$e^+e^- \to$ HADRONS AT NLO

R ratio : $\dfrac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} = \left[N_c \sum_f Q_f^2 \right] \left[1 + \dfrac{\alpha_s}{\pi} + O(\alpha_s^2) \right]$

$\uparrow$ LO $\quad \uparrow$ NLO

At LO, we identified $\sigma(e^+e^- \to \text{hadrons}) = \sigma(e^+e^- \to q\bar{q})$

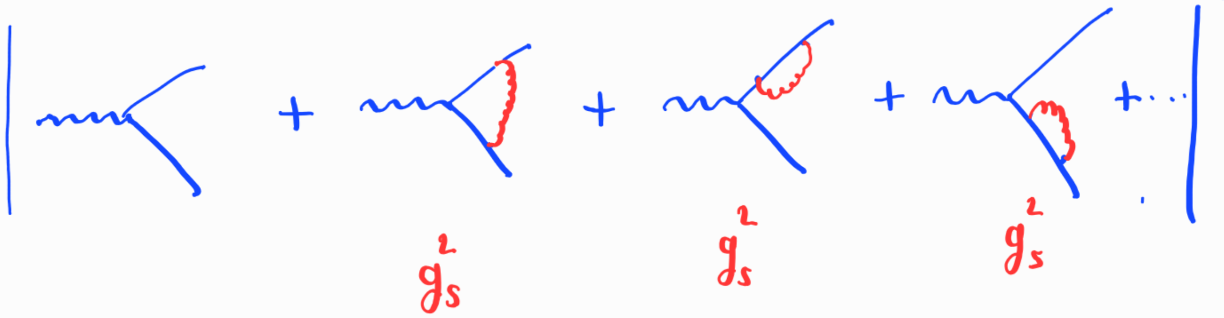
At NLO : identify all process that contribute to the cross section at order α_s and that have strongly interacting particles in the final state

a) LOOP DIAGRAMS

b) GLUON EMISSION IN THE FINAL STATE

• "anatomy" of a NLO computation

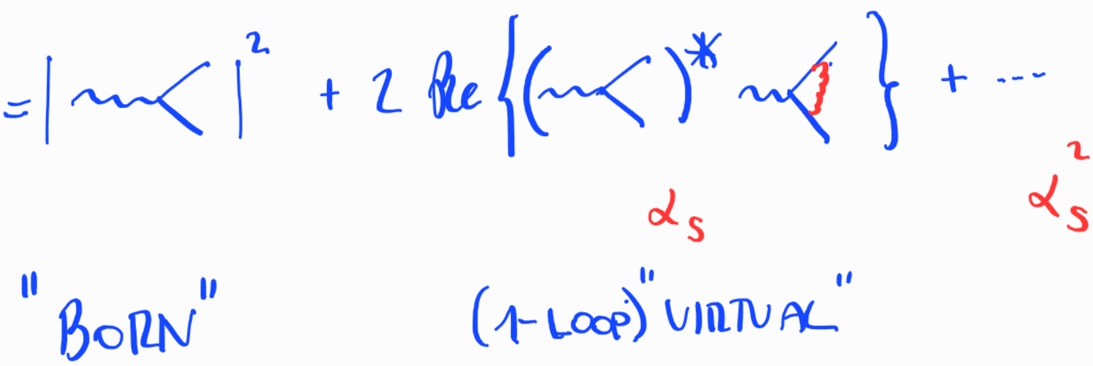
a) $|M_{q\bar{q}}|^2 =$



$$= |M_{\text{tree}} + M_{1\,\text{loop}} + M_{2\,\text{loop}} + \dots|^2$$

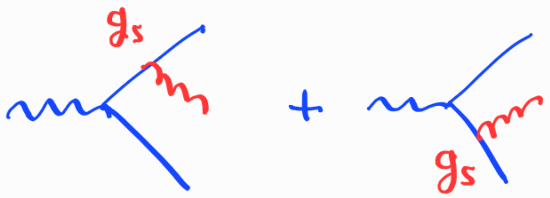
$M_{1\,\text{loop}} \sim g_s^2$

$$\Rightarrow |M_{q\bar{q}}| = |M_{\text{tree}}|^2 + 2\,\text{Re}(M_{\text{tree}}^* M_{1\,\text{loop}}) + |M_{1\,\text{loop}}|^2 + \dots$$



$$d\sigma = \frac{1}{2s} |M_{q\bar{q}}|^2 \, d\phi_2 = d\sigma_B + d\sigma_V$$

After integration : $\sigma \to \sigma_B + \sigma_V$

b) $|M_{q\bar{q}g}|^2 = \Big|$  $\Big|^2$ is of $O(\alpha_s)$

"REAL"

$$d\sigma = \frac{1}{2s} |M_{q\bar{q}g}|^2 \, d\phi_3 = d\sigma_R$$

- Different phase space (3 body final state)
- After integration : $\sigma \to \sigma_R$

At NLO : σ_V, σ_R

VIRTUAL CROSS SECTION : $\longrightarrow$ IR DIVERGENT

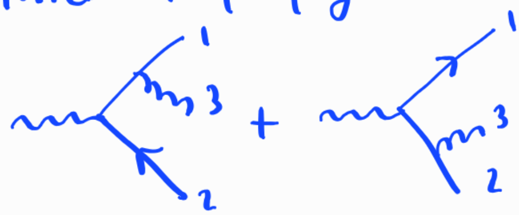
- dimensional regularization

$$\sigma_V = \sigma_B \, H(\varepsilon) \, C_F \frac{\alpha_s}{2\pi} \left[-\frac{2}{\varepsilon^2} - \frac{3}{\varepsilon} + \pi^2 - 8 \right]$$

- $\frac{1}{\varepsilon^2}$ certainly not from UV [renormalization @ 1 loop $\dots\to \frac{1}{\varepsilon}$]
- In fact (after renormalization) all poles are IR
- INFRARED POLES

REAL CROSS SECTION :

- internal propagators



NB I'll also use k for the gluon momentum and p_1, p_2 for the quarks...

First diagram: $\frac{\cancel{k_1}+\cancel{k_3}}{(k_1+k_3)^2} \sim \frac{1}{2k_1k_3}$

massless quarks $\frac{1}{2E_qE_g(1-\cos\theta_{qg})}$

- Emission of a gluon from external leg
  $\Rightarrow$ potentially divergent if

  $E_g \to 0 \quad \to$ SOFT DIVERGENCE

  $\left.\begin{array}{l}\theta_{qg} \to 0 \\ \theta_{\bar{q}g} \to 0\end{array}\right\}$ COLLINEAR DIVERGENCE

- To be more precise, I should check what happens at the x-section level (and not just at the amplitude level)

- $$|M_{q\bar{q}g}|^2 \sim \left[\frac{k_1k}{k_2k} + \frac{k_2k}{k_1k} + \frac{1}{2}\frac{(k_1+k_2)^2(k_1+k_2+k)^2}{(k_1k)(k_2k)}\right]$$

$\Rightarrow$ singularities when $\begin{array}{l}k_1k \to 0 \\ k_2k \to 0\end{array} \Leftrightarrow \left\{\begin{array}{l}E_g \to 0 \\ \theta_{qg} \to 0 \\ \theta_{\bar{q}g} \to 0\end{array}\right.$

- $\underline{d\phi_3}$: 5-dimensional $(3\times 3 - 4)$

  We are only interested in gluon kinematics with respect to q and $\bar{q}$

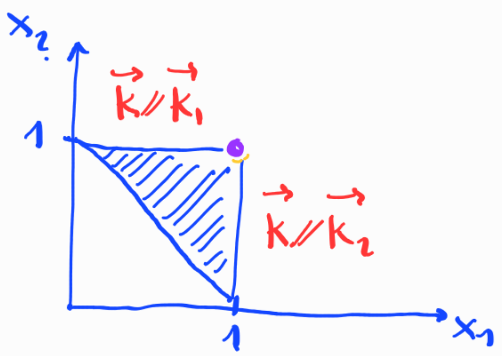
  $Q^\mu = k_1^\mu + k_2^\mu + k^\mu$

  CoM: $Q^\mu = \sqrt{s}(1,0,0,0)$

Only 2 variables needed :

$$x_1 = \frac{2E_q}{\sqrt{s}} \,, \quad x_2 = \frac{2E_{\bar{q}}}{\sqrt{s}} \,, \quad x_3 = \frac{2E_g}{\sqrt{s}}$$

with constraint $\begin{cases} x_3 = 2 - x_1 - x_2 \\ 0 < x_1 < 1 \\ 0 < x_2 < 1 \end{cases}$, $x_1 + x_2 \geq 1$



$$1 - x_1 = \frac{x_2 E_g}{\sqrt{s}} (1 - \cos\theta_{\bar{q}g})$$

$x_1 = 1$

$x_2 = 1$ } collinear limits

$x_1 + x_2 = 2 \to$ SOFT LIMIT

$$|M_{q\bar{q}g}|^2 d\phi_3 \sim \frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)} dx_1 dx_2$$

REAL CROSS SECTION : IR (soft/collinear) DIVERGENT

[notice: divergences are LOGARITHMICAL]

$\sigma_R + \sigma_V$ IS FINITE

Need to compute σ_R in d-dimension

$$d\phi_3 \sim (1-x_1)^{-\varepsilon} (1-x_2)^{-\varepsilon} (1-x_3)^{-\varepsilon} dx_1 dx_2$$

$$|M_{q\bar{q}g}|^2 \sim \frac{x_1^2 + x_2^2 - \varepsilon(2 - x_1 - x_2)}{(1-x_1)(1-x_2)}$$

It can be shown that

$$\sigma_R^{(d)} \simeq \sigma_B \, H(\varepsilon) \, C_F \frac{\alpha_s}{2\pi} \left[\frac{2}{\varepsilon^2} + \frac{3}{\varepsilon} + \frac{19}{2} - \pi^2 + O(\varepsilon) \right]$$

$$\sigma_R^{(d)} + \sigma_V^{(d)} \xrightarrow[\varepsilon \to 0]{} \sigma_B \, C_F \frac{\alpha_s}{2\pi} \left[\frac{19}{2} - 8 \right] = \sigma_B \frac{\alpha_s}{\pi}$$

This cancellation is NOT accidental

- soft and collinear poles cancelled because we computed a fully inclusive cross section
- KLN theorem [Kinoshita, Lee, Nauenberg]

Cross section obtained summing over degenerate final states are FINITE

- In our case,

$d\phi_3$ with $\begin{cases} \text{soft gluon} \\ \text{collinear } qg \ (\bar{q}g) \end{cases}$

are degenerate with final state without a gluon in the final state $(d\phi_2)$

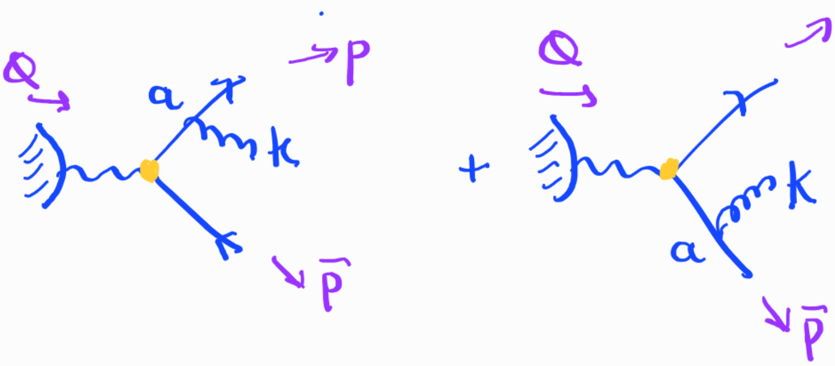
- If all contributions at order α_s $(\to R, V)$

$\Rightarrow$ FINITE RESULT

REAL AND VIRTUAL CORRECTIONS IN THE SOFT LIMIT

We can understand better the cancellation of divergencies in R+V by looking at the soft limit (in $d=4$)

- amplitude



$= M_1 + M_2$

$= -i\Gamma$, $M_B = \bar{u}(p)\,\Gamma\, v(\bar{p})$

M_1^μ , M_2^μ : amplitudes before contracting them with the gluon polarization $\epsilon_\mu(k)$

EX: show that, in the soft limit

$$M^\mu = (M_1 + M_2)^\mu = g\, \bar{u}\, \Gamma\, t^a\, v \left(\frac{\bar{p}^\mu}{\bar{p}k} - \frac{p^\mu}{pk} \right)$$

EX: show that

$$|M|^2 = \sum_{\substack{\text{pol.}\\ \text{color}}} \left| M^\mu \epsilon_\mu(k) \right|^2 = |M_B|^2\, C_F\, g^2\, \frac{2\, p\bar{p}}{(pk)(\bar{p}k)}$$

• Phase space:

$$d\phi_3 = \frac{d^3p}{(2\pi)^3 2p_0} \frac{d^3\bar{p}}{(2\pi)^3 2\bar{p}_0} \frac{d^3k}{(2\pi)^3 2k_0} (2\pi)^4 \delta^4(\quad)$$

$$\delta^4(Q-p-\bar{p}-k) \simeq \delta^4(Q-p-\bar{p})$$

$$\Rightarrow d\phi_3 = d\phi_2 \frac{d^3k}{(2\pi)^3 2k_0}$$

• $d\sigma_R$ in soft limit:

$$d\sigma_R^s \simeq d\sigma_B \; C_F (4\pi\alpha_s) \frac{2p\bar{p}}{(pk)(\bar{p}k)} \frac{d^3k}{(2\pi)^3 2k_0}$$

• FACTORIZATION:

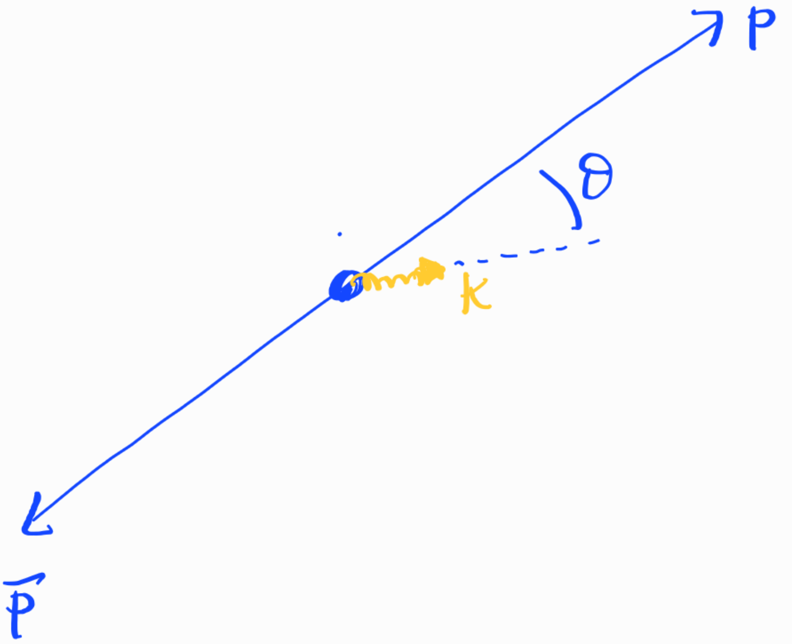
$$d\sigma_R^{soft} \simeq d\sigma_B \times \text{Soft factor}$$

Comments

a) Not too difficult to show that

$$d\sigma_R^s \simeq d\sigma_B \frac{2\alpha_s}{\pi} C_F \int_0 \frac{dk_0}{k_0} \int_{-1}^{1} \frac{d\cos\theta}{(1+\cos\theta)(1-\cos\theta)}$$

where:



- soft (and also collinear) divergences (logarithmic)

b) We know that total x-section is finite (KLN theorem)

$\Rightarrow$ necessarily

$$\sigma_V^s = -\sigma_B \frac{2\alpha_s C_F}{\pi} \int_0 \frac{dk_0'}{k_0'} \int_{-1}^{1} \frac{d\cos\theta}{(1-\cos\theta)(1+\cos\theta)}$$

[It's possible to obtain the above result directly, looking at  in soft limit (4 dim)

c) We've just discussed the soft limit.....

$$\Rightarrow \begin{cases} d\sigma_R = d\sigma_R^S + \text{"finite"} \\ \sigma_V = \sigma_V^S + \text{"finite"} \end{cases}$$

IR-SAFE OBSERVABLES

Generic observable at NLO, in soft limit:

- we've expressed σ_V with the same variables of $d\sigma_R$
- Integration boundaries are, in general, different

$\sigma(\mathcal{O}) = \sigma_{LO} + \sigma_{R+V} =$

$\sigma(\mathcal{O})$ = x-section for a generic observ.

$$= \int d\phi_2 \, |M_B|^2 \, U_2(p, \bar{p}) \; +$$

$$\int d\phi_2 \, |M_B|^2 \int \frac{d^3k}{(2\pi)^3 2E_k} \, 2g^2 C_F \frac{p\bar{p}}{(pk)(\bar{p}k)} \Big[\underset{\text{real}}{U_3(\phi_3)} - \underset{\text{virtual}}{U_2(\phi_2)} \Big]$$

- RESULT IS FINITE ONLY IF $U_3(\phi_3) \to U_2(\phi_2)$ when soft gluon collinear $qg/\bar{q}g$

i.e. $\mathcal{O}$ is INFRARED AND COLLINEAR SAFE

- if not:

  infinite results $\longleftrightarrow$ not summing over degenerate states at x-sec level [KLN theorem hypothesis]

- LHC observables (e.g. jets) are IR safe

  (generalized to n-body phase space)

$$U_{m+1}(\ldots, k_i, k_j, \ldots) \to U_m(\ldots, k_i + k_j, \ldots) \quad \text{if } \vec{k}_i \parallel \vec{k}_j$$

$$U_{m+1}(\ldots, k_i, \ldots) \to U_m(\ldots, \ldots) \quad \text{if } k_i \to 0$$

and similar properties if real emission is collinear to beam axis

$\rightsquigarrow$ pQCD cannot predict an arbitrary observable. There are theoretical constraints.

[have a look at Sterman-Weinberg jets in Nason's lectures.]