# How Current Loops and Solenoids Curve Spacetime [1]

## VSOP 28

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[1] André Füzfa - Phys. Rev. D 93, 024014

# Einstein-Maxwell Equations

Maxwell equation:

$$\nabla_\mu F^{\mu\nu} = \mu_0 J^\nu$$

where $J^\mu = (c\rho, \vec{j})$ is the four-current density, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the Faraday tensor of the electromagnetic field, and $A_\mu$ is the four-vector potential.

Einstein-Maxwell equation:

$$R_{\mu\nu} = -\frac{8\pi G}{c^4} T_{\mu\nu}$$

where $T_{\mu\nu} = -\frac{1}{\mu_0}\left(g^{\alpha\beta} F_{\mu\alpha} F_{\nu\beta} - \frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta}\right)$ is the Maxwell stress-energy tensor while $g_{\mu\nu}$ is the metric tensor.

# Weyl gauge

Current loop and solenoid both have one axis of symmetry $\rightarrow$ we choose Weyl gauge for the metric:

$$ds^2 = c^2 e^{\rho(r,z)} dt^2 - e^{\lambda(r,z)}(dr^2 + dz^2) - e^{-\rho(r,z)} r^2 d\varphi^2$$

In this basis, the four-vector potential only has one non-vanishing component $A_\varphi = a(r,z)/r$.

# Vector Potential

For the current loop of radius $l$ corresponds to a current density located on a infinitely thin ring such that $J \sim \delta(z) \cdot \delta(r - l)$:

$$a^{loop}(r, z) = \frac{\mu_0 I}{2\pi} \sqrt{z^2 + (l + r)^2} \left[ \frac{z^2 + l^2 + r^2}{z^2 + (l + r)^2} K(k^2) - E(k^2) \right]$$

where $k^2 = \frac{4lr}{z^2 + (l+r)^2}$ and where

$$K(k^2) = \int_0^{\pi/2} (1 - k^2 sin^2(\varphi))^{-1/2} d\varphi$$
$$E(k^2) = \int_0^{\pi/2} (1 - k^2 sin^2(\varphi))^{1/2} d\varphi$$

are the complete elliptic integrals of the first and second kind respectively.

# Vector Potential

For the solenoid of finite length $L$ and of radius $l$ corresponds to a current density located on an infinitely thin sheet at $r = l$ and $z \in [-L/2, L/2]$, one can find:

$$a^{sol}(r,z) = \frac{\mu_0 n I}{4\pi}\sqrt{lr}\left[\xi k\left(\frac{k^2+g^2-g^2k^2}{k^2g^2}K(k^2) - \frac{E(k^2)}{k^2} + \frac{g^2-1}{g^2}\Pi(g^2,k^2)\right)\right]_{\xi_-}^{\xi_+}$$

with $n$ being the number of wire loops per unit length, $k^2 = \frac{4rl}{(l+r)^2+\xi^2}$, $g^2 = \frac{4rl}{(r+l)^2}$, $\xi_\pm = z \pm \frac{L}{2}$, and

$$\Pi(g^2,k^2) = \int_0^{\pi/2}(1-g^2sin^2(\varphi))^{-1}(1-k^2sin^2(\varphi))^{-1/2}d\varphi$$

is the complete elliptic integral of the third kind.

# Field Equations

After some long calculations, we can find:

$$R_{11} = \frac{1}{2}\left(\lambda_{11} + \lambda_{22} + r^{-1}\lambda_2 + \rho_1^2\right) = \frac{4\pi G}{\mu_0 c^4}\frac{e^\rho}{r^2}(a_2^2 - a_1^2)$$

$$R_{22} = \frac{1}{2}\left(\lambda_{11} + \lambda_{22} - r^{-1}\lambda_2 - 2r^{-1}\rho_2 + \rho_2^2\right) = \frac{4\pi G}{\mu_0 c^4}\frac{e^\rho}{r^2}(a_1^2 - a_2^2)$$

$$R_{44} = -\frac{1}{2}e^{\rho-\lambda}\left(\rho_{11} + \rho_{22} + r^{-1}\rho_2\right) = -\frac{4\pi G}{\mu_0 c^4}\frac{e^{2\rho-\lambda}}{r^2}(a_1^2 + a_2^2)$$

$$R_{12} = \frac{1}{2}\left(\rho_1\rho_2 - r^{-1}\rho_1 - r^{-1}\lambda_1\right) = -\frac{8\pi G}{\mu_0 c^4}\frac{e^\rho}{r^2}(a_1 a_2)$$

$$-\mu_0 J = \frac{1}{r}\left[\left(a_{11} + a_{22} + \frac{a_2}{r}\right) - \frac{2a_2}{r} + a_1\rho_1 + a_2\rho_2\right]$$

Note that we let the subscript indices $1, 2$ indicate the partial derivatives with respect to $z$ and $r$, respectively.

# Field Equations

Rearrange the above equations, one arrives at:

$$\begin{aligned}
\nabla^2_{(r,z)}\rho &= \frac{8\pi G}{\mu_0 c^4}\frac{e^{\rho}}{r^2}\left((\partial_r a)^2 + (\partial_z a)^2\right) \\
\nabla^2_{(r,z)}\lambda + (\partial_z \rho)^2 &= \frac{8\pi G}{\mu_0 c^4}\frac{e^{\rho}}{r^2}\left((\partial_r a)^2 - (\partial_z a)^2\right) \\
\partial_z \lambda + \partial_z \rho &= r\partial_r \rho \partial_z \rho + \frac{16\pi G}{\mu_0 c^4}\frac{e^{\rho}}{r}\partial_r a \partial_z a \\
\nabla^2_{(r,z)} a - \frac{2}{r}\partial_r a &= -\left(\partial_r a \partial_r \rho + \partial_z a \partial_z \rho\right) - r\mu_0 J
\end{aligned}$$

where $\nabla^2_{(r,z)} := \partial_r^2 + \partial_z^2 + \frac{1}{r}\partial_r$

# Field Equations

If we take the non-relativistic limit of the last equation, that is for a flat Minkowski spacetime where $\rho = \lambda = 0$, we obtain the non-relativistic field $a_{nr}$ that satisfies

$$\nabla^2_{(r,z)} a_{nr} - \frac{2}{r} \partial_r a_{nr} = -r \mu_0 J$$

So if we write $a = a_{nr} + a_{rel}$, then we will get the following equation:

$$\nabla^2_{(r,z)} a_{rel} - \frac{2}{r} \partial_r a_{rel} = - \left[ (\partial_r a_{nr} + \partial_r a_{rel}) \partial_r \rho + (\partial_z a_{nr} + \partial_z a_{rel}) \partial_z \rho \right]$$

# Field Equations

Set $r = ul$, $z = vL$, and $a_{nr,rel} \to a_{nr,rel}/(\mu_0 Il)$ (for current loop), $a_{nr,rel} \to a_{nr,rel}/(\mu_0 nIlL)$ (for solenoid), the equations become a set of dimensionless equations

$$\nabla^2 \rho = \mathcal{C}_I \frac{L^2}{l^2} \frac{e^\rho}{u^2} \left( (\partial_u (a_{nr} + a_{rel}))^2 + \frac{l^2}{L^2} (\partial_v (a_{nr} + a_{rel}))^2 \right)$$

$$\nabla^2 \lambda + \frac{l^2}{L^2} (\partial_v \rho)^2 = \mathcal{C}_I \frac{L^2}{l^2} \frac{e^\rho}{u^2} \left( (\partial_u (a_{nr} + a_{rel}))^2 - \frac{l^2}{L^2} (\partial_v (a_{nr} + a_{rel}))^2 \right)$$

$$\nabla^2 a_{rel} - \frac{2}{u} \partial_u a_{rel} = - \left( \partial_u (a_{nr} + a_{rel}) \partial_u \rho + \frac{l^2}{L^2} \partial_v (a_{nr} + a_{rel}) \partial_v \rho \right)$$

$$\begin{aligned} 0 &= -\partial_v \lambda - \partial_v \rho + u \partial_u \rho \partial_v \rho \\ &\quad + 2\mathcal{C}_I \frac{e^\rho}{u} \partial_u (a_{nr} + a_{rel}) \partial_v (a_{nr} + a_{rel}) \end{aligned}$$

$\nabla^2 = \partial_u^2 + \frac{l^2}{L^2} \partial_v^2 + \frac{1}{u} \partial_u$, $\mathcal{C}_I^{loop} = \frac{8\pi G}{c^4} \mu_0 I^2$, $\mathcal{C}_I^{sol} = \frac{8\pi G}{c^4} \mu_0 I^2 n^2 l^2$

We need some boundary conditions!

# Boundary Conditions

We want our metric fields to have the asymptotic behaviors. Furthermore, spacetime must be smooth on the axis of symmetry

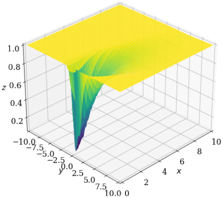
$$\partial_r \rho|_{r=0} = \partial_r \lambda|_{r=0} = \partial_r a|_{r=0} = 0$$

$$\rho \sim \frac{\mathcal{C}_l}{32} \frac{L^4}{l^4} v^2 \left( u^2 + \frac{L^2}{l^2} v^2 \right)^{-3}$$

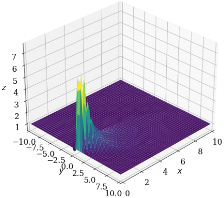
$$\lambda \sim \frac{\mathcal{C}_l}{16} \frac{L^4}{l^4} \left[ 2u^2 \left( u^2 + \frac{L^2}{l^2} v^2 \right)^{-3} - \frac{L^2}{l^2} \frac{v^2}{2} \left( u^2 + \frac{L^2}{l^2} v^2 \right)^{-3} - \frac{9u^4}{4} \left( u^2 + \frac{L^2}{l^2} v^2 \right)^{-3} \right]$$

# Solutions of Field Equations

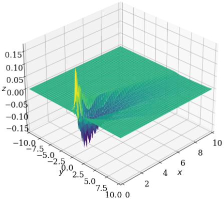
Current loop of $\mathcal{C}_I = 15$




$[e^{\rho}]$



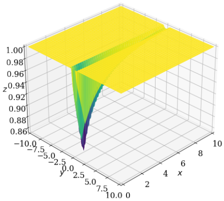

$[e^{\lambda}]$



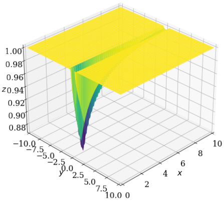

$[a_{rel}]$

# Solutions of Field Equations

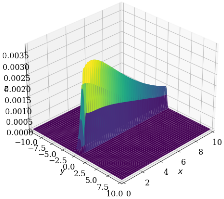
Solenoid of length $L = 10$ and $\mathcal{C}_I = 10$




$[e^{\rho}]$



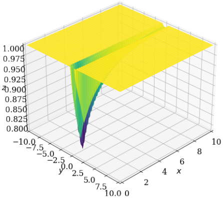

$[e^{\lambda}]$



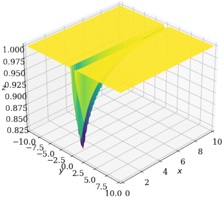

$[a_{rel}]$

# Solutions of Field Equations

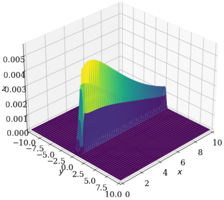
Solenoid of length $L = 10$ and $\mathcal{C}_I = 15$




$[e^{\rho}]$



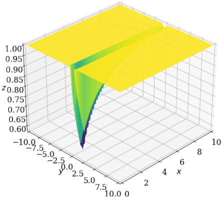

$[e^{\lambda}]$



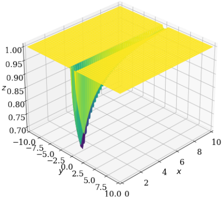

$[a_{rel}]$

# Solutions of Field Equations

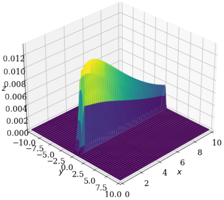
Solenoid of length $L = 15$ and $\mathcal{C}_I = 15$




$[e^{\rho}]$




$[e^{\lambda}]$




$[a_{rel}]$

# Current Loops and Solenoids Do Bend Light!

In GR, light follows the paths called geodesics:

$$\frac{d^2 x^\alpha}{ds^2} + \Gamma^\alpha_{\beta\gamma} \frac{dx^\beta}{ds} \frac{dx^\gamma}{ds} = 0$$

# Current Loops and Solenoids Do Bend Light!

If we restrict ourselves to the planar trajectories in the $(r, z)$-plane, then $\varphi = constant$ and $\frac{d\varphi}{ds} = 0$. We also want to obtain a set of dimensionless equations by setting $s = Sl, r = ul, z = vL$. We can derive the following equations:

$$\frac{d^2 v}{dS^2} + \frac{1}{2}\left(\frac{\partial \rho}{\partial v} + \frac{\partial \lambda}{\partial v}\right)\left(\frac{dv}{dS}\right)^2 + \frac{l^2}{2L^2}\left(\frac{\partial \rho}{\partial v} - \frac{\partial \lambda}{\partial v}\right)\left(\frac{du}{dS}\right)^2 + \frac{\partial \lambda}{\partial u}\frac{dv}{dS}\frac{du}{dS} = 0$$

$$\frac{d^2 u}{dS^2} + \frac{L^2}{2l^2}\left(\frac{\partial \rho}{\partial u} - \frac{\partial \lambda}{\partial u}\right)\left(\frac{dv}{dS}\right)^2 + \frac{1}{2}\left(\frac{\partial \lambda}{\partial u} + \frac{\partial \rho}{\partial u}\right)\left(\frac{du}{dS}\right)^2 + \frac{\partial \lambda}{\partial v}\frac{dv}{dS}\frac{du}{dS} = 0$$

# Current Loops and Solenoids Do Bend Light!

Let $\frac{du}{dS} = U$ and $\frac{dv}{dS} = V$. The above equations become:

$$\frac{du}{dS} = U$$

$$\frac{dv}{dS} = V$$

$$\frac{dU}{dS} = -\frac{L^2}{2l^2}\left(\frac{\partial \rho}{\partial u} - \frac{\partial \lambda}{\partial u}\right)V^2 - \frac{1}{2}\left(\frac{\partial \lambda}{\partial u} + \frac{\partial \rho}{\partial u}\right)U^2 - \frac{\partial \lambda}{\partial v}UV$$

$$\frac{dV}{dS} = -\frac{1}{2}\left(\frac{\partial \rho}{\partial v} + \frac{\partial \lambda}{\partial v}\right)V^2 - \frac{l^2}{2L^2}\left(\frac{\partial \rho}{\partial v} - \frac{\partial \rho}{\partial v}\right)U^2 - \frac{\partial \lambda}{\partial u}UV$$

We need some boundary conditions. Again.

# Current Loops and Solenoids Do Bend Light!

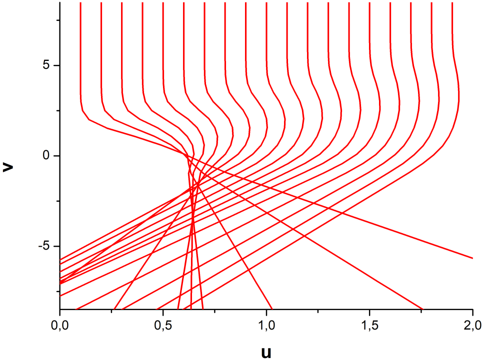
Let $s = ct$ where $t$ is the coordinate time at spatial infinity, we can choose the following boundary conditions:

$$\begin{aligned} u_i &= u_k, \quad k = 1, 2, 3, \ldots \\ v_i &= \infty \\ U_i &= 0 \\ V_i &= -1 \end{aligned}$$

where each $k = 1, 2, 3, \ldots$ corresponds to each specific path of light.

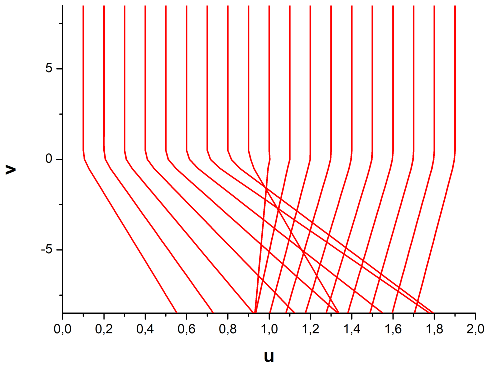
# Numerical Results

Current loop of $\mathcal{C}_I = 15$



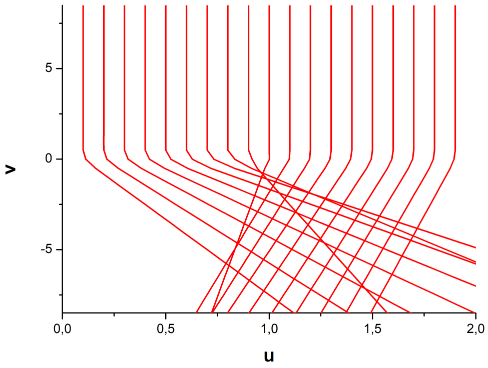
# Numerical Results

Solenoid of $L = 10, \mathcal{C}_I = 10$



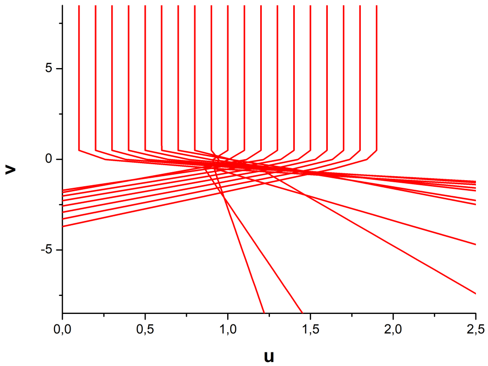
# Numerical Results

Solenoid of $L = 10, \mathcal{C}_I = 15$



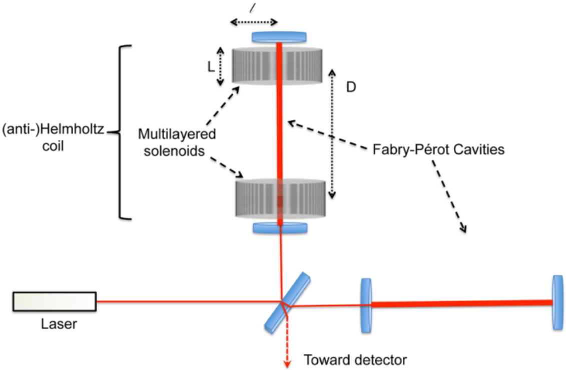
# Numerical Results

Solenoid of $L = 15, \mathcal{C}_I = 15$



# The Proposed Experiment

Figure from [Phys. Rev. D 93, 024014]. Schematic view of the proposed experimental set up.



# The Proposed Experiment

We prepare beforehand the ingredients as follows: we will have a set of 10 stacked anti-Helmholtz coils, each coil constituted by 2 superconducting solenoids of the same length $L = 2.5(m)$ carrying opposite direction, steady electric currents with a power of $I = 20(kA)$, and the spacing we choose is $D = 2.5(m)$. The external solenoids will have a radius of $l = 5(m)$ and the spacings between the shell of the 10 solenoids are all equal and have the value between $r = 1(m)$ and $r = 5(m)$. We choose the length of the interferometer arm to be $\mathcal{L} = 50(m)$.

# The Proposed Experiment

If the experiment is conducted for a really long time $T_{exp}$, say $T_{exp} \approx 200$ days, the phase shift that we will get would be $\Delta\Phi \approx -1.08 \times 10^{-11}$. Which is of the order of the gravitational wave phase shift. This is really amazing!

Thank you for listening!