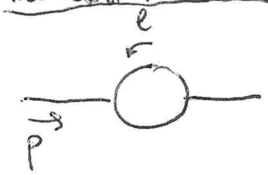
REGULARIZATION HOWTO



$$I_2 = \int \frac{d^4 \ell}{(2\pi)^4} \frac{1}{(\ell^2 - m^2 + i\varepsilon)((\ell+p)^2 - m^2 + i\varepsilon)}$$

$|\ell| \to \infty$ (UV) $\quad I_2 \sim \int d\Omega_3 \int_0^\infty |\ell|^3 \, d|\ell| \, \frac{1}{|\ell|^4} \sim \int_{|\ell_{min}|}^{\Lambda} \frac{|\ell|^3 \, d|\ell|}{|\ell|^4} \sim \log \Lambda$

- cutoff regularization not compatible with Lorentz and gauge invariance (Ward identities violated)
- The method of choice, in PERTURBATIVE QCD, is dimensional regularization
- dim reg: '72 t'Hooft-Veltman / Bollini Giambiagi

DIMENSIONAL REGULARIZATION

- $d = 4 - 2\varepsilon$ space-time dimensions [some authors: $d = 4 - \varepsilon$]
- Lorentz momenta, pol. vector, metric tensor $\to$ d-dim
- γ algebra (Clifford algebra) $\to$ d-dim
- divergences as POLES in $\frac{1}{\varepsilon}$
- it regulates both IR and UV divergences [we'll see that also tree-level amplitudes have to be computed in d-dim]
- Lagrangian is the same, except that
  - $g_s \to g_s \mu^\varepsilon$ $\hookrightarrow$ ARBITRARY MASS SCALE
  - loop measure $\int \frac{d^4 \ell}{(2\pi)^4} \to \int \frac{d^d \ell}{(2\pi)^d}$
- action is dimensionless

$$\Rightarrow \begin{matrix} [A] = \frac{d-2}{2} \\ [\psi] = \frac{d-1}{2} \end{matrix} \Rightarrow d^d x \, \mu^\rho g \bar{\psi} \psi A \Rightarrow \rho = \frac{4-d}{2} = \varepsilon$$

- Dirac algebra identities (DKS, BOHM, PESKIN)

$$\{\gamma^\mu, \gamma^\nu\} = 2 g^{\mu\nu} \quad \text{with } g^\mu{}_\mu = d$$

$$\gamma^\mu \gamma_\mu = d$$

$$\gamma^\lambda \gamma^\mu \gamma_\lambda = (2-d)\gamma^\mu$$

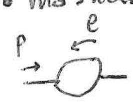
$$\mathrm{Tr}(\mu\nu\rho\sigma) = 4(g_{\mu\nu} g_{\rho\sigma} - g_{\mu\rho} g_{\nu\sigma} + g_{\mu\sigma} g_{\nu\rho}) \qquad \mathrm{Tr}(\mu\nu) = 4 g_{\mu\nu}$$

$$\gamma^\lambda [\gamma^{\mu_1} \dots \gamma^{\mu_{2n+1}}] \gamma_\lambda = -2 [\gamma^{\mu_{2n+1}} \dots \gamma^{\mu_1}] + (4-d) [\gamma^{\mu_1} \dots \gamma^{\mu_{2n+1}}]$$

$$\gamma^\lambda [\gamma^\alpha \gamma^\beta] \gamma_\lambda = 4 g^{\alpha\beta} - (4-d) \gamma^\alpha \gamma^\beta$$

SCALAR INTEGRALS @ 1LOOP

• massless bubble ($p^2 \neq 0$)



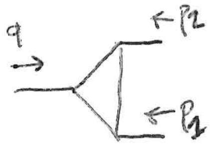
$$B_0(p^2) = \int \frac{d^d\ell}{(2\pi)^d} \frac{1}{(\ell^2+i\eta)((\ell+p)^2+i\eta)} = \frac{i}{(4\pi)^2}\frac{c_\Gamma}{1-2\varepsilon}\frac{1}{\varepsilon}\left(-p^2-i\eta\right)^{-\varepsilon}$$

$$c_\Gamma = (4\pi)^\varepsilon \frac{\Gamma(1+\varepsilon)\Gamma^2(1-\varepsilon)}{\Gamma(1-2\varepsilon)}$$

(can also be rewritten as

$$\frac{i}{(4\pi)^2}\mu^{-2\varepsilon}\frac{c_\Gamma}{1-2\varepsilon}\frac{1}{\varepsilon}\left(-\frac{p^2}{\mu^2}-i\eta\right)^{-\varepsilon} = \frac{i}{(4\pi)^2}\mu^{-2\varepsilon}\frac{c_\Gamma}{1-2\varepsilon}\frac{1}{\varepsilon}\left[1-\varepsilon\log\left(\frac{-p^2}{\mu^2}-i\eta\right)\right]$$

• massless triangle



$$C_0(q^2) = \int \frac{d^d\ell}{(2\pi)^d} \frac{1}{(\ell^2+i\eta)((\ell+q)^2+i\eta)[(\ell+q+p_1)^2+i\eta]} = \frac{i}{(4\pi)^2} c_\Gamma \frac{1}{q^2}\frac{1}{\varepsilon^2}\left(-q^2-i\eta\right)^{-\varepsilon}$$

• scale invariant integrals

$$\int d^dk \, (k^2)^{-\alpha} = 0 \quad (\alpha > 0)$$

- This can be seen as an axiom of dimens. reg.
- It can also be justified as follows: there is no dimensionful quantity (mass/external momenta) that can carry the dimension of the integral

- We can give a more detailed explanation, that will be useful in the following (argument from MUTA 2.5 / COLLINS eq. 3.28)
  - integral is divergent both in the UV and the IR, but the 2 poles exactly cancel

$$I = \frac{1}{(2\pi)^d}\int d^dk \, (-k^2)^{-\alpha} = \frac{i}{(2\pi)^d}\int d^dk_E \, (k_E^2)^{-\alpha} = \frac{i}{(2\pi)^d}\left[\int d^{d-1}\Omega\right]\int_0^\infty d|k_E| \, |k_E|^{d-1-2\alpha}$$

$$= \frac{i}{(2\pi)^d}\left[\frac{2\pi^{d/2}}{\Gamma(d/2)}\right]\int_0^\infty d|k_E| \, |k_E|^{d-1-2\alpha} = \left\{ |k_E|^2 = t \right\} =$$

$$= \frac{i}{(2\pi)^d}\frac{\pi^{d/2}}{\Gamma(d/2)}\int_0^\infty dt \, t^{d/2-1-\alpha}$$

- depending on d and α, the integral diverges if $t \to 0$ AND/OR $t \to \infty$

- split UV and IR

$$\int_0^\infty dt\, t^{d/2-1-\alpha} = \int_0^{\Lambda^2} dt\, t^{d/2-1-\alpha} + \int_{\Lambda^2}^{\infty} dt\, t^{d/2-1-\alpha} = I_{IR} + I_{UV}$$

$$I_{UV} = \frac{t^{d/2-\alpha}}{d/2-\alpha}\Bigg|_{\Lambda^2}^{\infty} = \frac{t^{2-\alpha-\varepsilon}}{2-\alpha-\varepsilon}\Bigg|_{\Lambda^2}^{\infty} = \frac{1}{2-\alpha-\varepsilon}\times(-)\left(\Lambda^2\right)^{2-\alpha-\varepsilon} \quad \leftarrow \begin{matrix} d_{UV} \\ \varepsilon_{UV} \end{matrix}$$

$$I_{IR} = \frac{t^{d/2-\alpha}}{d/2-\alpha}\Bigg|_{0}^{\Lambda^2} = \qquad = \frac{1}{2-\alpha-\varepsilon}\times\left(\Lambda^2\right)^{2-\alpha-\varepsilon} \quad \leftarrow \begin{matrix} d_{IR} \\ \varepsilon_{IR} \end{matrix}$$

- if we identify ε^{UV} and ε^{IR}, $\quad I_{UV} + I_{IR} = 0$

- For $\alpha = 2$

↳ $$I = \frac{1}{(2\pi)^d}\int d^dk\, (k^2)^{-2} = \frac{i}{(2\pi)^d}\frac{\pi^{d/2}}{\Gamma(d/2)}\left[\frac{1}{\varepsilon_{UV}} - \frac{1}{\varepsilon_{IR}}\right]$$

↳ If one is interested only in the $\overbrace{\text{UV}}^{\text{divergent}}$ behavior of I, one puts a cutoff on the IR part

$$\Rightarrow \frac{1}{(2\pi)^d}\int_{\Lambda^2}^{\infty}\frac{d^dk}{(k^2)^2} = \frac{i}{(2\pi)^d}\frac{\pi^{d/2}}{\Gamma(d/2)}\underbrace{\left[\frac{1}{\varepsilon} + \text{finite}\right]}_{\frac{1}{\varepsilon}\left(\Lambda^2\right)^{-\varepsilon} = \frac{1}{\varepsilon}\left[1 + O(\varepsilon)\right]} \qquad \text{(Collins 3.28)}$$

TENSOR INTEGRALS

$$\int \frac{d^d\ell}{(2\pi)^d} \frac{\ell^{\alpha_1}\cdots\ell^{\alpha_k}}{[\ell^2+i\eta][(\ell+p_1)^2+i\eta][(\ell+p_1+p_2)^2+i\eta]\cdots[(\ell+p_1+\cdots+p_{n-1})^2+i\eta]} = \sum_I (\text{Coeff})_I \, T_I^{\alpha_1\ldots\alpha_k}$$

$\{T_I^{\alpha_1\ldots\alpha_k}\}_I$ ← tensor built from metric & external momenta

- Above equation is the Passarino-Veltman reduction
  - in practice not always ideal: in the linear combination, there are Gram determinants → potentially singular in some kinematics regions
  - cumbersome with generic kinematics

For our own purposes, this is not a problem.

Here are some results

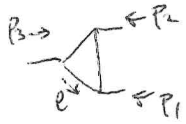
$$B^\mu(p) = \int \frac{d^d\ell}{(2\pi)^d} \frac{\ell^\mu}{\ell^2(\ell+p)^2} = B_{11}\, p^\mu \qquad B_{11} = -\frac{1}{2} B_0(p^2)$$

$$B^{\mu\nu}(p) = \int \frac{d^d\ell}{(2\pi)^d} \frac{\ell^\mu \ell^\nu}{\ell^2(\ell+p)^2} = B_{21}\, p^\mu p^\nu + B_{22}\, g^{\mu\nu}$$

$$B_{21} = \frac{d}{d-1}\frac{B_0(p^2)}{4}$$

$$B_{22} = \frac{-1}{d-1}\frac{p^2}{4} B_0(p^2)$$

$$C^\mu = \int [d^d\ell] \frac{\ell^\mu}{\ell^2(\ell+p_1)^2(\ell+p_1+p_2)^2} \qquad (\text{with } p_1^2 = p_2^2 = 0,\ p_3^2 = (p_1+p_2)^2 \neq 0)$$



$$= C_{11}\, p_1^\mu + C_{12}\, p_2^\mu$$

$$C_{11} = -C_0(p_3^2) - \frac{B_0(p_3^2)}{p_3^2}$$

$$C_{12} = \frac{B_0(p_3^2)}{p_3^2}$$

$$C^{\mu\nu} = \int [d^d\ell] \frac{\ell^\mu \ell^\nu}{\ell^2(\ell+p_1)^2(\ell+p_1+p_2)^2} = C_{21}\, p_1^\mu p_1^\nu + C_{22}\, p_2^\mu p_2^\nu + C_{23}\{p_1 p_2 + p_1 p_2\}^{\mu\nu} + C_{24}\, g^{\mu\nu}$$

$$C_{21} = C_0 + \frac{3}{2}\frac{B_0}{p_3^2} \qquad C_{22} = -\frac{1}{2}\frac{B_0}{p_3^2}$$

$$C_{23} = \frac{-d}{d-2}\frac{B_0}{2p_3^2} \qquad C_{24} = \frac{1}{2(d-2)} B_0$$

RENORMALIZATION

- higher order corrections involve loops, and loops are divergent (UV/IR)
- Remove UV divergences ⇝ important consequences
- QCD (& QED) are renormalizable
  - → define a FINITE NUMBER of "counterterms" that remove ALL UV-divergences of n-point functions
  - → Counterterms subtract UV divergences: need to REGULATE the theory
    - cutoff $\int^{\Lambda}$
    - dim-reg.
    - ⋮
  - → when regulators removed, parameters (and hence physical prediction) acquire a DEPENDENCE on a RENORMALIZATION SCALE
    - ⇒ Such dependence has very deep consequences
- If theory was <u>not</u> renormalizable, no predictivity (∞ number of conditions to make the theory finite)
- RENORMALIZABLE THEORY:
  - One only needs to fix the values of a FINITE NUMBER of parameters from data, then predictive
  - More mathematically: absorb all UV-div into an universal redefinition of a <u>finite number</u> of the BARE PARAMETERS

With dimensional regularization, with $\overline{MS}$ scheme:

- Counterterms ⟶ poles at $\varepsilon \to 0$ (with typical overall factor)

- $g_0 = g\mu^{\varepsilon} Z_g$, $Z_g = 1 - \frac{\alpha}{4\pi}\frac{S_\varepsilon}{\varepsilon}\left(\frac{11}{6}C_A - \frac{2}{3}T_R n_F\right)$

$\Rightarrow$

$$\alpha(\mu_2) = \frac{\alpha(\mu_1)}{1 + \alpha(\mu_1)\, b_0 \log\left(\frac{\mu_2}{\mu_1}\right)^2} \quad , \quad \alpha = \frac{g^2}{4\pi}$$

- α is the renormalized coupling.
  Physical (FINITE!) results are expressed with α
- α depends on the renormalization scale

- μ is the typical scale of the process one is computing ⟶ see notes for an explanation (QED renorm. with a cutoff)