vlasov-Poisson equations**. $f = f(\boldsymbol{x}, \boldsymbol{p}, t)$

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \frac{\boldsymbol{p}}{ma^2} \cdot \boldsymbol{\nabla}_x f - m(\boldsymbol{\nabla}_x \phi) \cdot \boldsymbol{\nabla}_p f = 0, \qquad \nabla_x^2 \phi = 4\pi G \bar{\rho}(t)\, a^2 \delta(\boldsymbol{x}, t),$$

$$\rho = \bar{\rho}(t)[1 + \delta(\boldsymbol{x}, t)], \qquad \rho = \int f(\boldsymbol{x}, \boldsymbol{p}, t)\, \mathrm{d}^3 p.$$

- **Wigner** quasi-probability distribution: **link the Schrödinger** wave function ψ to a function f in phase space.

$$f_{\mathrm{W}}(\vec{r}, \vec{v}) = \int \frac{d\vec{r}\,'}{(2\pi)^3} e^{i\vec{v}\cdot\vec{r}\,'} \psi\left(\vec{r} - \frac{\epsilon}{2}\vec{r}\,'\right) \psi^*\left(\vec{r} + \frac{\epsilon}{2}\vec{r}\,'\right),$$

- **Husimi** representation: **smoothing of the Wigner** distribution by a Gaussian filter of width σx and σp in x and p space.

$$f_{\mathrm{H}}(\vec{r}, \vec{v}) = \int \frac{d\vec{r}\,' d\vec{v}\,'}{(2\pi\epsilon)^3 \sigma_r^3 \sigma_v^3} e^{-(\vec{r}-\vec{r}\,')^2/(2\epsilon\sigma_r^2) - (\vec{v}-\vec{v}\,')^2/(2\epsilon\sigma_v^2)} \times f_{\mathrm{W}}(\vec{r}\,', \vec{v}\,'),$$

- Husimi is a positive-semidefinite function → No fast oscillations of Wigner

FDM comoving Vlasov equation

$$\frac{\partial f_{\mathrm{W}}}{\partial t} + \frac{\vec{p}}{a^2} \cdot \frac{\partial f_{\mathrm{W}}}{\partial \vec{x}} - \vec{\nabla}_x \varphi_{\mathrm{N}} \cdot \frac{\partial f_{\mathrm{W}}}{\partial \vec{p}} + \mathcal{O}(\epsilon) = 0.$$

Kaiser (1993)
C. Uhlemann,
M. Kopp, & T. Haugg (2014)

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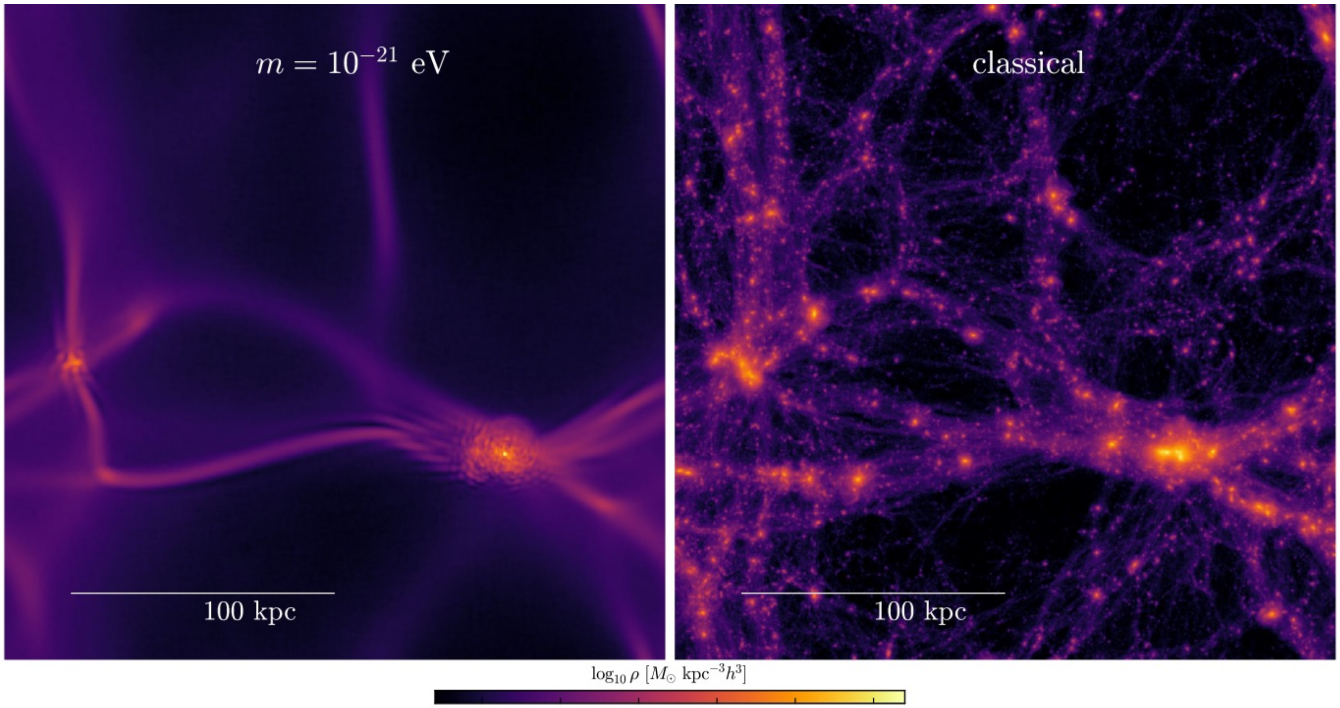

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Back up: Self-similar solutions for FDM

Cosmological Self-similar solutions

FLUID PICTURE

$$\rho = t^{-2} f\left(\frac{r}{\sqrt{t}}\right), \quad v = t^{-1/2} g\left(\frac{r}{\sqrt{t}}\right), \quad \Phi_{\rm N} = t^{-1} h\left(\frac{r}{\sqrt{t}}\right)$$

Continuity, Euler and Poisson

COSMOLOGICAL BACKGROUND

Einstein de-Sitter Universe: matter era & the scale factor : $a \propto t^{2/3}$

$$\bar{\rho} = \frac{1}{6\pi t^2}, \quad \bar{v} = \frac{2r}{3t}, \quad \bar{\Phi}_{\rm N} = \frac{r^2}{9t^2}.$$

Self-similar form ✓

Comoving spatial coordinates $\vec{x} = \vec{r}/a$

PERTURBATIONS AROUND THE EXPANDING BACKGROUND

$$\rho = \bar{\rho}(1+\delta), \quad \vec{v} = \bar{\vec{v}} + \vec{u}, \quad \Phi_{\rm N} = \bar{\Phi}_{\rm N} + \varphi_{\rm N},$$

Comoving Self-similar solutions: Scaling variable

$\rho = \bar{\rho}(1+\delta), \quad \vec{v} = \bar{\vec{v}} + \vec{u}, \quad \Phi_{\rm N} = \bar{\Phi}_{\rm N} + \varphi_{\rm N},$ → Substituting into the Euler, Poisson and continuity eq.

COMOVING FLUID EQUATIONS

$$\frac{\partial \delta}{\partial t} + \frac{1}{a}\nabla_x \cdot [(1+\delta)\vec{u}] = 0,$$
$$\frac{\partial \vec{u}}{\partial t} + \frac{1}{a}(\vec{u}\cdot\nabla_x)\vec{u} + H\vec{u} = -\frac{1}{a}\nabla_x(\varphi_{\rm N} + \Phi_{\rm Q}),$$
$$\nabla_x^2 \varphi_{\rm N} = \frac{2}{3}\frac{\delta}{a},$$

Quantum Pressure

$$\Phi_{\rm Q} = -\frac{\epsilon^2}{2a^2}\frac{\nabla_x^2\sqrt{\rho}}{\sqrt{\rho}}.$$

Spherical self-similar solutions will be of the form:

$$\delta(x,t) = \hat{\delta}(\eta), \quad u(x,t) = \epsilon^{1/2} t^{-1/2}\,\hat{u}(\eta),$$
$$\varphi_{\rm N}(x,t) = \epsilon t^{-1}\,\hat{\varphi}_{\rm N}(\eta), \quad \Phi_{\rm Q}(x,t) = \epsilon t^{-1}\,\hat{\Phi}_{\rm Q}(\eta),$$
$$\delta M(x,t) = \epsilon^{3/2} t^{-1/2}\,\delta\hat{M}(\eta),$$

Where the mass perturbation inside the radius r :

$$\delta M(r) = 4\pi\int_0^r dr\, r^2 \delta\rho(r) = \frac{2}{3}\int_0^x dx\, x^2 \delta(x),$$

Self-similar ansatz

$$v = t^{-1/2} g\left(\frac{r}{\sqrt{t}}\right)$$
$$\rho = t^{-2} f\left(\frac{r}{\sqrt{t}}\right) \quad \Phi_{\rm N} = t^{-1} h\left(\frac{r}{\sqrt{t}}\right)$$

SCALING VARIABLE

$$\eta = \frac{t^{1/6}x}{\epsilon^{1/2}} = \frac{r}{\sqrt{\epsilon t}}.$$

Scaling variable

Spherical self-similar solutions:

$$\delta(x,t) = \hat{\delta}(\eta), \quad u(x,t) = \epsilon^{1/2} t^{-1/2}\, \hat{u}(\eta),$$
$$\varphi_{\rm N}(x,t) = \epsilon t^{-1}\, \hat{\varphi}_{\rm N}(\eta), \quad \Phi_{\rm Q}(x,t) = \epsilon t^{-1}\, \hat{\Phi}_{\rm Q}(\eta),$$
$$\delta M(x,t) = \epsilon^{3/2} t^{-1/2}\, \delta\hat{M}(\eta),$$

SCALING VARIABLE

$$\eta = \frac{t^{1/6} x}{\epsilon^{1/2}} = \frac{r}{\sqrt{\epsilon t}}.$$

$$M = \bar{M} + \delta M = \epsilon^{3/2} t^{-1/2} \left[\frac{2}{9}\eta^3 + \delta\hat{M}(\eta) \right].$$

- The size **grows as** $\sim\sqrt{t}$ **in physical units** but more slowly than the scale factor, → shrink **as** $t^{-1/6}$ in comoving units.
- The associated **mass decreases as** $M \sim 1/\sqrt{t}$. (≠CDM: grow both in comoving size and in mass.)

Back up: Non-linear regime

Closed equation over δM

Euler equation in terms of η :

$$\frac{1}{6}(\hat{u}+\eta\hat{u}')+\hat{u}\hat{u}'+\hat{\varphi}'_{\rm N}+\hat{\Phi}'_{\rm Q}=0,$$

The fields in terms of η :

$$\hat{\Phi}_{\rm Q}=-\frac{1}{2\eta^2\sqrt{1+\hat{\delta}}}\frac{d}{d\eta}\left(\eta^2\frac{d}{d\eta}\sqrt{1+\hat{\delta}}\right) \qquad \hat{\delta}=\frac{3}{2\eta^2}\delta\hat{M}'$$

Poisson equation in terms of η :

$$\frac{1}{\eta^2}\frac{d}{d\eta}\left(\eta^2\frac{d\hat{\varphi}_{\rm N}}{d\eta}\right)=\frac{2}{3}\hat{\delta},$$

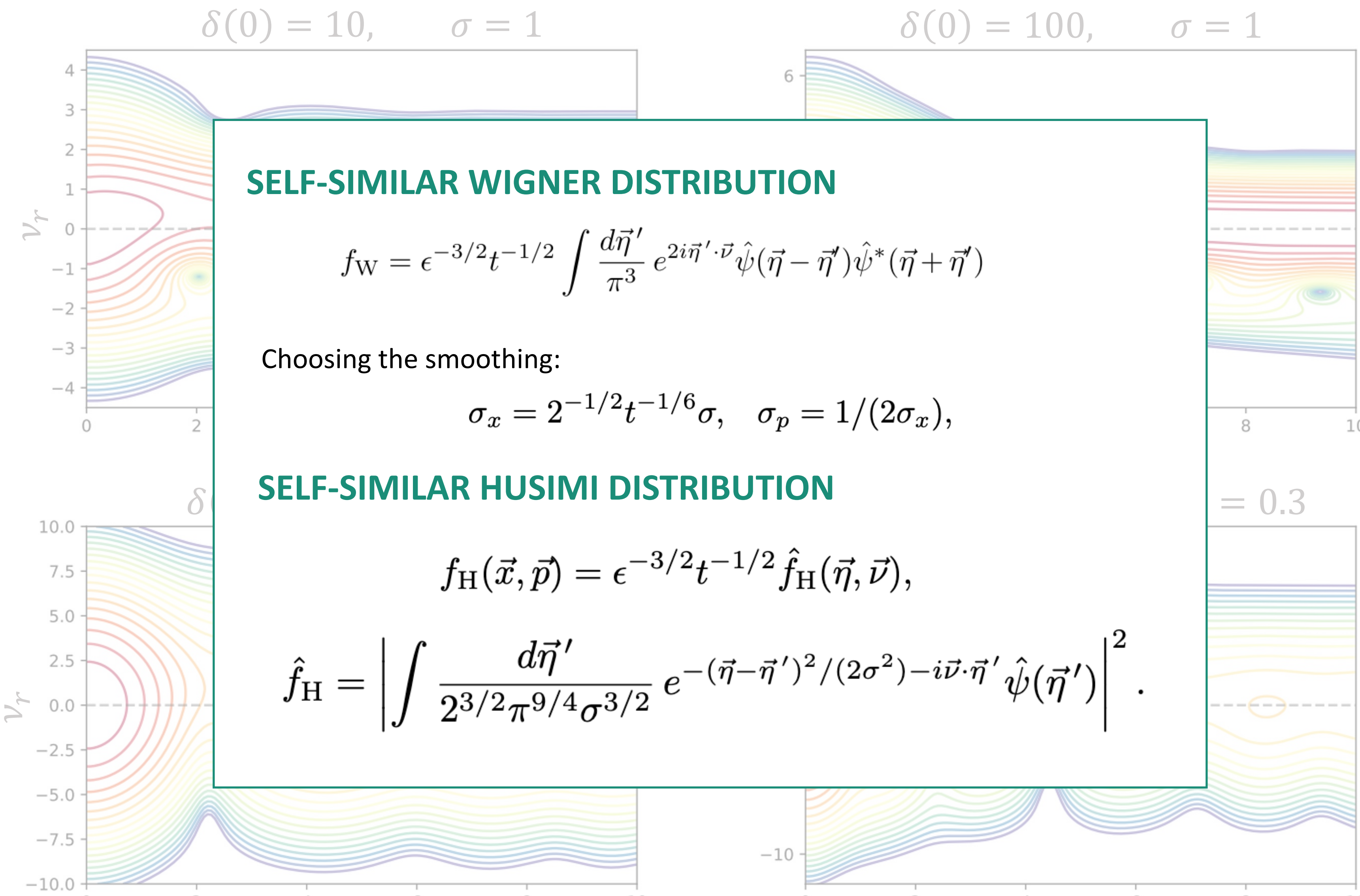
Integrating the continuity equation:

$$\hat{u}=\frac{3\delta\hat{M}-\eta\delta\hat{M}'}{4\eta^2+6\delta\hat{M}'}.$$

CLOSED NON LINEAR EQUATION OVER δM

$$\begin{aligned}&9(2\eta^3+3\eta\delta\hat{M}')^2\delta\hat{M}^{(4)}-(144\eta^5+216\eta^3\delta\hat{M}'+108\eta^4\delta\hat{M}''\\&+162\eta^2\delta\hat{M}'\delta\hat{M}'')\delta\hat{M}^{(3)}+(4\eta^8+288\eta^4+36\eta^5\delta\hat{M}\\&-216\eta^2\delta\hat{M}'+324\eta^3\delta\hat{M}''+81\eta^2\delta\hat{M}^2+81\eta^2\delta\hat{M}''^2)\delta\hat{M}''\\&-3(4\eta^7+96\eta^3+180\eta^4\delta\hat{M}+243\eta^2\delta\hat{M}\delta\hat{M}'-3\eta^3\delta\hat{M}'^2\\&+108\delta\hat{M}\delta\hat{M}'^2)\delta\hat{M}'-12\eta^3(7\eta^3-9\delta\hat{M})\delta\hat{M}=0.\end{aligned}$$

Self-similar Husimi phase-space distribution $\hat{f}_H\,(\eta; v_r; v_t = 0)$



SELF-SIMILAR WIGNER DISTRIBUTION

$$f_{\rm W} = \epsilon^{-3/2} t^{-1/2} \int \frac{d\vec{\eta}'}{\pi^3}\, e^{2i\vec{\eta}'\cdot\vec{\nu}} \hat{\psi}(\vec{\eta}-\vec{\eta}')\hat{\psi}^*(\vec{\eta}+\vec{\eta}')$$

Choosing the smoothing:

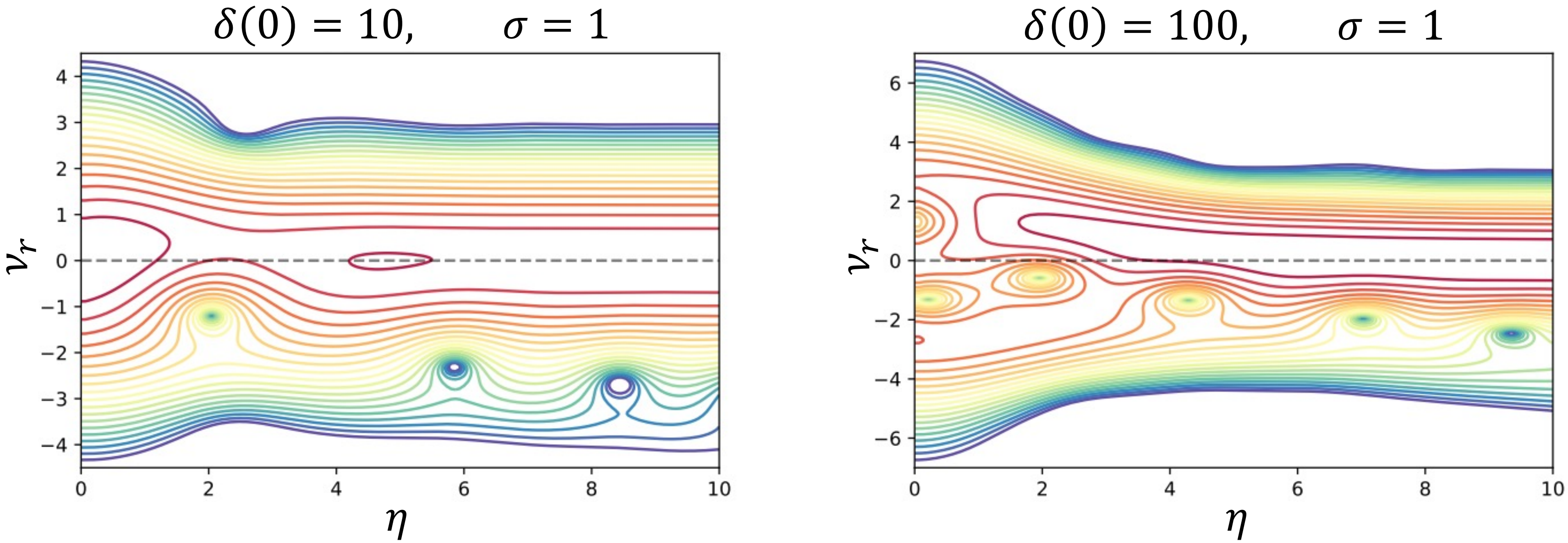
$$\sigma_x = 2^{-1/2} t^{-1/6} \sigma, \quad \sigma_p = 1/(2\sigma_x),$$

SELF-SIMILAR HUSIMI DISTRIBUTION

$$f_{\rm H}(\vec{x},\vec{p}) = \epsilon^{-3/2} t^{-1/2} \hat{f}_{\rm H}(\vec{\eta},\vec{\nu}),$$

$$\hat{f}_{\rm H} = \left| \int \frac{d\vec{\eta}'}{2^{3/2}\pi^{9/4}\sigma^{3/2}}\, e^{-(\vec{\eta}-\vec{\eta}')^2/(2\sigma^2) - i\vec{\nu}\cdot\vec{\eta}'} \hat{\psi}(\vec{\eta}') \right|^2 .$$

Radial Husimi phase-space distribution $\hat{f}_H\,(\eta; v_r; v_t = 0), \sigma = 1$

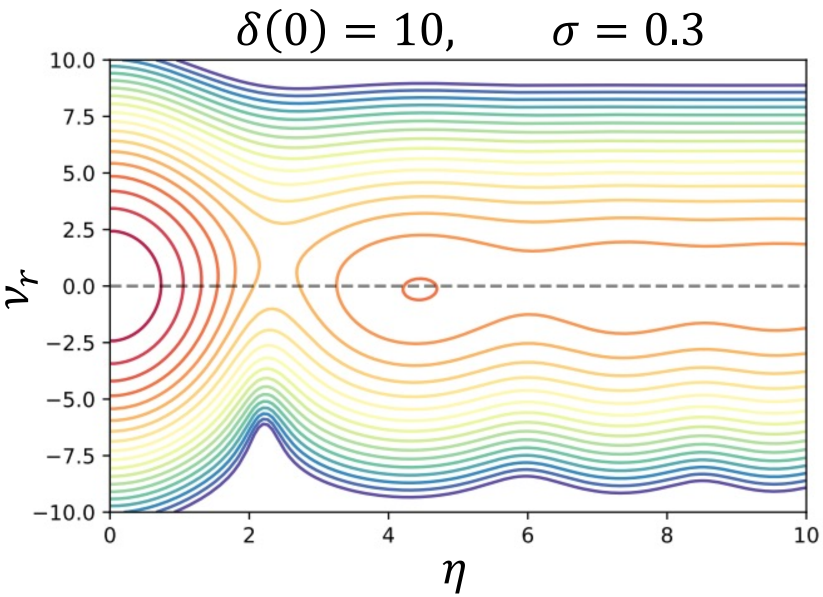


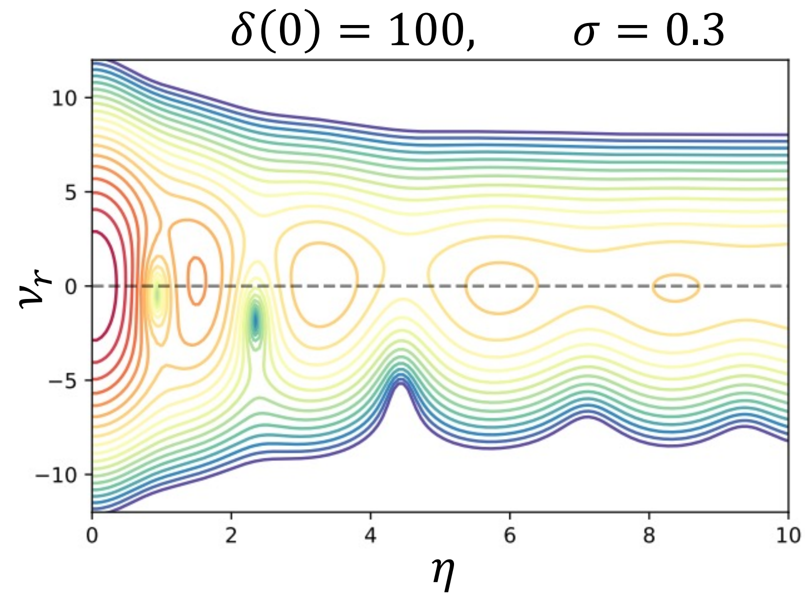
- As the spatial coarsening $\sigma_x \propto \sigma$ increases → the velocity coarsening, $\sigma_p \propto 1/\sigma$ decreases : **Heisenberg uncertainty principle.**

- **Velocity asymmetries but spatial profile is smoothed out.**
- At large distance, the profile → **cosmological BKGD.**

- **The coarsening $\sigma = 1$ is no longer sufficient to separate the first few peaks.**
- Artificial interferences between these peaks and to a Husimi distribution that is difficult to interpret and far from the semiclassical expectations

Radial Husimi phase-space distribution $\hat{f}_H\,(\eta; v_r; v_t = 0), \sigma = 0.3$





- **Well-defined peaks increasing** with δ(0) and preserves **signs of the density fluctuations.**
- **At large distance→ the cosmological BKGD.**

- More **faithful representation**: Sequence of scalar-field clumps
- **COST: This erases most of the information about the velocity field.**
- Different choices of σ→ different pictures →Difficult to relate to the underlying dynamics

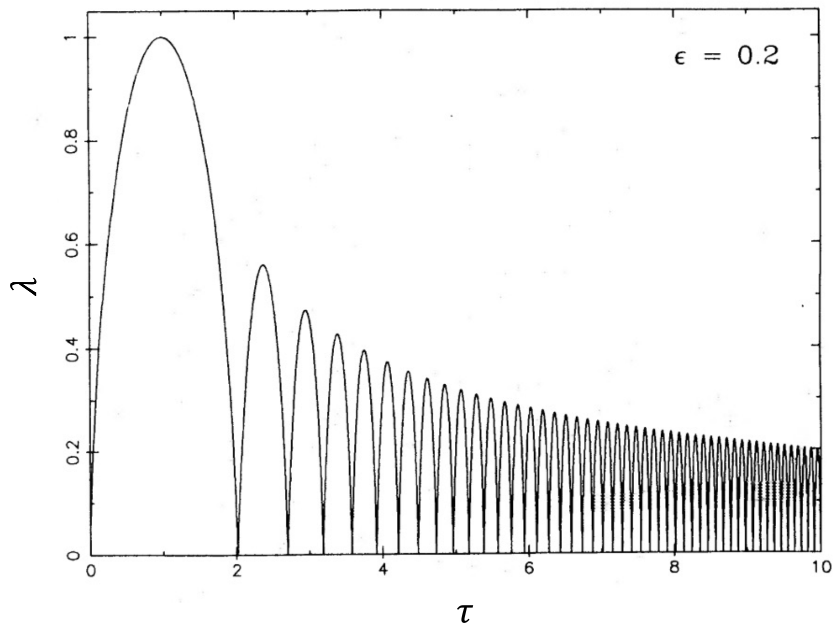
- **The hydrodynamical mapping clearer picture of the dynamics.**

Back up: CDM

CDM self-similar solutions vs FDM self-similar solutions

CDM spherical collapse

- **Density contrast grows from linear regime with a power-law profile → non linear regime**.
- Non-linear effects modify the shape of the profile in the inner regions→ **At small radii it takes again a power-law form.**
- Gravitational instability and increasingly distant shells collapse→ **Stabilize at a fixed fraction of the turn-around radius→ (gravity balanced by velocity dispersion). → Virial equilibrium in the inner nonlinear core.**
- **Mass and a radius that grow with time, both in physical and comoving coordinates**.



Bertschinger, E.(1985)
Fillmore, J. A. and Goldreich, P. (1983)

CDM self-similar solutions vs FDM self-similar solutions

- **Density amplitude does not grow with time.** → Balance φ_N vs Φ_Q → No transition from the linear to non-linear regime.→ **No gravitational collapse.**
- **Oscillations of the fields.**
- **The profile remains linear on all scales and at all times or non-linear at the center.**
- At large distance→ **oscillations around the BCKG**→ 2 additional linear modes.
- **Acoustic-like oscillations**→ transport information from small to large scales.
- Matter moves though clumps : **Gravitational cooling**.
- **FDM self-similar solutions disappear in the semi classical limit $\epsilon \to 0$, as their size and mass decrease as $\epsilon^{1/2}$and $\epsilon^{3/2}$**