RENORMALIZATION THROUGH COUNTERTERMS [discussion ~ Collins]

- Change (redefine) fields and parameters of the original "bare" $\mathcal{L}$

$$A_0^\mu = \sqrt{Z_A}\, A^\mu$$
$$\psi_0 = \sqrt{Z_\psi}\, \psi$$
$$\eta_0 = \sqrt{Z_\eta}\, \eta$$

BARE FIELDS

RENORM. FIELDS

$$\mathcal{L}(A_0, \psi_0, \eta_0, m_0, g_0) = Z_\psi \bar{\psi}(i\partial\!\!\!/ - m_0)\psi - \frac{Z_A}{4}\left(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a\right)^2$$
$$- Z_\psi \sqrt{Z_A}\, g_0 (\bar{\psi} t^a \gamma^\mu \psi) A_\mu^a$$
$$+ Z_A^{3/2} \frac{g_0}{2} f^{abc} (\partial_\mu A_\nu^a - \partial_\nu A_\mu^a) A^{\mu b} A^{\nu c}$$
$$+ \ldots$$

$$\mathcal{L} = \mathcal{L}_{free} + \mathcal{L}_{int} + \mathcal{L}_{counterterms}$$

$$\mathcal{L}_{free} = \bar{\psi}(i\partial\!\!\!/ - m)\psi - \frac{1}{4}\left(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a\right)^2 + \ldots$$

$$\mathcal{L}_{int} = - g\mu^\varepsilon (\bar{\psi} t^a \gamma^\mu \psi) A_\mu^a$$
$$+ \frac{1}{2} g\mu^\varepsilon f^{abc} (\partial_\mu A_\nu^a - \partial_\nu A_\mu^a) A^{\mu b} A^{\nu c} + \ldots$$

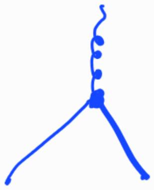
$$\mathcal{L}_{count} = (Z_\psi - 1)\bar{\psi}\, i\partial\!\!\!/\, \psi - (g_0 Z_\psi Z_A^{1/2} - g\mu^\varepsilon)\bar{\psi} A\!\!\!/^a t^a \psi + \ldots$$

g : renormalized coupling , $m_0 = Z_m m$

$\mathcal{L}_{int}$: interactions with FINITE RENORMALIZED COUPLINGS

$\mathcal{L}_{count}$: to cancel divergences

- $\mathcal{L}_{counter}$: extra vertexes $\to$ interactions
  $\to$ 2-pt functions



$$= \left(g_0 Z_\psi \sqrt{Z_A} - g\mu^\varepsilon\right)(-i)\gamma^\mu t^a$$

$$= \left(g_0 Z_A^{3/2} - g\mu^\varepsilon\right) \times -f^{abc}\left[g^{\alpha\beta}(p_a - p_b)^\gamma + \dots\right]$$

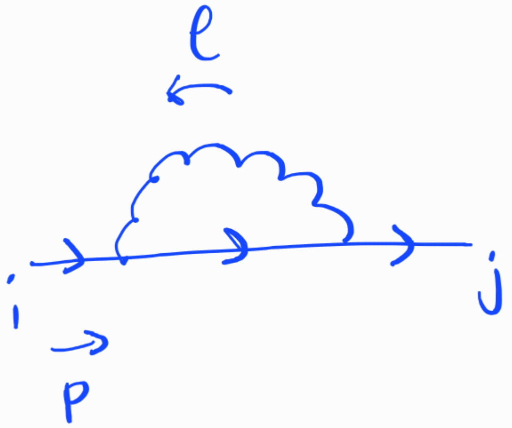
$$= i\left(Z_A - 1\right)\left(-g_{\mu\nu}p^2 + p_\mu p_\nu\right)$$

$$= i\left[\not{p}\,(Z_\psi - 1) - (Z_\psi Z_m - 1)m\right]$$

WAVE FUNCTION RENORMALIZATION [quark self energy]

- We want to compute radiative corrections + counterterms
  - ↳ find Z factor removing UV-divergencies, according to $\overline{MS}$ scheme

- Reported in many textbooks
- Here as quickly as possible → focus only on Z_ψ
  Feynman gauge
  neglect mass in denomin.



$$I = \int \frac{d^d \ell}{(2\pi)^d} \left(-ig\mu^\varepsilon t^a_{jk} \gamma^\mu\right) \frac{i(\not\ell + \not p)}{(\ell+p)^2} \left(-ig\mu^\varepsilon t^a_{ki} \gamma_\mu\right) \frac{-i}{\ell^2}$$

$$= \left\{ \gamma^\mu \gamma^\alpha \gamma_\mu = (2-d)\gamma^\alpha, \quad (t^a t^a)_{jk} = C_F \delta_{jk} \right\}$$

$$= (d-2) g^2 \mu^{2\varepsilon} C_F \delta_{ij} \gamma_\mu \int \frac{d^d \ell}{(2\pi)^d} \frac{\ell^\mu + p^\mu}{\ell^2 (\ell+p)^2}$$

$$= g^2 \mu^{2\varepsilon} C_F \delta_{ij} (d-2) \gamma_\mu \left[\frac{-B_0(p^2)}{2} p^\mu + p^\mu B_0(p^2) \right]$$

$$= g^2 \mu^{2\varepsilon} C_F \, \delta_{ij} \, \not p \, \frac{d-2}{2} \, B_0(p^2)$$

$$\underset{UV}{\simeq} g^2 \mu^{2\varepsilon} C_F \, \delta_{ij} \, \not p \, (1-\varepsilon) \left[\frac{i}{(4\pi)^2} \frac{C_\Gamma}{1-2\varepsilon} \left(\frac{1}{\varepsilon} + \dots \right) \right]$$

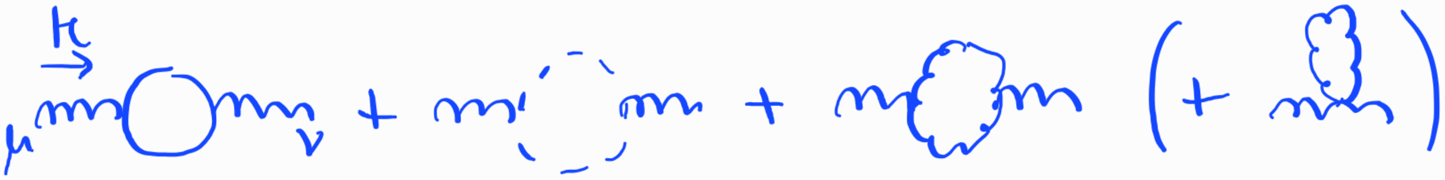
UV pole ($\overline{MS}$) $\longrightarrow (i \not p) \cdot \frac{S_\varepsilon}{\varepsilon} C_F \cdot \frac{\alpha}{4\pi}$

$-\!\!\times\!\!- = i \not p \, (Z_\psi - 1)$ HAS TO CANCEL THE ABOVE POLE

$$\Rightarrow Z_\psi - 1 = - \frac{S_\varepsilon}{\varepsilon} C_F \frac{\alpha}{4\pi}$$

$$Z_\psi = 1 - C_F \frac{\alpha}{4\pi} \frac{S_\varepsilon}{\varepsilon}$$

WAVE FUNCTION RENORM. [gluon self energy]



UV pole ($\overline{MS}$) $\longrightarrow i \frac{\alpha}{4\pi} \left[-\frac{4}{3} n_F T_F \frac{S_\varepsilon}{\varepsilon} \right] (k^2 g^{\mu\nu} - k^\mu k^\nu) +$

$$i \frac{\alpha}{4\pi} \left[C_A \frac{S_\varepsilon}{\varepsilon} \right] \left(\frac{k^2}{12} g^{\mu\nu} + \frac{1}{6} k^\mu k^\nu \right) +$$

$$i \frac{\alpha}{4\pi} \left[C_A \frac{S_\varepsilon}{\varepsilon} \right] \left(\frac{19}{12} k^2 g^{\mu\nu} - \frac{11}{6} k^\mu k^\nu \right)$$

Notice that corrections are transverse : ghost diagram CRUCIAL

UV pole ($\overline{MS}$) $\longrightarrow \frac{S_\varepsilon}{\varepsilon}\left(k^2 g^{\mu\nu} - k^\mu k^\nu\right)\left[\frac{5}{3}C_A - \frac{4}{3}n_F T_F\right]\frac{\alpha}{4\pi}$

$$\sim\!\!\times\!\!\sim \; = i\left(Z_A - 1\right)\left(-g^{\mu\nu}k^2 + k^\mu k^\nu\right)$$

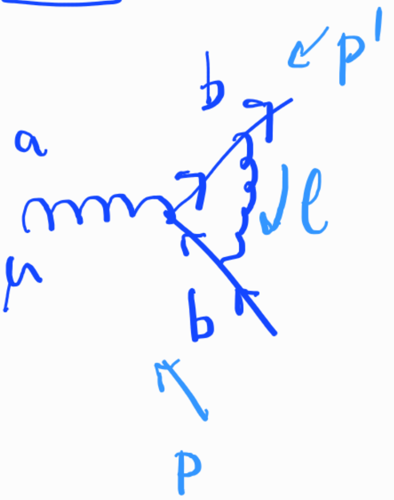
UV POLE CANCELLATION

$$\Rightarrow Z_A = 1 + \left(\frac{5}{3}C_A - \frac{4}{3}n_F T_F\right)\frac{\alpha}{4\pi}\,\frac{S_\varepsilon}{\varepsilon}$$

QUARK GLUON VERTEX

2 diagrams:  +  $= M_a + M_b$

M_a



$$\int \frac{d^d \ell}{(2\pi)^d} \left(-ig\mu^{\varepsilon} t^b \gamma^\nu\right) \frac{i(\ell-p')}{(\ell-p')^2} \left(-ig\mu^{\varepsilon} t^a \gamma^\mu\right) \cdot \frac{i(\ell+p)}{(\ell+p)^2}$$

$$\cdot \left(-ig\mu^{\varepsilon} \gamma_\nu t^b\right) \frac{-i}{\ell^2}$$

$$= (-)\left(C_F - \frac{C_A}{2}\right) t^a g^3 \mu^{3\varepsilon} \cdot$$

$$\int \frac{d^d \ell}{(2\pi)^d} \left[\gamma^\nu (\ell-p') \gamma^\mu (\ell+p) \gamma_\nu\right] \frac{1}{\ell^2 (\ell-p')^2 (\ell+p)^2}$$

EXTRACT UV behaviour $\to$ neglect extand momenta

- $[\dots] = \ell^\alpha \ell^\beta [\gamma_\nu \gamma_\alpha \gamma^\mu \gamma_\beta \gamma^\nu] = \ell^\alpha \ell^\beta (2-d) \gamma_\alpha \gamma^\mu \gamma_\beta$

- $\displaystyle\int \frac{d^d\ell}{(2\pi)^d} \frac{\ell^\alpha \ell^\beta}{(\ell^2)^3} = A g^{\alpha\beta} \quad \Rightarrow A = \frac{1}{d} \int \frac{d^d\ell}{(2\pi)^d} \left(\frac{1}{\ell^2}\right)^2$

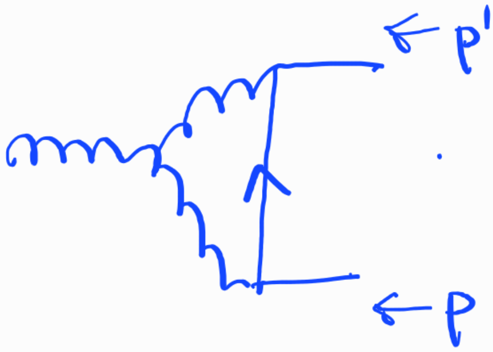
- $\displaystyle A_{UV} = \frac{1}{d} \frac{i}{(2\pi)^d} \frac{\pi^{d/2}}{\Gamma(d/2)} \frac{1}{\varepsilon} = \frac{i}{d} \frac{1}{16\pi^2} \frac{1}{1-\varepsilon} \frac{S_\varepsilon}{\varepsilon}$

Putting everything together:

$$M_a : (-)\left(C_F - \frac{C_A}{2}\right) t^a g^3 \mu^{3\varepsilon} (2-d)^2 A \gamma^\mu$$

UV pole ($\overline{MS}$) $\to -ig t^a \gamma^\mu \mu^\varepsilon \frac{\alpha}{4\pi} \left(C_F - \frac{C_A}{2}\right) \frac{S_\varepsilon}{\varepsilon}$

M_b]

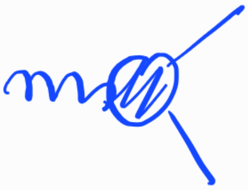


$= \ldots\ldots$

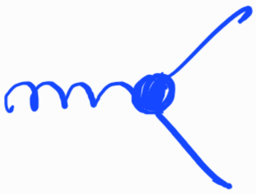
UV pole ($\overline{MS}$) $\longrightarrow -ig\, t^a \gamma^\mu \mu^\varepsilon \frac{\alpha}{4\pi}\left(\frac{C_A}{2}\right)\cdot 3\, \frac{S_\varepsilon}{\varepsilon}$

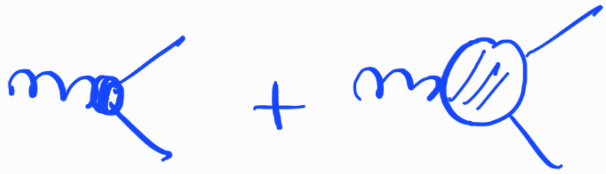
[3 comes from Dirac matrices]

Putting everything together:

UV pole ($\overline{MS}$) $\Big\{$  $\Big\}$ $= -ig\, t^a \gamma^\mu \mu^\varepsilon \cdot \frac{\alpha}{4\pi}\left(C_F + C_A\right)\frac{S_\varepsilon}{\varepsilon}$

Counterterm:

 $= -i\gamma^\mu t^a \left(g_0 Z_\psi Z_A^{1/2} - g\mu^\varepsilon\right)$

- Substitute Z_ψ and Z_A
- Sum  , cancel UV poles

$$g_0 = g\mu^{\varepsilon}\left[1 - \frac{\alpha}{4\pi}\frac{S_\varepsilon}{\varepsilon}\left(\frac{11}{6}C_A - \frac{2}{3}T_F n_F\right) + O(\alpha^2)\right]$$

$$g_0 = g\mu^{\varepsilon} Z_g \quad , \quad Z_g = 1 - \frac{\alpha}{4\pi}\frac{S_\varepsilon}{\varepsilon}\left(\frac{11}{6}C_A - \frac{2}{3}n_F T_F\right)$$

Comments:

- it is crucial that one gets the same Z_g for the renormalization of all vertexes. Otherwise DIFFERENT COUPLINGS after renormalization.
  Gauge invariance would be lost
- SLAVNOV TAYLOR identities (BRST)
- Z_g for ggg vertex is the same!

$$\frac{g_0}{g\mu^{\varepsilon}} = \frac{Z_{A\psi\psi}}{Z_A^{1/2}\, Z_\psi} = \frac{Z_{A\eta\eta}}{Z_A^{1/2}\, Z_\eta} = \frac{Z_{AAA}}{Z_A^{3/2}}$$

SINGLE RENORM. FACTOR

β FUNCTION AND ASYMPTOTIC FREEDOM

$$g_0 = Z_g \mu^{\varepsilon} g$$

$$\alpha_0 = Z_g^2 \mu^{2\varepsilon} \alpha$$

"BARE COUPLING"

"RENORMALIZED COUPLING"

α_0 cannot depend on μ

We can expect α to depend on μ
(μ introduced to keep g dimensionless)

dependence MUST compensate

log derivative, $\varepsilon \neq 0$, limit at the end

$$0 = \mu^2 \frac{d\alpha_0}{d\mu^2} = \mu^2 \left\{ 2Z \frac{dZ}{d\mu^2} \mu^{2\varepsilon} \alpha + Z^2 \varepsilon \mu^{2(\varepsilon-1)} \alpha + Z^2 \mu^{2\varepsilon} \frac{d\alpha}{d\mu^2} \right\}$$

$$\frac{dZ}{d\mu^2} = \frac{\partial Z}{\partial \alpha} \frac{d\alpha}{d\mu^2}$$

$$\Rightarrow 0 = \mu^{2\varepsilon}\left\{\left(2Z\frac{\partial Z}{\partial \alpha}\alpha + Z^2\right)\mu^2\frac{d\alpha}{d\mu^2} + Z^2\varepsilon\alpha\right\} \qquad (*)$$

<u>definition</u>:

$$\boxed{\beta = \mu^2\frac{d\alpha}{d\mu^2}}$$

$$\frac{\partial Z}{\partial \alpha} = -\frac{1}{4\pi}\frac{S_\varepsilon}{\varepsilon}\left(\frac{11}{6}C_A - \frac{2}{3}T_F n_F\right) + O(\alpha^2)$$

$$b_0 = \frac{1}{12\pi}\left(11C_A - 2n_F\right)$$

Eq (*) gives

$$\left[Z + 2\alpha\frac{\partial Z}{\partial \alpha}\right]\beta = -\varepsilon\alpha Z$$

$$\left[\left(1 - \frac{\alpha}{2}\frac{S_\varepsilon}{\varepsilon}b_0 + O(\alpha^2)\right) + 2\alpha\left(-\frac{1}{2}\frac{S_\varepsilon}{\varepsilon}b_0 + O(\alpha)\right)\right]\beta = -\alpha\left[\varepsilon - \frac{\alpha}{2}S_\varepsilon b_0 + \dots\right]$$

$$\Rightarrow\left[1 - \frac{3}{2}\alpha\frac{S_\varepsilon}{\varepsilon}b_0 + O(\alpha^2)\right]\beta = -\alpha\varepsilon + \frac{\alpha^2}{2}S_\varepsilon b_0 + O(\alpha^3)$$

Necessarily we have $\beta(\alpha,\varepsilon) = a\varepsilon + b$

$$\begin{cases} a = -\alpha \\ -\frac{3}{2}\alpha S_\varepsilon b_0 a + b = \frac{\alpha^2}{2}S_\varepsilon b_0 \end{cases}$$

$$\beta(\alpha,\varepsilon) = -\alpha\varepsilon - \alpha^2 b_0 S_\varepsilon$$

Now I can take the $\varepsilon \to 0$ limit

$$\beta(\alpha) = \mu^2 \frac{d\alpha}{d\mu^2} = -b_0 \alpha^2 \qquad \left(b_0 = \frac{1}{2\pi}\left(\frac{11}{6} C_A - \frac{2}{3} n_F T_F\right) \right)$$

- α here is the RENORMALIZED COUPLING
- The renormalized coupling is NOT CONSTANT but it's a function of the renormalization scale

~/—

- Solution:

$$-b_0 \frac{d\mu^2}{\mu^2} = \frac{d\alpha}{\alpha^2} \quad \Rightarrow -b_0 \log \frac{\mu^2}{\mu_0^2} = \frac{1}{\alpha(\mu_0)} - \frac{1}{\alpha(\mu)}$$

multiply by $\alpha(\mu_0)$, solve:

$$\alpha(\mu) = \frac{\alpha(\mu_0)}{1 + \alpha(\mu_0)\, b_0 \log \mu^2/\mu_0^2}$$

In QCD: $C_A = 3$, $n_f = 5$ (or 6)

$$\Rightarrow b_0 > 0 \qquad \boxed{\frac{\alpha(\mu)}{\alpha(\mu_0)} < 1 \quad \text{if} \quad \mu > \mu_0}$$

- We have obtained that QCD is ASYMPTOTICALLY FREE

- Another useful way of expressing α:

$$\mu^2 \frac{d\alpha}{d\mu^2} = \beta(\alpha) \Rightarrow \frac{d\mu^2}{\mu^2} = \frac{d\alpha}{\beta(\alpha)} \Rightarrow \log \frac{\mu^2}{\Lambda^2} = \int_{\infty}^{\alpha(\mu)} \frac{d\alpha}{\beta(\alpha)}$$

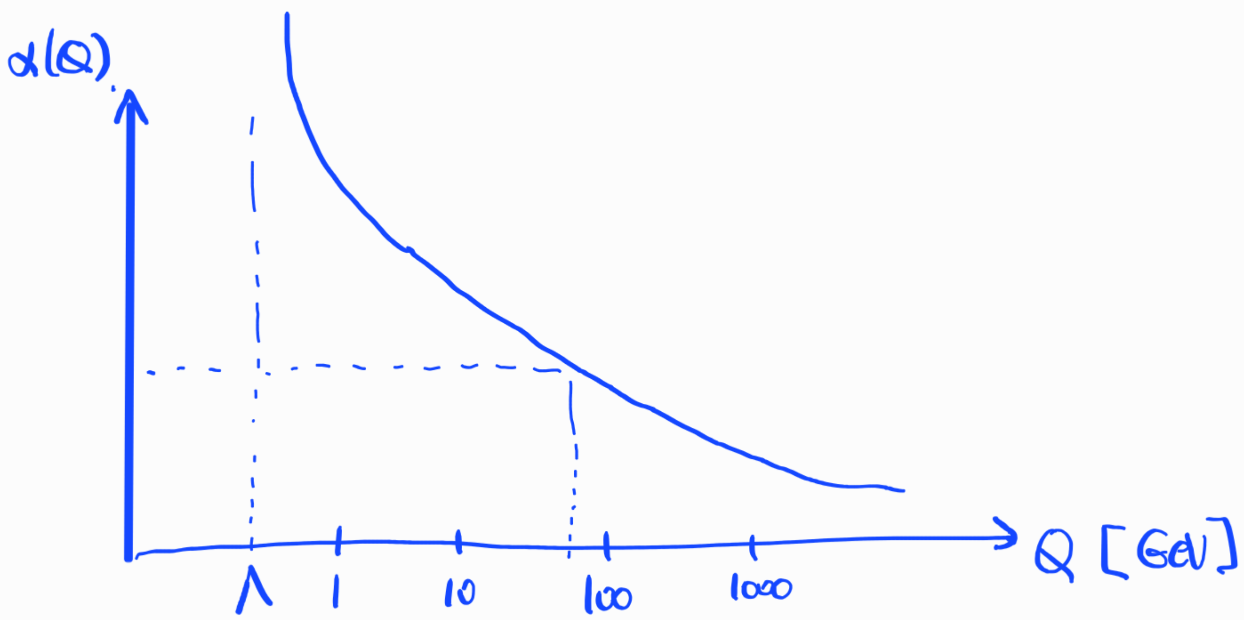
At 1 loop: $\log \frac{\mu^2}{\Lambda^2} = \int_{\infty}^{\alpha} -\frac{d\alpha}{b_0 \alpha^2} \Rightarrow \alpha(\mu) = \frac{1}{b_0 \log \frac{\mu^2}{\Lambda^2}}$

Current status:

$$\beta(\alpha) = -\alpha\left(b_0 \alpha + b_1 \alpha^2 + b_2 \alpha^3 + b_3 \alpha^4 + \cdots\right)$$

Gross-Wilczek Politzer

4-loop ('97)



- measure at one energy, then predictive
- data-theory ✓
- $\Lambda \simeq 200$ MeV ← At small energy, theory is strongly coupled
- $\alpha(M_Z) = 0.118$ ← $\alpha_{EM} \simeq 1/137$