Anomalous Magnetic Moment of the Muon in Quantum Electrodynamics

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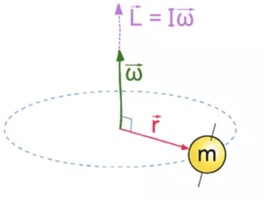
Contents

Classical result

Let's assume that a charged lepton like a muon as a charged rigid body rotates around the $z-axis$ with an angular velocity.
The magnetic moment can be obtain as

$$\vec{M} = \frac{e}{2m}\vec{L}. \quad (1)$$



Classical result

From here, Dirac made his prediction from the Dirac equation in the electromagnetic field.

$$(i\gamma^\mu \partial_\mu - m)\psi - e\gamma^\mu A_\mu \psi = 0. \tag{2}$$

Following by the non-relativistic Schrödinger-Pauli Hamiltonian equation that can explain the interaction of spin one-half particles:

$$H = \frac{1}{2m}(\vec{p} - e\vec{A})^2 - \frac{e}{m}\vec{S}\cdot\vec{B} + e\phi, \tag{3}$$

Taking out the magnetic moment component:

$$\vec{M} = \frac{e}{m}\vec{S} = g\frac{e}{2m}\vec{S} \tag{4}$$

where g is called the Landé g-factor.

Dirac equation's prediction

Compared to the classical result g must equal to 2. This holds true not only for muons but for any one-half spin elementary particles.

In higher-order correction, however, QED predicts that $g = 2 + \mathcal{O}(\alpha)$. The anomalous magnetic moment we are looking for refers to this small correction.

$$a = \frac{g-2}{2}. \tag{5}$$

Anomalous magnetic moment from scattering amplitude

We will calculate the muon anomalous magnetic moment through the scattering matrix (**S-matrix**) of an imaginary process:

$$\mu^-(p_1, s_1) \longrightarrow \mu^-(p_2, s_2) + \gamma(p_3, \lambda_3), \tag{6}$$

The QED Lagrangian density of this system is given by

$$\begin{aligned} \mathcal{L} &= \mathcal{L}_{Dirac} + \mathcal{L}_{photon} + \mathcal{L}_{int} \\ &= \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} - e\bar{\psi}\gamma^\mu \psi A_\mu, \end{aligned} \tag{7}$$

Definition of the initial state and the final state is

$$|i\rangle = \sqrt{2E_1} a_{p_1}^{s_1 \dagger} |0\rangle, \qquad |f\rangle = \sqrt{4E_2 E_3} a_{p_2}^{s_2 \dagger} c_{p_3}^{\lambda_3 \dagger} |0\rangle. \tag{8}$$

S-matrix

The scattering amplitude is defined

$$A = \langle f|S|i\rangle, \tag{9}$$

where the total scattering amplitude $A = A_0 + A_1 + A_2 + ...$ is the sum of all leading terms in each order of perturbation theory, and the scattering matrix S can be expanded to Dyson's series as follows:

$$S = \sum_{n=0}^{\infty} \frac{i^n}{n!} \int_{-\infty}^{\infty} d^4x_1 ... \int_{-\infty}^{\infty} d^4x_n \, T\{\mathcal{L}_{int}(x_1)...\mathcal{L}_{int}(x_n)\}. \tag{10}$$

Form factors

We expand the S-matrix and only keep the orders that have contribution to the scattering matrix, from which we can derive the muon's anomalous magnetic moment in QED at NLO.

$$\langle f|S|i\rangle = -ie\bar{u}(p_2)\left\{\gamma^{\mu}F_1(q^2) + \frac{i\sigma^{\mu\nu}}{2m}(p_1 - p_2)_{\nu}F_2(q^2)\right\}u(p_1)\epsilon^{*}_{\mu}, \tag{11}$$

where form factor F_1 and factor F_2 contain the influence of the EM field on the muon.
The g-factor now is defined

$$g = 2[F_1(0) + F_2(0)]. \tag{12}$$

The leading 2 is prediction of Dirac equation that we proved above.

$\Longrightarrow$ **Find these form factors ?**

Leading order

The Feynman diagram corresponds to LO:

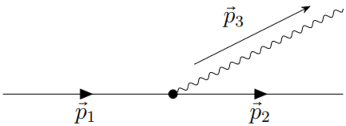


Figure 1: Tree-level Feynman diagram.

The scattering amplitude at LO can be obtained quickly as

$$A_1 = \bar{u}(p_2)(-ie\gamma^{\mu})u(p_1)\epsilon^{*}_{\mu}. \tag{13}$$

At the LO, $F_1 = 1$ and $F_2 = 0$ followed by g-factor is equal to 2 $\rightarrow$ satisfies the classical result.
The anomalous magnetic moment at LO:

$$a_{\mu} = \frac{g_{\mu} - 2}{2} = \frac{2(1+0) - 2}{2} = 0. \tag{14}$$

Next-to-leading order

The Feynman diagrams correspond to NLO:

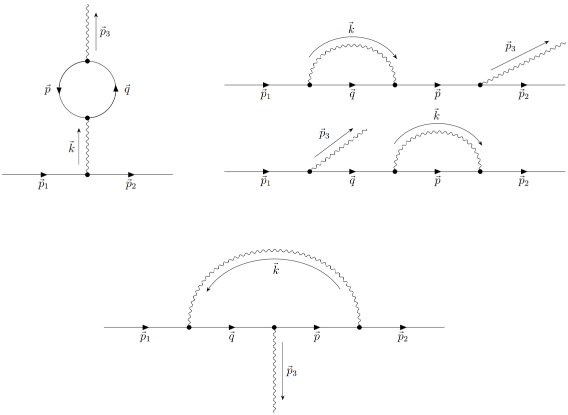


Figure 2: NLO connected Feynman diagrams.

Amputated Feynman diagram

Due to the on-shell condition, we only keep the amputated diagram.

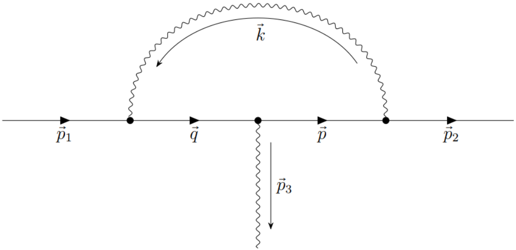


Figure 3: Amputated vertex correction.

$$i\mathcal{M}_v = -e^3\, \bar{u}_{p_2} \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{\gamma^\nu(\not{p}_2 + \not{k} + m)\gamma^\mu(\not{p}_1 + \not{k} + m)\gamma_\nu}{(k^2 - \lambda^2 + i\epsilon)[(p_1 + k)^2 - m^2 + i\epsilon][(p_2 + k)^2 - m^2 + i\epsilon]}\, u_{p_1}\, \epsilon^*_\mu. \tag{15}$$

UV and IR-divergence

This integral diverges at

- $x \longrightarrow \infty$: UV-divergence.
  $\Longrightarrow$ We need to use the dimensional regularization and renormalization method to cancel UV-divergence.
- $x \longrightarrow 0$: IR-divergence.
  $\Longrightarrow$ Cancelled thanks to the on-shell condition.

D-dimensional regularization

Calculation of loop integrals in $D = 4 - 2\epsilon$ dimension with ϵ is called dimensional regularization parameter.

$$\int \frac{\mathrm{d}^4 \boldsymbol{q}}{(2\pi)^4} = \mu^{4-D} \int \frac{\mathrm{d}^D \boldsymbol{q}}{(2\pi)^D}. \tag{16}$$

where the renormalization scale μ has dimension of energy.

Conventions

Conventions		
Space	4-Dimension	D-Dimension
Metric Tensor $g^{\mu\nu}$	$\mu, \nu = 0, 1, 2, 3$ $g^{\mu}_{\mu} = \delta^{\mu}_{\mu} = 4$	$\mu, \nu = 0, 1, ..., D-1$ $g^{\mu}_{\mu} = \delta^{\mu}_{\mu} = D$
Dirac Matrices	dim(γ^{μ})=4 $\gamma^{\mu}\gamma_{\mu} = 4 \cdot \mathbf{I}_4$ $\gamma^{\mu}\gamma_{\nu}\gamma_{\mu} = -2\gamma_{\nu}$ $\{\gamma^{\mu}, \gamma^{\nu}\} = 2g^{\mu\nu} \cdot \mathbf{I}_4$ $Tr(\mathbf{I}_4) = 4$	dim(γ^{μ})=$2^{D/2}$ $\gamma^{\mu}\gamma_{\mu} = D \cdot \mathbf{I}_D$ $\gamma^{\mu}\gamma_{\nu}\gamma_{\mu} = (2-D)\gamma_{\nu}$ $\{\gamma^{\mu}, \gamma^{\nu}\} = 2g^{\mu\nu} \cdot \mathbf{I}_D$ $Tr(\mathbf{I}_D) = 4$ (per definition)
Properties	Preserve Lorentz invariant, Gauge invariant	

Table 1: Conventions in D-dimension compared to 4-dimension.

Counterterms

The Lagrangian that we used at the tree level is not sufficient to describe the divergences that appear in the Feynman diagrams, so we need a tiny impact to modify the Lagrangian with quantities called "bare quantities" (denoted by subscript 0).

$$\mathcal{L}_0 = -\frac{1}{4}(\partial^\mu A_0^\nu - \partial^\nu A_0^\mu)^2 + \overline{\psi}_0(i\not{\partial} - m_0)\psi_0 - e_0\bar{\psi}_0\gamma^\mu\psi_0 A_{0\mu}. \quad (17)$$

The bare quantities are conventionally associated with the finite renormalized quantities by renormalization parameters Z_i as

$$\psi_0 = \sqrt{Z_2}\psi \equiv (1+\delta_2)\psi, \qquad A_0^\mu = \sqrt{Z_3}A^\mu \equiv (1+\delta_3)A^\mu. \quad (18)$$

$$m_0 = Z_m m \equiv m + \delta_m, \qquad e_0 = Z_e e \equiv e + \delta_e. \quad (19)$$

Counterterm Lagrangian

Applying these expansions into the bare Lagrangian, we derive the Lagrangian for renormalized perturbation theory:

$$\begin{aligned}\mathcal{L}_0 =& -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \bar{\psi}(i\not{\partial} - m)\psi - e\bar{\psi}\gamma^{\mu}\psi A_{\mu} \\ & -\frac{1}{4}\delta_3 F^{\mu\nu}F_{\mu\nu} + \delta_2\bar{\psi}(i\not{\partial} - m)\psi - \delta_m\bar{\psi}\psi - (\delta_e + \delta_2 + \frac{1}{2}\delta_3)e\bar{\psi}\gamma^{\mu}\psi A_{\mu} \\ =& \mathcal{L}_R + \delta\mathcal{L}.\end{aligned} \tag{20}$$

- By rescaling the Lagrangian density, we can split the Lagrangian density into two terms, which are the renormalized Lagrangian ($\mathcal{L}_{\mathcal{R}}$) and the counterterm Lagrangian ($\delta\mathcal{L}$).
- The renormalization parameters are fixed by the on-shell renormalization condition ($p^2 = m^2$).

Feynman rules

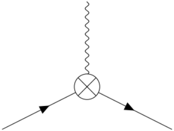
Feynman rules for counterterm vertex:

$$= -i\delta_3 k^2 g^{\mu\nu}.$$

Figure 4: Photon counterterm vertex.

$$= i\delta_2(\not{p} - m) - i\delta_m.$$

Figure 5: Fermion counterterm vertex.

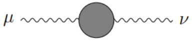


$$= -ie\gamma^\mu(\delta_2 + \delta_e + \tfrac{1}{2}\delta_3).$$

Figure 6: Photon-fermion counterterm vertex.

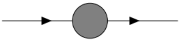
Renormalized functions

Let's go to the renormalized vertex function (denoted by a hat) definition:



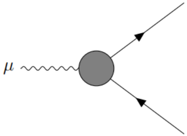
$$= \hat{\Gamma}^{\mu\nu}(k)$$

$$= \Gamma^{\mu\nu}(k) + \delta\Gamma^{\mu\nu}(k) = -ik^2 g^{\mu\nu} \underbrace{+ i\Pi^{\mu\nu}(k) - ik^2 g^{\mu\nu}\delta_3}_{i\hat{\Pi}^{\mu\nu}(k)} .$$



$$= \hat{\Gamma}^{ff}(p)$$

$$= \Gamma^{ff}(p) + \delta\Gamma(p) = i(\not{p} - m) \underbrace{- i\Sigma(p) + i\delta_2(\not{p} - m) - i\delta_m}_{-i\hat{\Sigma}(p)} .$$



$$= \hat{\Gamma}^{Aff}(p, p')$$

$$= \Gamma^{Aff}(p, p') + \delta\Gamma^{Aff}(p, p') = -ie\gamma^\mu \underbrace{- ie\Lambda^\mu(p, p') - ie\gamma^\mu(\delta_e + \delta_2 + \frac{1}{2}\delta_3)}_{-ie\hat{\Lambda}^\mu(p,p')} .$$

Renormalization conditions

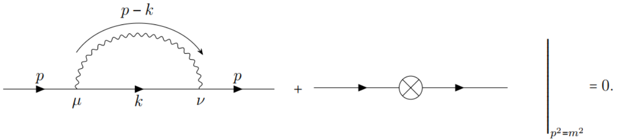
By imposing renormalization conditions, the renormalization constants are fixed to cancel out the UV-divergence.

- Condition 1 (Dirac equation):

$$\hat{\Sigma}(p)u(p) = 0$$

$$\Longrightarrow \frac{\delta_m}{m} = -\Sigma_V(m^2) - \Sigma_S(m^2). \tag{21}$$

- Condition 2:

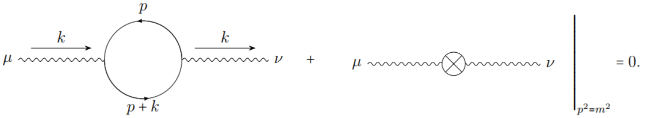


$$\left[\lim_{p^2 \to m^2} \frac{\not{p} + m}{p^2 - m^2}\hat{\Sigma}(p)\right] u(p) \overset{!}{=} 0$$

$$\Longrightarrow \delta_2 = \Sigma_V(m^2) + 2m^2\left(\Sigma_V^{'}(m^2) + \Sigma_S^{'}(m^2)\right). \tag{22}$$

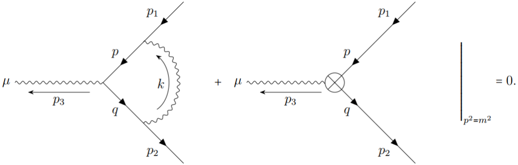
Renormalization conditions

- Condition 3:



$$\left[\lim_{k^2\to 0}\frac{-ig_{\nu\rho}}{k^2}\hat{\Pi}^{\mu\nu}(k)\right]\epsilon_\mu(k)\stackrel{!}{=}0$$
$$\Longrightarrow \delta_3 = \Pi_T'(0). \tag{23}$$

- Condition 4:



$$\bar{u}(p)\hat{\Lambda}^\mu(p,p)u(p) = 0$$
$$\Longrightarrow \delta_e = -\frac{1}{2}\delta_3. \tag{24}$$

Summary

Renormalization procedure

Define the new parameters → Separate the bare parameters into renormalized parameters and counterterm parameters → Choose renormalization conditions to fix the counterterms.

Vertex correction

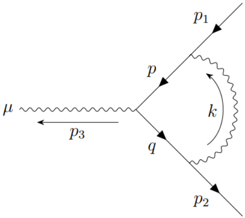


Figure 7: Vertex correction.

$$
\begin{aligned}
&\Lambda^{\mu}(p_1, p_2) \\
=& - ie^2\mu^{4-D} \int \frac{\mathrm{d}^D k}{(2\pi)^D} \frac{\gamma^{\nu}(\not{p}_2 + \not{k} + m)\gamma^{\mu}(\not{p}_1 + \not{k} + m)\gamma_{\nu}}{(k^2 - \lambda^2 + i\epsilon)[(p_1 + k)^2 - m^2 + i\epsilon][(p_2 + k)^2 - m^2 + i\epsilon]} \\
=& \frac{e^2}{(4\pi)^2} \frac{(2\pi\mu)^{4-D}}{i\pi^2} \int \mathrm{d}^D k \, \frac{\gamma^{\nu}(\not{p}_2 + \not{k} + m)\gamma^{\mu}(\not{p}_1 + \not{k} + m)\gamma_{\nu}}{(k^2 - \lambda^2 + i\epsilon)[(p_1 + k)^2 - m^2 + i\epsilon][(p_2 + k)^2 - m^2 + i\epsilon]} .
\end{aligned}
\tag{25}
$$

Let's consider this sophisticated numerator with gamma matrices properties in D-dimension:

$$\gamma^{\nu}\gamma^{\mu}\gamma_{\nu} = (2-D)\gamma^{\mu}, \tag{26}$$

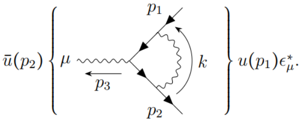
$$\gamma^{\nu}\gamma^{\mu}\gamma^{\rho}\gamma_{\nu} = 4g^{\mu\rho} + (D-4)\gamma^{\mu}\gamma^{\rho}, \tag{27}$$

$$\gamma^{\nu}\gamma^{\rho}\gamma^{\mu}\gamma^{\sigma}\gamma_{\nu} = -2\gamma^{\sigma}\gamma^{\mu}\gamma^{\rho} - (D-4)\gamma^{\rho}\gamma^{\mu}\gamma^{\sigma}. \tag{28}$$

Expanding the numerator, we got

$$\begin{aligned}
\mathcal{N}^{\mu} =& \gamma^{\nu}\not{p}_2\gamma^{\mu}\not{p}_1\gamma_{\nu} + \gamma^{\nu}\not{p}_2\gamma^{\mu}\not{k}\gamma_{\nu} + m\gamma^{\nu}\not{p}_2\gamma^{\mu}\gamma_{\nu} \\
&+ \gamma^{\nu}\not{k}\gamma^{\mu}\not{p}_1\gamma_{\nu} + \gamma^{\nu}\not{k}\gamma^{\mu}\not{k}\gamma_{\nu} + m\gamma^{\nu}\not{k}\gamma^{\mu}\gamma_{\nu} \\
&+ m\gamma^{\nu}\gamma^{\mu}\not{p}_1\gamma_{\nu} + m\gamma^{\nu}\gamma^{\mu}\not{k}\gamma_{\nu} + m^2\gamma^{\nu}\gamma^{\mu}\gamma_{\nu} \\
=& -2(\not{p}_1\gamma^{\mu}\not{p}_2 + \not{k}\gamma^{\mu}\not{p}_2 + \not{p}_1\gamma^{\mu}\not{k}) - (D-4)(\not{p}_2\gamma^{\mu}\not{p}_1 + \not{p}_2\gamma^{\mu}\not{k} + \not{k}\gamma^{\mu}\not{p}_1) \\
&+ 4m(p_1 + p_2 + 2k)^{\mu} + (D-4)m(\not{p}_2\gamma^{\mu} + \not{k}\gamma^{\mu} + \gamma^{\mu}\not{p}_1 + \gamma^{\mu}\not{k}) \\
&+ (2-D)\not{k}\gamma^{\mu}\not{k} + (2-D)m^2\gamma^{\mu}.
\end{aligned} \tag{29}$$

Consequently, sandwich the vertex function:



$$\bar{u}(p_2)\big(-ie\Lambda^\mu(p_1,p_2)\big)u(p_1)\epsilon^*_\mu$$
$$=\bar{u}(p_2)(-ie\gamma^\mu)u(p_1)\epsilon^*_\mu\frac{\alpha}{4\pi}\Big\{4B_0(m^2,0,m)-3B_0(p_3^2,m,m)+(4m^2-2p_3^2)C_0-2\Big\}$$
$$+\bar{u}(p_2)(-ie)u(p_1)\epsilon^*_\mu(p_1+p_2)^\mu\frac{2m}{4m^2-k^2}\frac{\alpha}{4\pi}\Big\{2B_0(p_3^2,m,m)-2B_0(m^2,0,m)+2\Big\}. \tag{30}$$

$\Longrightarrow$ Of course, this is not UV-convergence $\rightarrow$ we need to add the counterterm diagram.

Next-to-leading order result

Applying $p_3 = 0$ due to on-shell condition to renormalized vertex function:

$$\begin{aligned}
&\bar{u}(p_2)\hat{\Gamma}^{Aff}(p_3, -p_1, p_2)\Big|_{p_3=0} u(p_1)\epsilon^*_\mu \\
=& -ie\bar{u}(p_2)\hat{\Lambda}^\mu(p_1, p_2)\Big|_{p_3\to 0} u(p_1)\epsilon^*_\mu \\
=& -ie\bar{u}(p_2)\Lambda^\mu(p_1, p_2)\Big|_{p_3\to 0} u(p_1)\epsilon^*_\mu - ie\bar{u}(p_2)\gamma^\mu\delta_2 u(p_1)\epsilon^*_\mu \\
=& -ie\bar{u}(p_2)\frac{i\sigma^{\mu\nu}}{2m}(p_1 - p_2)_\nu u(p_1)\epsilon^*_\mu \frac{\alpha}{4\pi}\Big\{\underbrace{4B_0(m^2, 0, m) - 4B_0(0, m, m) - 6}_{=2}\Big\} \\
=& -ie\bar{u}(p_2)\Big\{\frac{i\sigma^{\mu\nu}}{2m}(p_1 - p_2)_\nu \frac{\alpha}{2\pi}\Big\} u(p_1)\epsilon^*_\mu .
\end{aligned} \tag{31}$$

Anomalous magnetic moment at NLO

Total scattering amplitude at NLO:

$$\begin{aligned} A^{\mathrm{NLO}} =& A_1 + A_3 \\ =& -ie\bar{u}(p_2)\left\{\gamma^\mu 1 + \frac{i\sigma^{\mu\nu}}{2m}(p_1 - p_2)_\nu \frac{\alpha}{2\pi}\right\}u(p_1)\epsilon^*_\mu. \end{aligned} \tag{32}$$

At NLO, $F1 = 1$ and $F_2 = \alpha/2\pi$, then the Landé g-factor that matches the QED prediction as

$$g_\mu = 2 + \frac{\alpha}{\pi} + \mathcal{O}(\alpha^2). \tag{33}$$

Anomalous magnetic moment of muon at NLO

$$a_\mu = \frac{g_\mu - 2}{2} = \frac{2F_2(0)}{2} = \frac{\alpha}{2\pi} \approx 0.0011614. \tag{34}$$

Conclusions

Here are some results of the anomalous magnetic moment of the muon:

$$\begin{aligned}
a_\mu^{\text{LO}} &= 0, && \text{(LO)} \\
a_\mu^{\text{NLO}} &= \frac{\alpha}{2\pi} \approx 116140973 \times 10^{-11}, && \text{(NLO)} \\
a_\mu^{\text{BNL}} &= 116592089(54)(33) \times 10^{-11}, && \text{(experimental value)} \\
a_\mu^{\text{FNAL}} &= 116592061(41) \times 10^{-11}. && \text{(experimental value)}
\end{aligned} \tag{35}$$

Where the fine-structure constant $\alpha \approx 1/137.035999084$ is available on Particle Data Group.
a_μ^{BNL} is the result from E821 experiment at Brookhaven National Lab (BNL).
a_μ^{FNAL} is the result from he Fermilab National Accelerator Laboratory (FNAL).

Higher-order correction

The anomalous magnetic moment in QED has been computed up to five loops

$$a_\mu^{\text{QED}[1]} = 0.5\frac{\alpha}{\pi} + 0.765857420\left(\frac{\alpha}{\pi}\right)^2 + 24.05050985(23)\left(\frac{\alpha}{\pi}\right)^3 + 130.8785(60)\left(\frac{\alpha}{\pi}\right)^4 + 751.0(9)\left(\frac{\alpha}{\pi}\right)^5 + \dots \quad (36)$$

The anomalous magnetic moment in higher-order correction:

$$\begin{aligned} a^{NNLO} &= 116554190 \times 10^{-11}, \\ a^{N^3LO} &= 116584332 \times 10^{-11}, \\ a^{N^4LO} &= 116584713 \times 10^{-11}, \\ a^{N^5LO} &= 116584718 \times 10^{-11}. \end{aligned} \quad (37)$$

[1] hep-ph/0507249,arXiv:0706.3496v2,hep-ph/0512330

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Thank you for your listening!