Chapter 1

Mathematical tools

1.1 The Feynman parametrization

The Feynman parametrization is given by the following formula

$$\prod_{i=1}^{n} \frac{1}{A_i^{c_i}} = \frac{\Gamma(c)}{\prod_{i=1}^{n} \Gamma(c_i)} \int_0^1 \prod_{i=1}^{n} \alpha_i^{c_i-1} d\alpha_i \, \frac{\delta\left(1 - \sum_{i=1}^{n} \alpha_i\right)}{\left(\sum_{k=1}^{n} \alpha_k A_k\right)^c} \tag{1.1}$$

where

$$c = \sum_{i=1}^{n} c_i \,. \tag{1.2}$$

The proof of this equation is done following a few steps. First of all, we demonstrate it by induction when all the c_i are equal to 1. The case with $n = 2$ is trivial: by a direct inspection

$$\begin{aligned} I_2 &\equiv \int_0^1 d\alpha_1 \, d\alpha_2 \, \frac{\delta(1 - \alpha_1 - \alpha_2)}{(\alpha_1 A_1 + \alpha_2 A_2)^2} = \int_0^1 d\alpha_1 \frac{1}{(\alpha_1 A_1 + (1-\alpha_1) A_2)^2} \\ &= -\frac{1}{A_1 - A_2} \left[\frac{1}{\alpha_1 A_1 + (1-\alpha_1) A_2} \right]_0^1 = \frac{1}{A_1 A_2} \end{aligned} \tag{1.3}$$

Supposing now that formula (1.1) is valid for $(n-1)$

$$\begin{aligned} I_{n-1} &\equiv \frac{1}{A_1 \dots A_{n-1}} = (n-2)! \int_0^1 \prod_{i=1}^{n-1} d\alpha_i \, \frac{\delta\left(1 - \sum_{i=1}^{n-1} \alpha_i\right)}{\left(\sum_{k=1}^{n-1} \alpha_k A_k\right)^{n-1}} \\ &= \int_0^1 \prod_{i=1}^{n-2} d\alpha_i \, \frac{(n-2)!}{\left(A_{n-1} + \sum_{k=1}^{n-2} \alpha_k \left(A_k - A_{n-1}\right)\right)^{n-1}} \end{aligned} \tag{1.4}$$

where in the last line

$$0 \leq \sum_{k=1}^{n-2} \alpha_k \leq 1 \tag{1.5}$$

we show, with some algebra, that it is true also for n

$$\begin{aligned} I_n \quad &\equiv \quad \frac{1}{A_1 \dots A_n} = (n-1)! \int_0^1 \prod_{i=1}^{n} d\alpha_i \, \frac{\delta(1-\sum_{i=1}^n \alpha_i)}{\left(\sum_{k=1}^n \alpha_k A_k\right)^n} \\ &= \int_0^1 \prod_{i=1}^{n-1} d\alpha_i \, \frac{(n-1)!}{\left(A_n + \sum_{k=1}^{n-1} \alpha_k \left(A_k - A_n\right)\right)^n} \end{aligned} \tag{1.6}$$

with

$$0 \leq \sum_{k=1}^{n-1} \alpha_k \leq 1\,. \tag{1.7}$$

In fact, integrating (1.6) in α_{n-1} between 0 and $\left(1-\sum_{i=1}^{n-2} \alpha_i\right)$, we find

$$\begin{aligned} I_n \quad &= \quad -\frac{(n-2)!}{(A_{n-1}-A_n)} \int_0^1 \prod_{i=1}^{n-2} d\alpha_i \, \frac{1}{\left(A_{n-1} + \sum_{k=1}^{n-2} \alpha_k \left(A_k - A_{n-1}\right)\right)^{n-1}} \\ &\quad +\frac{(n-2)!}{(A_{n-1}-A_n)} \int_0^1 \prod_{i=1}^{n-2} d\alpha_i \, \frac{1}{\left(A_n + \sum_{k=1}^{n-2} \alpha_k \left(A_k - A_n\right)\right)^{n-1}} \end{aligned} \tag{1.8}$$

and using (1.4)

$$I_n \quad = \quad \frac{1}{\left(A_{n-1}-A_n\right) A_1 \dots A_{n-2}} \left[\frac{1}{A_n} - \frac{1}{A_{n-1}}\right] = \frac{1}{A_1 \dots A_n} \tag{1.9}$$

so equation (1.6) is indeed an identity.

To complete the demonstration of eq. (1.1), we derive c_n times both members of the first line of (1.6) with respect to A_n. On the left-hand side we have

$$\frac{\partial^{c_n}}{\partial A_n^{c_n}} \left(\frac{1}{A_1 \dots A_n}\right) = \frac{(-1)^{c_n}\,(c_n)!}{A_1 \dots A_{n-1} A_n^{c_n+1}}\,, \tag{1.10}$$

while on the right-hand side, the derivation gives

$$\frac{\partial^{c_n}}{\partial A_n^{c_n}} I_n = \int_0^1 \prod_{i=1}^{n} d\alpha_i \, \delta\left(1 - \sum_{i=1}^{n} \alpha_i\right) \frac{(-1)^{c_n}\,(n+c_n-1)!\,\alpha_n^{c_n}}{\left(\sum_{k=1}^n \alpha_k A_k\right)^{n+c_n}} \tag{1.11}$$

Comparing the last two equations one can see that

$$\frac{1}{A_1 \dots A_{n-1} A_n^{c_n}} = \frac{(n+c_n-2)!}{(c_n-1)!} \int_0^1 \prod_{i=1}^{n} d\alpha_i \, \delta\left(1 - \sum_{i=1}^{n} \alpha_i\right) \frac{\alpha_n^{c_n-1}}{\left(\sum_{k=1}^n \alpha_k A_k\right)^{n+c_n-1}}\,. \tag{1.12}$$

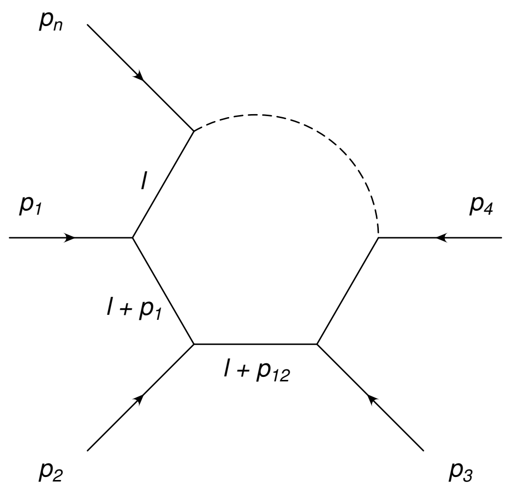
Repeating now the derivation with respect to a generic A_k, we get

$$\prod_{i=1}^{n}\frac{1}{A_i^{c_i}} = \frac{(c-1)!}{\prod_{i=1}^{n}(c_i-1)!}\int_0^1\prod_{i=1}^{n}\alpha_i^{c_i-1}d\alpha_i\;\frac{\delta\left(1-\sum_{i=1}^{n}\alpha_i\right)}{\left(\sum_{k=1}^{n}\alpha_k A_k\right)^c}$$
$$= \frac{\Gamma(c)}{\prod_{i=1}^{n}\Gamma(c_i)}\int_0^1\prod_{i=1}^{n}\alpha_i^{c_i-1}d\alpha_i\;\frac{\delta\left(1-\sum_{i=1}^{n}\alpha_i\right)}{\left(\sum_{k=1}^{n}\alpha_k A_k\right)^c}\,. \tag{1.13}$$

This proves eq. (1.1).

1.2 The scalar one-loop integrals

In this section we want to introduce all the principal mathematical tools useful to calculate d-dimensional scalar one-loop Feynman integrals. These integrals are built up with the propagators of n massive particles, with masses m_i, connecting $n+1$ vertexes of interaction with other external particles, each carrying momentum p_i.[1]



The integral can be written in this general form (notice that $\sum_{i=1}^{n}p_i=0$ for momentum conservation)

$$I=\int\frac{d^d\ell}{(2\pi)^d}\frac{1}{\left[(\ell+p_1)^2-m_1^2+i\eta\right]\left[(\ell+p_{12})^2-m_2^2+i\eta\right]\ldots\left[(\ell+p_{12\ldots n})^2-m_n^2+i\eta\right]}\,, \tag{1.14}$$

[1] All momenta incoming.

where we have introduced a small imaginary part $i\eta$ according to the Feynman prescription for the T–ordered propagator and we have used the shortcut $p_{12} = p_1 + p_2$, and similar ones.

Using the Feynman parametrization (1.1) we can write

$$I = \Gamma(n) \int_0^1 \prod_{i=1}^n d\alpha_i \int \frac{d^d\ell}{(2\pi)^d} \frac{\delta(1 - \sum_{i=1}^n \alpha_i)}{\left(\sum_{k=1}^n \alpha_k A_k\right)^n} \tag{1.15}$$

The sum in the denominator can then be rewritten as

$$\begin{aligned} \sum_{k=1}^n \alpha_k A_k &= \sum_{k=1}^n \alpha_k \left[(\ell + p_{1\ldots k})^2 - m_k^2 + i\eta\right] = \\ &= \ell^2 + 2\ell \cdot \left(\sum_{k=1}^n \alpha_k p_{1\ldots k}\right) + \sum_{k=1}^n \alpha_k \left(p_{1\ldots k}^2 - m_k^2 + i\eta\right) = \\ &\equiv \ell^2 + 2\ell \cdot P + K^2 + i\eta\,. \end{aligned} \tag{1.16}$$

The integral (1.15) becomes

$$\begin{aligned} I &= \Gamma(n) \int_0^1 \prod_{i=1}^n d\alpha_i\, \delta\left(1 - \sum_{i=1}^n \alpha_i\right) \int \frac{d^d\ell}{(2\pi)^d} \frac{1}{(\ell^2 + 2\ell \cdot P + K^2 + i\eta)^n} \qquad (\ell \to \ell + P) \\ &= \Gamma(n) \int_0^1 [d\alpha]_n \int \frac{d^d\ell}{(2\pi)^d} \frac{1}{(\ell^2 - m^2 + i\eta)^n} \end{aligned} \tag{1.17}$$

where we used the shorthand notation

$$[d\alpha]_n \equiv \prod_{i=1}^n d\alpha_i\, \delta\left(1 - \sum_{i=1}^n \alpha_i\right), \qquad m^2 \equiv P^2 - K^2 \tag{1.18}$$

Notice that in the last line η is not the same one defined previously but it plays the same role again picking the poles away from the path of the integration as the Feynman prescription requires.

The integral over the loop momentum l can be performed once and for all. We first perform the integral over l_0. In Fig. 1.1 we have promoted the real variable l_0 into a complex variable and we have plotted the two poles

$$\ell^2 - m^2 + i\eta \equiv \ell_0^2 - |\ell|^2 - m^2 + i\eta = 0 \qquad \Longrightarrow \qquad l_0 = \pm\sqrt{|\ell|^2 + m^2} \mp i\eta\,. \tag{1.19}$$

The integration over l_0 is along the real axis. Exploiting the fact that the Feynman integrals are analytic functions, we interpret the integration along the real axis as part of the integration over the closed path in the figure. Using the residue theorem, we know that the integral along that closed path is zero, since the poles of the integral are outside the integration path. So we can write

$$0 = \int_{-\infty}^{+\infty} d\ell_0 \ldots + \int_{+\infty}^{-\infty} i d\ell_0^E \ldots \qquad \Longrightarrow \qquad \int_{-\infty}^{+\infty} d\ell_0 \ldots = i \int_{-\infty}^{+\infty} d\ell_0^E \ldots \tag{1.20}$$

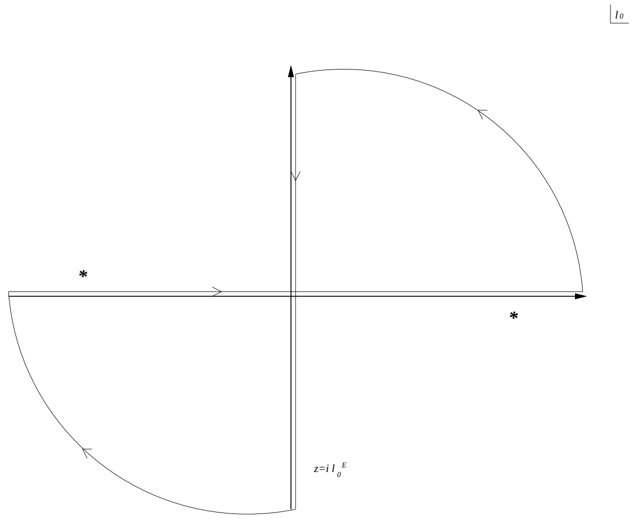


Figure 1.1: Wick rotation.

since the contribution from the circular parts of the path goes to zero as we radius goes to infinity. We have indicated with ℓ_0^E the new integration variable, reminiscent of the fact that now we are using an Euclidean notation and no longer a Minkoskian one. The integral I then becomes

$$I = i\Gamma(n) \int_0^1 [d\alpha]_n \int_{-\infty}^{+\infty} \frac{d\ell_0^E \; d^{(d-1)}\ell}{(2\pi)^d} \frac{1}{\left(-\left(\ell_0^E\right)^2 - |\ell|^2 - m^2 + i\eta\right)^n} \tag{1.21}$$

and using spherical coordinates, defining $\ell_E^2 \equiv (\ell_0^E)^2 + |\ell|^2$ (please notice that the integral over the loop momentum is now perfectly defined and we could set $\eta = 0$. We keep it, since it will be useful in the integration over the Feynman parameters α_i, yet to be done)

$$\begin{aligned}
I &= \frac{(-1)^n i\Gamma(n)}{(2\pi)^d} \int_0^1 [d\alpha]_n \int d^d\Omega \; d\ell_E \frac{(\ell_E)^{d-1}}{(\ell_E^2 + m^2 - i\eta)^n} \\
&\qquad \Longrightarrow \; t = \frac{\ell_E^2}{m^2} \\
&= \frac{(-1)^n i\Gamma(n)\Omega_d}{2(2\pi)^d} \int_0^1 [d\alpha]_n \left(m^2 - i\eta\right)^{\frac{d}{2}-n} \int_0^\infty dt \; t^{\frac{d}{2}-1} (t+1)^{-n} \\
&\qquad \Longrightarrow \; x = \frac{1}{1+t} \\
&= \frac{(-1)^n i\Gamma(n)\Omega_d}{2(2\pi)^d} \int_0^1 [d\alpha]_n \left(m^2 - i\eta\right)^{\frac{d}{2}-n} \int_0^1 dx \; x^{n-\frac{d}{2}-1} (1-x)^{\frac{d}{2}-1} \\
&= \frac{(-1)^n i\Gamma(n)\Omega_d}{2(2\pi)^d} \beta\left(\frac{d}{2}, n - \frac{d}{2}\right) \int_0^1 [d\alpha]_n \; \left(m^2 - i\eta\right)^{\frac{d}{2}-n} \\
&= \frac{(-1)^n i\Gamma\left(n - \frac{d}{2}\right)\Gamma\left(\frac{d}{2}\right)\Omega_d}{2(2\pi)^d} \int_0^1 [d\alpha]_n \; \left(m^2 - i\eta\right)^{\frac{d}{2}-n}
\end{aligned} \tag{1.22}$$

where Ω_d is the total angle in d dimensions

$$\Omega_d = \frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)} \,. \tag{1.23}$$

Finally, the scalar integral (1.15) takes this form

$$\begin{aligned} I &= (-1)^n \frac{i}{(4\pi)^{\frac{d}{2}}} \Gamma\left(n - \frac{d}{2}\right) \int_0^1 [d\alpha]_n \ \left(m^2 - i\eta\right)^{\frac{d}{2}-n} \\ &= (-1)^n \frac{i}{(4\pi)^{\frac{d}{2}}} \Gamma\left(n - \frac{d}{2}\right) \int_0^1 [d\alpha]_n \ \left(-\sum_{i>j}^{n} \alpha_i \alpha_j \, p^2_{j+1\ldots i} + \sum_{i=1}^{n} \alpha_i m_i^2 - i\eta\right)^{\frac{d}{2}-n} \end{aligned} \tag{1.24}$$

where in the last line we used

$$\begin{aligned} m^2 &= P^2 - K^2 = \left(\sum_{i=1}^{n} \alpha_i p_{1\ldots i}\right)^2 - \sum_{i=1}^{n} \alpha_i \left(p^2_{1\ldots i} - m_i^2 + i\eta\right) \\ &= \sum_{i=1}^{n} \alpha_i^2 p^2_{1\ldots i} + 2 \sum_{i>j}^{n} \alpha_i \alpha_j p_{1\ldots i} p_{1\ldots j} - \sum_{i=1}^{n} \alpha_i p^2_{1\ldots i} + \sum_{i=1}^{n} \alpha_i m_i^2 - i\eta \\ &= -\sum_{i=1}^{n} \alpha_i \sum_{j\neq i} \alpha_j p^2_{1\ldots i} + 2 \sum_{i>j}^{n} \alpha_i \alpha_j p_{1\ldots i} p_{1\ldots j} + \sum_{i=1}^{n} \alpha_i m_i^2 - i\eta \\ &= -\sum_{i>j}^{n} \alpha_i \alpha_j p^2_{1\ldots i} - \sum_{i>j}^{n} \alpha_i \alpha_j p_{1\ldots i} p_{1\ldots j} \\ &\quad - \sum_{j>i}^{n} \alpha_i \alpha_j p^2_{1\ldots i} - \sum_{j>i}^{n} \alpha_j \alpha_i p_{1\ldots j} p_{1\ldots i} + \sum_{i=1}^{n} \alpha_i m_i^2 - i\eta \\ &= -\sum_{i>j}^{n} \alpha_i \alpha_j p_{1\ldots i} p_{j+1\ldots i} + \sum_{j>i}^{n} \alpha_i \alpha_j p_{1\ldots i} p_{i+1\ldots j} + \sum_{i=1}^{n} \alpha_i m_i^2 - i\eta \\ &= -\sum_{i>j}^{n} \alpha_i \alpha_j p^2_{j+1\ldots i} + \sum_{i=1}^{n} \alpha_i m_i^2 - i\eta \,. \end{aligned} \tag{1.25}$$

In summary

$$\begin{aligned} I &= \int \frac{d^d\ell}{(2\pi)^d} \frac{1}{\left[(\ell+p_1)^2 - m_1^2 + i\eta\right]\left[(\ell+p_{12})^2 - m_2^2 + i\eta\right] \ldots \left[(\ell+p_{12\ldots n})^2 - m_n^2 + i\eta\right]} \\ &= (-1)^n \frac{i}{(4\pi)^{\left(\frac{d}{2}\right)}} \Gamma\left(n - \frac{d}{2}\right) \int_0^1 \frac{[d\alpha]_n}{D^{n-\frac{d}{2}}} \,, \end{aligned} \tag{1.26}$$

where

$$D = -\sum_{i>j} \alpha_i \, \alpha_j \, s_{ij} + \sum_{i=1}^{n} \alpha_i \, m_i^2 - i\eta \,, \tag{1.27}$$

and s_{ij} is the square of the momentum flowing through the i-j cut of the diagram representing I.

1.2.1 The one-point function (tadpole)

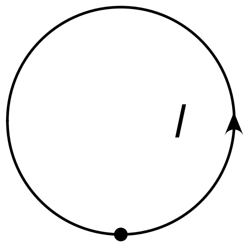


Figure 1.2: One-point function (tadpole).

The one-point function is given by

$$\begin{aligned} A_0\left(m^2\right) &= \int \frac{d^d\ell}{(2\pi)^d} \frac{1}{\ell^2 - m^2 + i\eta} = \frac{-i\Gamma\left(\frac{2-d}{2}\right)}{(4\pi)^{\frac{d}{2}}} \int_0^1 d\alpha \;\, \delta\left(1-\alpha\right)\left(\alpha m^2 - i\eta\right)^{\frac{d-2}{2}} \\ &= \frac{-i\Gamma\left(\frac{2-d}{2}\right)}{(4\pi)^{\frac{d}{2}}} \left(m^2 - i\eta\right)^{\frac{d-2}{2}} \end{aligned} \tag{1.28}$$

where m is the mass of the particle propagating in the loop. Please notice that

$$m = 0 \qquad \Longrightarrow \qquad A_0 = 0 \tag{1.29}$$

since if the mass is zero, there are not dimensional variables that carry the dimension of A_0 after the integration over the loop momentum. So the integral must be zero.

If $m \neq 0$, with the usual definition $d = 4 - 2\epsilon$, we have

$$A_0\left(m^2\right) = \frac{-i\Gamma\left(\epsilon - 1\right)}{(4\pi)^{2-\epsilon}} \left(m^2 - i\eta\right)^{1-\epsilon} = \frac{i}{(4\pi)^2} \frac{(4\pi)^{\epsilon}\,\Gamma\left(1+\epsilon\right)}{\epsilon\left(1-\epsilon\right)} \left(m^2 - i\eta\right)^{1-\epsilon}\,, \tag{1.30}$$

that shows that A_0 diverges as $1/\epsilon$ when $\epsilon \to 0$.

1.2.2 The two-point function (bubble) with $m_1 = m_2 = 0$

We now consider the integral corresponding to the two-point function with massless propagators, i.e. $m_1 = m_2 = 0$. The external momentum p must then have $p^2 \neq 0$ otherwise, as for A_0 with $m^2 = 0$, if also the external particles are massless, the integral vanishes. The integral is given by

$$\begin{aligned} B_0\left(p^2\right) &= \int \frac{d^d\ell}{(2\pi)^d} \frac{1}{\left[\ell^2 + i\eta\right]\left[\left(\ell+p\right)^2 + i\eta\right]} = \frac{i\Gamma\left(\frac{4-d}{2}\right)}{(4\pi)^{\frac{d}{2}}} \int_0^1 \left[d\alpha\right]_2 \frac{1}{\left(-\alpha_1\alpha_2\, p^2 - i\eta\right)^{\frac{4-d}{2}}} \\ &= \frac{i\Gamma\left(\frac{4-d}{2}\right)}{(4\pi)^{\frac{d}{2}}} \int_0^1 d\alpha_1 \, \left(\alpha_1\left(1-\alpha_1\right)\left(-p^2 - i\eta\right)\right)^{\frac{d-4}{2}} = \frac{i\Gamma\left(\frac{4-d}{2}\right)}{(4\pi)^{\frac{d}{2}}} \left(-p^2 - i\eta\right)^{\frac{d-4}{2}} \frac{\Gamma^2\left(\frac{d-2}{2}\right)}{\Gamma\left(d-2\right)} \end{aligned} \tag{1.31}$$

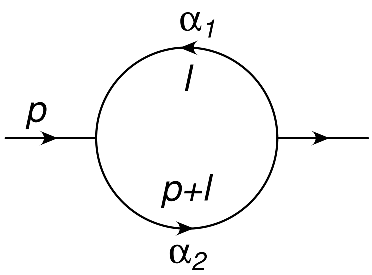


Figure 1.3: Two-point function (bubble).

With $d = 4 - 2\epsilon$ we have

$$B_0(p^2) = \frac{i}{(4\pi)^{2-\epsilon}} \frac{\Gamma(\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(2-2\epsilon)} \left(-p^2 - i\eta\right)^{-\epsilon} = \frac{i}{(4\pi)^2} \frac{C_\Gamma}{\epsilon(1-2\epsilon)} \left(-p^2 - i\eta\right)^{-\epsilon} \quad (1.32)$$

where we have defined

$$C_\Gamma = (4\pi)^\epsilon \frac{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \quad (1.33)$$

Since we are interested in an expansion in ϵ of B_0, we have to deal with

$$\left(-p^2 - i\eta\right)^{-\epsilon} = 1 - \epsilon \log\left(-p^2 - i\eta\right) + \mathcal{O}\left(\epsilon^2\right) \quad (1.34)$$

If $p^2 < 0$, then the logarithm is perfectly defined and no imaginary part is needed to give meaning to it. If instead $p^2 > 0$, then $-p^2 - i\eta$ is a complex negative number with a small imaginary part, so that it is below the typical cut for the definition of the logarithm. In this case, we have

$$\left(-p^2 - i\eta\right)^{-\epsilon} = 1 - \epsilon \log\left(-p^2 - i\eta\right) + \mathcal{O}\left(\epsilon^2\right) = 1 - \epsilon\left[\log\left(p^2\right) - i\pi\right] + \mathcal{O}\left(\epsilon^2\right) \quad (1.35)$$

In the kinematic region $p^2 > 0$ we then have

$$B_0(p^2) = \frac{i}{(4\pi)^2} \frac{C_\Gamma}{(1-2\epsilon)} \left[\frac{1}{\epsilon} - \log(p^2) + i\pi + \mathcal{O}(\epsilon)\right] \quad (1.36)$$

This integral is divergent as $1/\epsilon$ in the limit $\epsilon \to 0$.

1.2.3 The two-point function (bubble) with $m_1 = m$, $m_2 = 0$

Left as exercise.

1.2.4 The two-point function (bubble) with $m_1 = m_2 = m$

Left as exercise.

Check that you get

$$B_0\left(p^2,m^2,m^2\right) \equiv \int \frac{d^d l}{(2\pi)^d}\,\frac{1}{l^2-m^2}\,\frac{1}{(l+p)^2-m^2} \tag{1.37}$$

in the kinematic region $p^2 \geq 4m^2$

$$B_0\left(p^2,m^2,m^2\right) = \frac{i}{(4\pi)^2} C_\Gamma \left(m^2\right)^{-\epsilon} \left\{\frac{1}{\epsilon} + 2 + (x_+ - x_-)\log\frac{x_-}{x_+} + i\,\pi\,(x_+ - x_-) + \mathcal{O}(\epsilon)\right\} \tag{1.38}$$

where

$$x_\pm = \frac{1}{2}\left(1 \pm \sqrt{1-\frac{4m^2}{p^2}}\right) \pm i\eta\,. \tag{1.39}$$

1.2.5 The three-point function (triangle) with $m_1 = m_2 = m_3 = 0$

We consider the simplified case where three-point function has all the propagators massless.

Triangle with one external massive leg

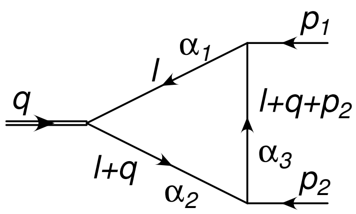


Figure 1.4: The three-point function (triangle). The double line denote the massive leg.

In the notation of Fig. 1.4, we have $q^2 \neq 0$, $p_1^2 = p_2^2 = 0$. We have only one independent invariant, i.e. q^2. Any other relativistic invariant can be written in terms of q^2. The Feynman diagram corresponding to Fig. 1.4 is given by

$$\begin{aligned} C_0\big(q^2\big) &= \int\frac{d^d\ell}{(2\pi)^d}\,\frac{1}{\left[(\ell)^2+i\eta\right]\left[(\ell+q)^2+i\eta\right]\left[(\ell+q+p_2)^2+i\eta\right]} = \\ &= \frac{-i\Gamma\left(\frac{6-d}{2}\right)}{(4\pi)^{\frac{d}{2}}}\int_0^1 \frac{[d\alpha]_3}{\left(-\alpha_1\alpha_2\,q^2 - i\eta\right)^{\frac{6-d}{2}}}\,, \end{aligned} \tag{1.40}$$

where we used $p_i^2 = 0$ for $i = 1, 2$. We can integrate over α_3 immediately, using the δ function. This gives $\alpha_3 = 1-\alpha_1-\alpha_2$. Since the range of integration of α_3 is from 0 to 1, this means that $0 \leq 1-\alpha_1-\alpha_2 \leq 1$, that implies that $\alpha_2 \leq 1-\alpha_1$. Performing now the integration on the Feynman parameters, we have

$$
\begin{aligned}
C_0\left(q^2\right) &= \frac{-i\Gamma\left(\frac{6-d}{2}\right)}{(4\pi)^{\frac{d}{2}}} \int_0^1 d\alpha_1 \int_0^{1-\alpha_1} d\alpha_2 \frac{1}{\left(-\alpha_1\alpha_2\, q^2 - i\eta\right)^{\frac{6-d}{2}}} \\
&= \frac{-i\Gamma\left(1+\epsilon\right)}{(4\pi)^{2-\epsilon}} \left(-q^2 - i\eta\right)^{-(1+\epsilon)} \int_0^1 d\alpha_1 \int_0^{1-\alpha_1} d\alpha_2\ \left(\alpha_1\alpha_2\right)^{-(1+\epsilon)} \\
&= \frac{i\Gamma\left(1+\epsilon\right)}{(4\pi)^{2-\epsilon}\,\epsilon} \left(-q^2 - i\eta\right)^{-(1+\epsilon)} \int_0^1 d\alpha_1 \left(\alpha_1\right)^{-(1+\epsilon)} \left[\alpha_2^{-\epsilon}\right]_0^{1-\alpha_1} = \\
&= \frac{i\Gamma\left(1+\epsilon\right)}{(4\pi)^{2-\epsilon}\,\epsilon} \left(-q^2 - i\eta\right)^{-(1+\epsilon)} \int_0^1 d\alpha_1 \left(\alpha_1\right)^{-(1+\epsilon)} \left(1-\alpha_1\right)^{-\epsilon} = \\
&= \frac{i\Gamma\left(1+\epsilon\right)}{(4\pi)^{2-\epsilon}\,\epsilon} \left(-q^2 - i\eta\right)^{-(1+\epsilon)} \frac{\Gamma\left(-\epsilon\right)\Gamma\left(1-\epsilon\right)}{\Gamma\left(1-2\epsilon\right)} \qquad (1.41)
\end{aligned}
$$

While this integral can be performed as done before, it is important to keep in mind also the following trick to restore the integration boundaries between 0 and 1. We make the change of variable $\alpha_2 = (1-\alpha_1)x$

$$
\begin{aligned}
\int_0^1 d\alpha_1\, \alpha_1^{-(1+\epsilon)} \int_0^{1-\alpha_1} d\alpha_2\, \alpha_2^{-(1+\epsilon)} &= \int_0^1 d\alpha_1\, \alpha_1^{-(1+\epsilon)} \int_0^1 dx\, (1-\alpha_1)(1-\alpha_1)^{-(1+\epsilon)} x^{-(1+\epsilon)} \\
&= \int_0^1 d\alpha_1\, \alpha_1^{-(1+\epsilon)} \left(1-\alpha_1\right)^{-\epsilon} \int_0^1 dx\, x^{-(1+\epsilon)} \\
&= B(-\epsilon, 1-\epsilon) B(-\epsilon, 1) = \frac{\Gamma(-\epsilon)\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} \frac{\Gamma(-\epsilon)\Gamma(1)}{\Gamma(1-\epsilon)} \\
&= \frac{1}{\epsilon^2} \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \qquad (1.42)
\end{aligned}
$$

where we have used the definition of the B function in eq. (A.9). We finally get

$$
C_0\left(q^2\right) = \frac{i}{(4\pi)^2} \frac{C_\Gamma}{q^2} \left(-q^2 - i\eta\right)^{-\epsilon} \frac{1}{\epsilon^2} \qquad (1.43)
$$

where C_Γ is given in equation (1.33). We refer to Sec. 1.2.2 for the expansion of the previous expression in the kinematic regions where $q^2 < 0$ or $q^2 > 0$.

Triangle with two external massive legs

Consider now the triangle with two massive external legs. In the notation of Fig. 1.5, we have $p^2 = 0$, $p_1^2 \neq 0$ and $p_2^2 \neq 0$. We compute this integral with the further hypothesis that $q_2^2 > 0$. The sign of q_1^2 is arbitrary. The integral corresponding to this Feynman graph is

$$
\begin{aligned}
C_0\left(q_1^2, q_2^2\right) &= \int \frac{d^d\ell}{(2\pi)^d} \frac{1}{\left[(\ell)^2 + i\eta\right]\left[(\ell+p)^2 + i\eta\right]\left[(\ell+p+q_2)^2 + i\eta\right]} = \\
&= \frac{-i\Gamma\left(\frac{6-d}{2}\right)}{(4\pi)^{\frac{d}{2}}} \int_0^1 \frac{[d\alpha]_3}{\left(-\alpha_1\alpha_3 q_1^2 - \alpha_2\alpha_3 q_2^2 - i\eta\right)^{\frac{6-d}{2}}}\,. \qquad (1.44)
\end{aligned}
$$

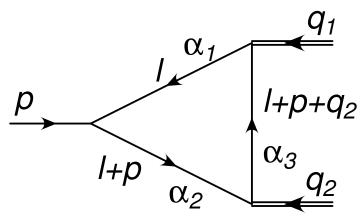


Figure 1.5: The three-point function (triangle). The double line denote the massive leg.

Factorizing out $-q_2^2$ with the right $i\eta$ prescription and using $d = 4 - 2\epsilon$, and defining

$$r = \frac{q_1^2}{q_2^2} + i\eta \tag{1.45}$$

we have

$$\begin{aligned} C_0\left(q_1^2, q_2^2\right) &= \frac{-i\Gamma\left(1+\epsilon\right)}{\left(4\pi\right)^{2-\epsilon}} \frac{1}{\left(-1-i\eta\right)^{1+\epsilon}\left(q_2^2\right)^{1+\epsilon}} \int_0^1 \frac{[d\alpha]_3}{\alpha_3^{1+\epsilon}\left(\alpha_1\, r+\alpha_2\right)^{1+\epsilon}} = \\ &= \frac{-i\Gamma\left(1+\epsilon\right)}{\left(4\pi\right)^{2-\epsilon}} \frac{e^{i\pi\epsilon}}{\left(q_2^2\right)^{1+\epsilon}} \int_0^1 d\alpha_3 \int_0^{1-\alpha_3} d\alpha_1 \frac{1}{\alpha_3^{1+\epsilon}\left[\alpha_1\left(r-1\right)+1-\alpha_3\right]^{1+\epsilon}} \end{aligned}$$

Integrating first over α_1 we have

$$\begin{aligned} C_0\left(q_1^2, q_2^2\right) &= \frac{ie^{i\pi\epsilon}\Gamma\left(1+\epsilon\right)}{\left(4\pi\right)^{2-\epsilon}\left(q_2^2\right)^{1+\epsilon}} \frac{1}{\epsilon\left(r-1\right)} \int_0^1 \frac{d\alpha_3}{\alpha_3^{1+\epsilon}} \left|\left[\alpha_1\left(r-1\right)+1-\alpha_3\right]^{-\epsilon}\right|_0^{1-\alpha_3} \\ &= \frac{ie^{i\pi\epsilon}\Gamma\left(1+\epsilon\right)}{\left(4\pi\right)^{2-\epsilon}\epsilon\left(q_2^2\right)^{1+\epsilon}\left(r-1\right)} \frac{\left(1-r^\epsilon\right)}{r^\epsilon} \int_0^1 d\alpha_3\, \alpha_3^{-(1+\epsilon)}\left(1-\alpha_3\right)^{-\epsilon} \\ &= \frac{-ie^{i\pi\epsilon}\Gamma\left(1+\epsilon\right)}{\left(4\pi\right)^2} \frac{C_\Gamma}{\epsilon^2} \frac{1}{\left(q_2^2\right)^{1+\epsilon}} \frac{\left(1-r^\epsilon\right)}{\left(r-1\right)r^\epsilon} \end{aligned} \tag{1.46}$$

By making a (partial) Laurent expansion in ϵ we have

$$C_0\left(q_1^2, q_2^2\right) = \frac{ie^{i\pi\epsilon}\Gamma\left(1+\epsilon\right)}{\left(4\pi\right)^2} \frac{C_\Gamma}{\epsilon} \frac{\log\left(\frac{q_1^2}{q_2^2}+i\eta\right)}{\left(q_1^2-q_2^2\right)\left(q_1^2+i\eta\right)^\epsilon} \tag{1.47}$$

1.2.6 The four-point function (box) with $m_i = 0$

Box with

$$p_1 + p_2 = p_3 + p_4, \qquad p_i^2 = 0, \qquad s = (p_1+p_2)^2 > 0, \qquad t = (p_1-p_3)^2 < 0 \tag{1.48}$$

Left as exercise.

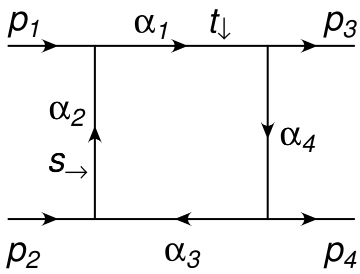


Figure 1.6: Four-point function.

1.3 The tensor one-loop integrals and the Passarino–Veltman reduction formula

We are ready to look at more complicated numerator structures. As previously stated, in QCD (and more in general in the Standard Model), this happens when we have one (or more) fermion legs in the loop or in the presence of triple and quartic gluon vertexes. In the following, we will deal only with massless propagators, to simplify the calculations and the notation. No conceptual problems arise in case of massive propagators.

For example, a massless fermionic n-point loop function is given by

$$\begin{aligned} I_n(\{p_i\}) &= \int \frac{d^d\ell}{(2\pi)^d}\, \not{\ell}(\not{\ell}+\not{p}_1)(\not{\ell}+\not{p}_1+\not{p}_2)\ldots(\not{\ell}+\not{p}_1+\ldots+\not{p}_{n-1}) \\ &\times \frac{1}{[\ell^2+i\eta]\,[(\ell+p_1)^2+i\eta]\,[(\ell+p_1+p_2)^2+i\eta]\ldots[(\ell+p_1+\ldots+p_{n-1})^2+i\eta]} \end{aligned}$$

The gamma matrix structure can be extracted from this integral and we can write $I_n(\{p_i\})$ as

$$\begin{aligned} I_n(\{p_i\}) &= \gamma_{\mu_n}\gamma_{\mu_1}\gamma_{\mu_2}\ldots\gamma_{\mu_{n-1}} \int \frac{d^d\ell}{(2\pi)^d}\, \ell^{\mu_n}(\ell+p_1)^{\mu_1}(\ell+p_{12})^{\mu_2}\ldots(\ell+p_{1\ldots n-1})^{\mu_{n-1}} \\ &\times \frac{1}{[\ell^2+i\eta]\,[(\ell+p_1)^2+i\eta]\,[(\ell+p_{12})^2+i\eta]\ldots[(\ell+p_{1\ldots n-1})^2+i\eta]} \end{aligned}$$

The Feynman integral with tensor components of the loop momentum in the numerator is called **tensor integral**.

$$\begin{aligned} I_n^{\mu_1\mu_2\ldots\mu_k}(\{p_i\}) &\equiv \int \frac{d^d\ell}{(2\pi)^d}\, \ell^{\mu_1}\ell^{\mu_2}\ldots\ell^{\mu_k} \\ &\times \frac{1}{[\ell^2+i\eta]\,[(\ell+p_1)^2+i\eta]\,[(\ell+p_{12})^2+i\eta]\ldots[(\ell+p_{1\ldots n-1})^2+i\eta]} \end{aligned} \tag{1.49}$$

The purpose of this section is to show how to compute this integral. We notice first that all the Lorentz structure of a tensor integral has to be carried by the external momenta $\{p_i\}$ or by the $g^{\mu\nu}$ tensor. The first step is the to write the more general linear combination of tensors of order k constructed with the components of the n external momenta and of the $g^{\mu\nu}$ tensor. The symmetry under permutation of Lorentz indices reduces the allowed tensor structure. In fact, $I_n^{\mu_1\mu_2\cdots\mu_k}(\{p_i\})$ must be totally symmetric with respect to the k indices $(\mu_1, \mu_2, ..., \mu_k)$.

The procedure to compute the tensor integrals has been outlined for the first time by Passarino and Veltman (PV).

We illustrate this procedure with a few examples.

1.3.1 The tensor two-point function $\mathcal{B}^\mu$ $(p^2 \neq 0)$

We start computing $\mathcal{B}^\mu(p)$. Here the tensor decomposition is trivial because only p can bring the index μ of the integral. In order for the integral to be different from zero, we must have $p^2 \neq 0$. We have to compute

$$\mathcal{B}^\mu(p) \equiv \int \frac{d^d\ell}{(2\pi)^d} \frac{\ell^\mu}{\ell^2(\ell+p)^2} = B_{11}\, p^\mu\,. \tag{1.50}$$

In order to compute the coefficient B_{11}, we contract both side of the previous equation with p_μ and use

$$\ell\cdot p = \frac{1}{2}[(\ell+p)^2 - \ell^2 - p^2]\,. \tag{1.51}$$

We have

$$\begin{aligned} p^2\, B_{11} &= \frac{1}{2}\int \frac{d^d\ell}{(2\pi)^d}\left[\frac{1}{\ell^2} - \frac{1}{(\ell+p)^2} - \frac{p^2}{\ell^2(\ell+p)^2}\right] \\ &= -\frac{p^2}{2} B_0(p^2) \end{aligned}$$

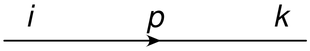
from which

$$B_{11} = -\frac{1}{2} B_0(p^2) \tag{1.52}$$

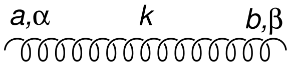
This very easy example illustrates the whole strategy of the PV reduction: the first thing to do is to write down the most general linear combination of tensors using the xternal momenta and the metric tensor. Then one has to contract with some tensor structure both sides of this decomposition and, by making use of identities like (1.51), simplify at least one propagator in the denominator. In this way one transforms a tensor integral into a scalar integral or a tensor integral of type I_n to a tensor integral of type I_{n-1}, as we will see in the following. By using different tensor structures to make the contraction, one obtains a set of linear equations[2] to be resolved with respect to the unknown factors B_{ij}, C_{ij},...

[2] The contraction with different elements of a tensor basis ensures to have a set of independent linear equations.

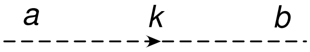
2.3.6 QCD Feynman rules



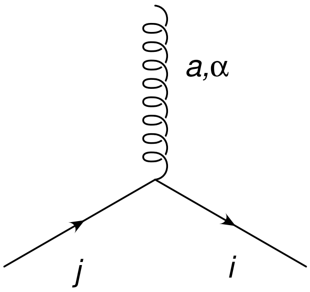
$$= \delta^{ik} \frac{i}{\not{p} - m + i\epsilon}$$



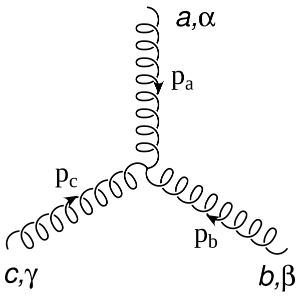
$$= \delta^{ab} \frac{i}{k^2 + i\epsilon} \left(-g^{\alpha\beta} + (1-\lambda) \frac{k^\alpha k^\beta}{k^2} \right)$$



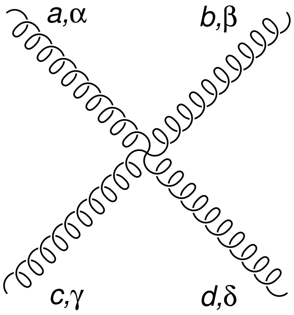
$$= \delta^{ab} \frac{i}{k^2 + i\epsilon}$$



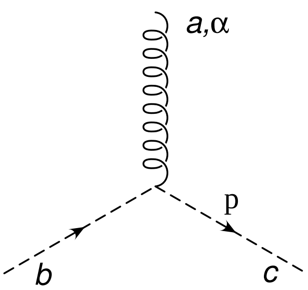
$$= -ig\gamma^\alpha t^a_{ij}$$



$$= -gf^{abc} \left[g^{\alpha\beta} (p_a - p_b)^\gamma + g^{\beta\gamma} (p_b - p_c)^\alpha + g^{\gamma\alpha} (p_c - p_a)^\beta \right]$$



$$= -ig^2 \left[f^{eac} f^{ebd} \left(g^{\alpha\beta} g^{\gamma\delta} - g^{\alpha\delta} g^{\gamma\beta} \right) + f^{ead} f^{ebc} \left(g^{\alpha\beta} g^{\gamma\delta} - g^{\alpha\gamma} g^{\beta\delta} \right) \right.$$
$$\left. + f^{eab} f^{ecd} \left(g^{\alpha\gamma} g^{\beta\delta} - g^{\alpha\delta} g^{\beta\gamma} \right) \right]$$



$$= gf^{abc} p^\alpha$$

Chapter 3

Color algebra

3.1 SU(3) algebra

The $SU(3)$ group is the group of 3×3 unitary matrices U with unit determinant

$$U^\dagger U = UU^\dagger = 1, \qquad \det U = e^{\mathrm{Tr}\{\log U\}} = 1. \tag{3.1}$$

One can always write

$$U = e^{i\omega_a t^a}, \qquad a = 1, \ldots, N^2 - 1 \tag{3.2}$$

with ω_a reals and matrices t^a hermitian and traceless

$$t^a = (t^a)^\dagger, \qquad \mathrm{Tr}\{t^a\} = 0 \tag{3.3}$$

Quark fields ψ are in the fundamental representation ($\mathbf{3}$), anti-quarks in the anti-fundamental ($\bar{\mathbf{3}}$) and gluons in the adjoint ($\mathbf{8}$). Matter fields transform under $SU(3)$ according to

$$\psi'(x) = U(x)\psi(x) \tag{3.4}$$
$$\bar{\psi}'(x) = \bar{\psi}(x)U(x)^\dagger, \tag{3.5}$$

color singlets can thus be formed out of a quark-antiquark pair via

$$\sum_i \psi_i^* \psi_i \to \sum_{i,j,k} U_{ij}^* \psi_j^* U_{ik} \psi_k = \sum_{j,k} \left(\sum_i U_{ji}^\dagger U_{ik} \right) \psi_j^* \psi_k = \sum_k \psi_k^* \psi_k \tag{3.6}$$

but it's also possible to form color singlet from three quarks (or anti quarks) using

$$\sum_{i,j,k} \epsilon^{ijk} \psi_i \psi_j \psi_k \to \sum_{i,j,k,l,m,n} \epsilon^{ijk} U_{il} U_{jm} U_{kn} \psi_l \psi_m \psi_n = \sum_{l,m,n} \det U \epsilon^{lmn} \psi_l \psi_m \psi_n \tag{3.7}$$

In this way one can accommodate all observed hadrons and mesons in color invariant states. Furthermore, since in a system with n_q quarks and $n_{\bar{q}}$ antiquarks it's possible to form color singlet only if

$$n_q - n_{\bar{q}} \mod 3 = 0, \tag{3.8}$$

it is easy to see that all these invariant states must have integer electric charge, provided the usual charges assignments : $\frac{2}{3}e$ for up type quarks and $-\frac{1}{3}e$ for down type ones. With these choices the QCD Lagrangian can be written as

$$\mathcal{L} = \mathcal{L}_G + \mathcal{L}_{G.F.} + \mathcal{L}_{F.P.} + \mathcal{L}_F \tag{3.9}$$

where the pure gauge Lagrangian is

$$\mathcal{L}_G = -\frac{1}{4} F^a_{\mu\nu} F^{a\ \mu\nu}, \ F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu, \tag{3.10}$$

the gauge-fixing part is

$$\mathcal{L}_{G.F.} = -\frac{1}{2\lambda} \left(\partial^\mu A^a_\mu\right)^2 \tag{3.11}$$

and the Faddeev-Popov one is

$$\mathcal{L}_{F.P.} = \partial^\mu \bar{\chi}^a D^{ab}_\mu \chi^b \quad \text{with} \quad D^{ab}_\mu = \delta^{ab}\partial_\mu + igf^{abc}A^c_\mu. \tag{3.12}$$

Finally the fermion Lagrangian reads

$$\mathcal{L}_F = \sum_{flavour} \bar{\psi}^i_f \left(i \not{D}^{ij}_\mu - m_f \delta^{ij}\right) \psi^j_f \qquad \text{with} \quad D^{ij}_\mu = \delta^{ij}\partial_\mu + igt^a_{ij}A^a_\mu \tag{3.13}$$

where the $SU(3)$ algebra tells us that

$$\left[t^a, t^b\right] = i\ f^{abc} t^c \tag{3.14}$$

and we chose the convention

$$\mathrm{Tr}\{t^a t^b\} = T_{\mathrm{F}} \delta^{ab}, \quad T_{\mathrm{F}} = \frac{1}{2}. \tag{3.15}$$

One can show that in this way the structure constants f are always reals and antisymmetric. For example taking the complex conjugate of (3.14) one has

$$-i \left(f^{abc}\right)^* (t^c)^\dagger = \left[\left(t^b\right)^\dagger, (t^a)^\dagger\right] = -\left[t^a, t^b\right] \tag{3.16}$$

because of hermiticity of t's. Thus $\left(f^{abc}\right)^* = f^{abc}$. In the same way taking the trace of

$$i f^{abc} t^c t^d = \left[t^a, t^b\right] t^d \tag{3.17}$$

one gets

$$i f^{abc} T_{\mathrm{F}} \delta^{cd} = \mathrm{Tr}\{\left[t^a, t^b\right] t^d\} \tag{3.18}$$

$$f^{abc} = -2i \mathrm{Tr}\{\left[t^a, t^b\right] t^c\}. \tag{3.19}$$

that shows that f is antisymmetric.

We generalize now to the $SU(n)$ group: the generic hermitian $n \times n$ matrix M can be written as

$$M = n^a t^a + n^0 \, \mathbb{I}_{n\times n} \tag{3.20}$$

with n^0 fixed by the trace to be $n^0 = \mathrm{Tr}\{M\}/n$. In the same way

$$M t^b = n^a t^a t^b + n^0 t^b \tag{3.21}$$

with n^a now fixed to $n^a = 2\mathrm{Tr}\{M t^a\}$. Thus

$$M = 2\mathrm{Tr}\{M t^a\} t^a + \frac{1}{n}\mathrm{Tr}\{M\}\mathbb{I}_{n\times n} \tag{3.22}$$

Taking $M = \left[t^a, t^b\right]$ one can re-derive the formula for f^{abc}

$$\left[t^a, t^b\right] = 2\mathrm{Tr}\{\left[t^a, t^b\right] t^c\} t^c \tag{3.23}$$

$$i f^{abc} t^c = 2\mathrm{Tr}\{\left[t^a, t^b\right] t^c\} t^c \tag{3.24}$$

$$f^{abc} = -2i\mathrm{Tr}\{\left[t^a, t^b\right] t^c\}. \tag{3.25}$$

Using Jacobi identities it's also possible to define the adjoint representation by means of matrices T, made by structure constants

$$\left(T^b\right)_{ac} = i f^{abc} \tag{3.26}$$

such that they satisfy $\left[T^a, T^b\right] = i f^{abc}\, T^c$. Defining now

$$T^2_{ik} = T^a_{ij} T^a_{jk} \tag{3.27}$$

one can show that

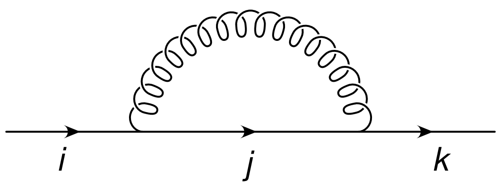
$$\left[T^b, T^2\right] = T^b T^a T^a - T^a T^a T^b = -\left[T^a, T^b\right] T^a - T^a \left[T^a, T^b\right] \tag{3.28}$$

$$= -i f^{abc} T^c T^a - T^a i f^{abc} T^c = -i f^{abc}\{T^c, T^a\} = 0 \tag{3.29}$$

T^2 is a Casimir of the representation and by Schur's lemma it must be proportional to the identity.

3.2 Color coefficients

Provided that the most important difference between QCD and QED is the non abelianity of the former, it worths to separate the non abelian part evaluating color coefficients of sequences of t matrices and then proceed as in usual QED computations. For example the color coefficient for the fermion self energy correction is defined to be



$$t^a_{ij} t^a_{jk} \equiv C_{\mathrm{F}} \mathbb{I}_{ik} \tag{3.31}$$

Chapter 4

QED renormalization

1. To be done even if the quantum corrections were **finite**!
2. the **same procedure** cancels all the divergences at **all orders**

In the first chapters we analyzed many aspects of gauge theories. Time is ready to put all those informations together and to look at the most important consequences that a quantum field theory has on our knowledge of physics.

In this section we start the analysis of perturbative corrections to amplitudes in the simplest context of QED. The bare Lagrangian is

$$\mathcal{L} = \bar{\psi}_B i\not{\partial}\psi_B - \frac{1}{4}F_B^{\mu\nu}F_{\mu\nu}^B - g_B\bar{\psi}_B\not{A}_B\psi_B - m_B\bar{\psi}_B\psi_B$$

The abelianity of the theory implies, as we have seen, that there is only one kind of vertex. This point, though simplifying many calculations, does not exclude the possibility to fix the principles governing renormalization and its main consequences.

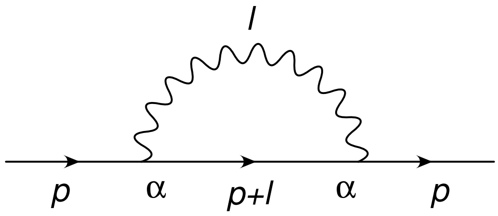
As we have just seen in section 1, as soon as loop integrals are concerned, one has to use a regularization technique in order to prevent the amplitude to diverge. In gauge theories, dimensional regularization, though very mathematical, is a natural choice since it preserves both gauge and Lorentz invariance. Other techniques are more physically based. Here we adopt one of these, consisting in the introduction of an UV cutoff Λ in the loop integral. This cutoff, in the Wilsonian way of thinking at quantum field theories, can be seen as the last energy scale at which our theory is valid: we can look at the theory as an effective field theory that makes sense up to Λ scale.

Dimensional regularization will be used in the next chapter, when we will study 1-loop corrections to QCD amplitudes.

The basic blocks we need to *renormalize the theory* (at one loop), i.e. to extract finite predictions from mathematical divergent quantities, are the computation of the fundamental divergent Feynman diagrams: the fermion and the photon self energy and the vertex corrections. Moreover, in all the computations we will assume for simplicity massless fermions.

4.1 Fermion propagator

We consider the first order corrections to the propagator of a fermion. The Feynman integral we have to calculate is The corresponding value[1], in the Feynman gauge and neglecting the



fermion mass, is given by

$$\begin{aligned} M &= \int^{\Lambda} \frac{d^4\ell}{(2\pi)^4} \, \frac{-i}{\ell^2} \, (-ie_B\gamma^\alpha) \, \frac{i}{\not{p}+\not{\ell}} \, (-ie_B\gamma_\alpha) \\ &= -e_B^2 \int^{\Lambda} \frac{d^4\ell}{(2\pi)^4} \, \frac{1}{\ell^2} \, \gamma^\alpha \frac{1}{\not{p}+\not{\ell}} \, \gamma_\alpha \\ &= 2e_B^2 \int^{\Lambda} \frac{d^4\ell}{(2\pi)^4} \, \frac{1}{\ell^2} \, \frac{1}{\not{p}+\not{\ell}} \end{aligned} \tag{4.1}$$

where Λ is the cutoff. In the last line we used the identity

$$\gamma^\alpha \, \gamma_\beta \, \gamma_\alpha = -2\gamma_\beta \tag{4.2}$$

The proof is a direct consequence[2] of the Clifford algebra $\{\gamma_\alpha, \gamma_\beta\} = 2g_{\alpha\beta}$.

We observe that M has the dimension of an energy. Having fixed the mass m of the fermion to be 0, the only dimensionful parameter in the integral (apart the cutoff, that plays a different role) is the momentum p_μ and since M is Lorentz-invariant, we can write

$$M = A\not{p} \tag{4.3}$$

The parameter A comes from the result of the loop integral and, since from (4.3) it has to be dimensionless, we expect it to diverge at most logarithmically with Λ. We also note that (4.1) does not diverge in the infrared region ($\ell \to 0$) because there is a $\not{p}$ in the denominator. In order to find A, we derive (4.1) and (4.3) with respect to p_μ. Using the relation

$$\partial_{p_\mu} \left(\frac{1}{\not{p}+\not{\ell}} \right) = -\frac{1}{\not{p}+\not{\ell}} \, \gamma_\mu \, \frac{1}{\not{p}+\not{\ell}} \tag{4.4}$$

[1] Having to calculate an amplitude, there would be the usual spinors $\bar{u}(p)$ and $u(p)$ at the extrema of our expression. Actually we are not calculating an amplitude but a self-energy diagram so we do not need to saturate polarization indexes with spinors.

[2] Note that (4.2) as it stands is true in four-dimension Minkowski spacetime.

which is a consequence of $s^{-1}s = 1$, with $s = (\not p + \not\ell)$, we approde to the identity

$$
\begin{aligned}
A\gamma_\mu &= -2e_B^2 \int^\Lambda \frac{d^4\ell}{(2\pi)^4} \frac{1}{\ell^2} \frac{1}{\not p + \not\ell} \gamma_\mu \frac{1}{\not p + \not\ell} \\
&= -2e_B^2 \int^\Lambda \frac{d^4\ell}{(2\pi)^4} \frac{(p+\ell)^\alpha (p+\ell)^\beta}{\ell^2 \left[(p+\ell)^2\right]^2} \gamma_\alpha \gamma_\mu \gamma_\beta
\end{aligned}
\tag{4.5}
$$

Up to now the computation is exact. At this point we neglect all the p momentum dependence in the integral ($\ell \gg p$) since we are now interested in the high momentum behavior of the theory: in other words we want to extract the leading singularity of the integral in the UV limit. After this assumptions the previous formula becomes

$$
\begin{aligned}
A\gamma_\mu &\simeq -2e_B^2 \int^\Lambda \frac{d^4\ell}{(2\pi)^4} \frac{\ell^\alpha \ell^\beta}{(\ell^2)^3} \gamma_\alpha \gamma_\mu \gamma_\beta \\
&= -\frac{e_B^2}{2} \gamma_\alpha \gamma_\mu \gamma^\alpha \int^\Lambda \frac{d^4\ell}{(2\pi)^4} \frac{1}{(\ell^2)^2}
\end{aligned}
\tag{4.6}
$$

where the Lorentz dependence of the integrand can be extracted by replacing, under the integral, $\ell^\alpha \ell^\beta$ with $\ell^2 g^{\alpha\beta}/4$. Using again (4.2) on the right hand side of (4.6) we have

$$
A \simeq e_B^2 \int^\Lambda \frac{d^4\ell}{(2\pi)^4} \frac{1}{(\ell^2)^2}
\tag{4.7}
$$

Perform this integral is now an easy task: passing in Euclidean time and remembering that the surface of a 4–dimensional sphere of radius one is $2\pi^2$ (see equation (1.23)), we get:

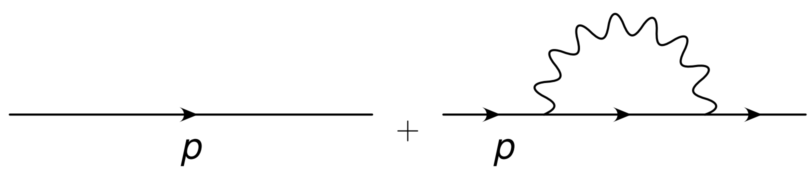
$$
\int^\Lambda \frac{d^4\ell}{(2\pi)^4} \frac{1}{(\ell^2)^2} = i\frac{\pi^2}{(2\pi)^4} \log\left(\frac{\Lambda^2}{\mu^2}\right)
\tag{4.8}
$$

So A becomes

$$
A = \frac{i\alpha_B}{4\pi} \log\left(\frac{\Lambda^2}{\mu^2}\right)
\tag{4.9}
$$

We observe that A diverges logarithmically with Λ as expected. Moreover, we have introduced an arbitrary new energy scale μ. At this level the use of μ is required only to maintain the argument of the logarithm dimensionless. Strictly speaking, the integral (4.8), as it stands, would diverge also in the IR region but, as we pointed out earlier, this expression comes from an integral that was free of IR divergences. For this reason the scale μ^2 has not a meaning deeper than that of being a generic scale obtained from the external momentum p.

We now consider the sum of the graphs relative to the fermion propagator and its first order correction. They are given by



$$
+ \ \mathcal{O}\left(\alpha_B^2\right) \tag{4.10}
$$

and the related expression M is

$$\begin{aligned} M &= \frac{i}{\not p} + \frac{i}{\not p} A \not p \frac{i}{\not p} \\ &= \frac{i}{\not p}(1 + iA) \\ &\equiv \frac{i}{\not p} Z_2 \end{aligned} \tag{4.11}$$

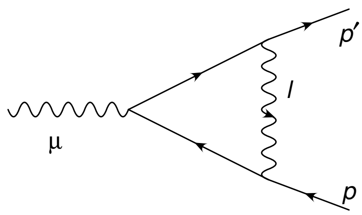
where the renormalization constant Z_2 (or the 1–loop correction to the propagator $\delta_2 \equiv Z_2 - 1$) is defined by

$$\begin{aligned} Z_2 &= 1 - \frac{\alpha_B}{4\pi} \log \frac{\Lambda^2}{\mu^2} + \mathcal{O}\left(\alpha_B^2\right) \\ \delta_2 &= -\frac{\alpha_B}{4\pi} \log \frac{\Lambda^2}{\mu^2} + \mathcal{O}\left(\alpha_B^2\right) \end{aligned} \tag{4.12}$$

Please notice that eq. (4.11) implies that the photon remains **massless**.

4.2 Vertex corrections

We now consider the first order correction to the QED vertex.



(4.13)

Doing computation as before in the Feynman gauge, the graph of fig. (4.13) corresponds to the following[3]:

$$\begin{aligned} M^\mu &= \int^\Lambda \frac{d^4\ell}{(2\pi)^4} (-ie_B\gamma^\alpha) \frac{i}{\not p' + \not \ell} (-ie_B\gamma^\mu) \frac{i}{\not p + \not \ell} (-ie_B\gamma^\beta) \frac{-ig_{\alpha\beta}}{\ell^2} \\ &= -e_B^3 \int^\Lambda \frac{d^4\ell}{(2\pi)^4} \gamma^\alpha \frac{1}{\not p' + \not \ell} \gamma^\mu \frac{1}{\not p + \not \ell} \gamma_\alpha \frac{1}{\ell^2} \end{aligned} \tag{4.14}$$

As in the fermion propagator loop, since we are interested in the UV behavior of the amplitude, we can neglect p and p' in the fermion propagators: collecting all the gamma matrices outside the integral, we are left with

$$\begin{aligned} M^\mu &\simeq -e_B^3 \gamma^\alpha \gamma^\gamma \gamma^\mu \gamma^\delta \gamma_\alpha \left[\int^\Lambda \frac{d^4\ell}{(2\pi)^4} \frac{\ell_\gamma \ell_\delta}{(\ell^2)^3} \right] \\ &= -e_B^3 \gamma^\alpha \gamma^\gamma \gamma^\mu \gamma^\delta \gamma_\alpha \left[\int^\Lambda \frac{d^4\ell}{(2\pi)^4} \frac{g_{\gamma\delta}}{4} \frac{1}{(\ell^2)^2} \right] \end{aligned} \tag{4.15}$$

[3]In (4.13) the momenta p and p' flow out of the graph.

where the Lorentz dependence of the integral within squared brackets can be extracted multiplying it with $g^{\gamma\delta}$. By repeated use of (4.2) we write

$$M^\mu = -e_B^3 \gamma^\mu \int^\Lambda \frac{d^4\ell}{(2\pi)^4} \frac{1}{(\ell^2)^2}$$

The integral is like that in the previous computation, so at the end we have

$$M^\mu = -i \frac{e_B^3}{16\pi^2} \gamma^\mu \log\left(\frac{\Lambda^2}{\mu^2}\right)$$

The divergence is logarithmic, as power counting shows in eq. (4.14) and the meaning of μ^2 is the same discussed in the previous paragraph.

In order to calculate the vertex renormalization constant Z_1 at 1–loop, we have to sum this graph with the tree level vertex, obtaining

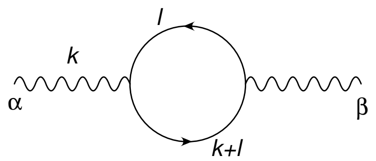
$$-ie_B\gamma^\mu Z_1^{-1} \equiv (-ie_B\gamma^\mu) + \left(-i\frac{e_B^3}{16\pi^2}\gamma^\mu \log\left(\frac{\Lambda^2}{\mu^2}\right)\right)$$

from which at the end we read ($Z_1 \equiv 1 + \delta_1$)

$$\begin{aligned} Z_1^{-1} &= 1 + \frac{\alpha_B}{4\pi} \log\left(\frac{\Lambda^2}{\mu^2}\right) + \mathcal{O}(\alpha_B^2) \\ \delta_1 &= -\frac{\alpha_B}{4\pi} \log\left(\frac{\Lambda^2}{\mu^2}\right) + \mathcal{O}(\alpha_B^2) \end{aligned} \tag{4.16}$$

4.3 Photon propagator

The one loop contribution to the photon propagator in the Feynman gauge is



$$= -\left(-ie_B\right)^2 \mathrm{Tr} \int^\Lambda \frac{d^4\ell}{(2\pi)^4} \gamma^\alpha \frac{i}{\not\ell} \gamma^\beta \frac{i}{\not\ell + \not k} \tag{4.17}$$

which seems to diverge quadratically. We will show that this divergence is instead logarithmic. The contribution to the propagator coming out (4.17) is only transverse, due to Ward identity, which means that if we contract this integral with k_α or k_β we do obtain zero. This means that the structure of the integral can be summarized as follows

$$-\left(-ie_B\right)^2 \mathrm{Tr} \int^\Lambda \frac{d^4\ell}{(2\pi)^4} \gamma^\alpha \frac{i}{\not\ell} \gamma^\beta \frac{i}{\not\ell + \not k} = B\left(k^\alpha k^\beta - k^2 g^{\alpha\beta}\right) \tag{4.18}$$

Since we want to calculate (4.17) this is equivalent to calculate B in (4.18). We also note that since B is dimensionless it can only depend on the ratio Λ^2/k^2. We contract with $g^{\alpha\beta}$ and use the identity (4.2) finding

$$2e_B^2 \mathrm{Tr} \int^\Lambda \frac{d^4\ell}{(2\pi)^4} \frac{1}{\not\ell} \frac{1}{\not\ell + \not k} = -3Bk^2 \tag{4.19}$$

The term which gives the dimensionful external scale is k and it appears in both the two sides of (4.19). If we would now to simplify the calculation of the integral we should eliminate k in the second side leaving it only in the integral. This can be achieved acting on the equation with the derivative ∂_{k_α} and ∂_{k_β}, using (4.4) we find

$$\begin{aligned}
-3B\partial_{k_\alpha}\partial_{k_\beta}k^2 &= 2e_B^2\mathrm{Tr}\int^\Lambda \frac{d^4\ell}{(2\pi)^4}\frac{1}{\not{\ell}}\partial_{k_\alpha}\partial_{k_\beta}\frac{1}{\not{\ell}+\not{k}} \\
-6Bg^{\alpha\beta} &= 2e_B^2\mathrm{Tr}\int^\Lambda \frac{d^4\ell}{(2\pi)^4}\frac{1}{\not{\ell}}\left(\frac{1}{\not{\ell}+\not{k}}\gamma_\beta\frac{1}{\not{\ell}+\not{k}}\gamma_\alpha\frac{1}{\not{\ell}+\not{k}}+\frac{1}{\not{\ell}+\not{k}}\gamma_\alpha\frac{1}{\not{\ell}+\not{k}}\gamma_\beta\frac{1}{\not{\ell}+\not{k}}\right)
\end{aligned}$$

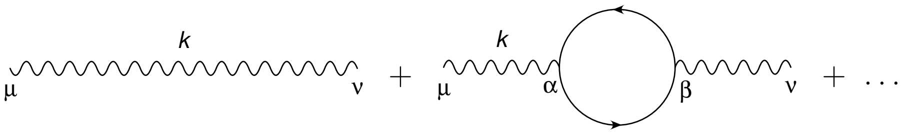
At this point we contract the equation with $g_{\alpha\beta}$ obtaining

$$\begin{aligned}
-24B = 4e_B^2\mathrm{Tr}\int^\Lambda \frac{d^4\ell}{(2\pi)^4}\frac{1}{\not{\ell}}\left(\frac{1}{\not{\ell}+\not{k}}\gamma^\alpha\frac{1}{\not{\ell}+\not{k}}\gamma_\alpha\frac{1}{\not{\ell}+\not{k}}\right) \\
B = \frac{e_B^2}{3}\mathrm{Tr}\int^\Lambda \frac{d^4\ell}{(2\pi)^4}\frac{1}{\not{\ell}}\left(\frac{1}{\not{\ell}+\not{k}}\right)^3
\end{aligned} \tag{4.20}$$

We can again make the approximation $\ell \gg k$. Recalling that $\not{\ell}^{-1}\not{\ell}^{-1} = 1/(\ell^2)$ and after the trace, one gets

$$B = \frac{4}{3}e_B^2\int^\Lambda \frac{d^4\ell}{(2\pi)^4}\frac{1}{(\ell^2)^2} \tag{4.21}$$

where the integral is again the same as in the previous calculations. This one loop integral contributes to the photon propagator when we sum it to the first piece. We have the structure



(4.22)

which corresponds to

$$-i\frac{g^{\mu\nu}}{k^2} + \left(-i\frac{g^{\mu\alpha}}{k^2}\right)B(k_\alpha k_\beta - k^2 g_{\alpha\beta})\left(-i\frac{g^{\beta\nu}}{k^2}\right) + \ldots \tag{4.23}$$

Having instead made the calculation in Lorentz gauge we would have used the tree level propagator with terms containing $k^\mu k^\nu(1-\lambda)$ and this would have given in (4.23) terms proportional to $k^\mu k^\nu$. Nevertheless, the result would be the same since these terms do not give any contribution because of gauge invariance. In fact the sum (4.21) will be connected to an external conserved current or to a polarization vector. In both case we will hit the $k_\mu\ k_\nu$ term with a current or a polarization vector and in both cases this will give a null contribution. We thus obtain from (4.23)

$$-i\frac{g^{\mu\nu}}{k^2}(1+iB) \equiv -i\frac{g^{\mu\nu}}{k^2}Z_3 \tag{4.24}$$

Using (4.8) and applying it to (4.21) we get ($Z_3 \equiv 1 + \delta_3$)

$$\begin{aligned} Z_3 &= 1 - \frac{4}{3}\frac{\pi^2}{(2\pi)^4}\, e_B^2 \log\left(\frac{\Lambda^2}{\mu^2}\right) = 1 - \frac{2}{3}\frac{\alpha_B}{2\pi}\log\left(\frac{\Lambda^2}{\mu^2}\right) \\ \delta_3 &= -\frac{2}{3}\frac{\alpha_B}{2\pi}\log\left(\frac{\Lambda^2}{\mu^2}\right) \end{aligned} \tag{4.25}$$

For the photon propagator it is important to take a look also at the result that we obtain if we try to sum all the radiative corrections. This corresponds to sum all the graphs made by connecting more and more 1–particle irreducible Feynman graphs. We obtain a geometric series like

$$-i\frac{g^{\mu\nu}}{k^2}\sum_{n=0}^{+\infty}(iB)^n = -i\frac{g^{\mu\nu}}{k^2}\frac{1}{1-iB} \tag{4.26}$$

where B is the 1–PI graph and at one loop it is exactly our old B.

Despite its obviousness, equation (4.26) shows a fundamental property of QED: the photon remains exactly massless even after higher order quantum corrections are considered. In fact the pole is again at $k^2 = 0$, i.e. it is not displaced by radiative corrections. This is is an example of a fundamental property of Quantum Field Theories: an exact symmetry of the Lagrangian (here the gauge symmetry) has deep consequences also on the way radiative corrections manifest themselves. As we have just seen the local $U(1)$ of QED forbids the photon to acquire a mass after quantum corrections, forcing the propagator structure to be transverse and lowering the degree of divergence from 2 to 0. Another well known example is the (global) chiral symmetry of the Dirac massless Lagrangian that forces the fermion field to stay massless even after loop corrections: from this argument follows that if the mass is present in $\mathcal{L}_{Dirac}$ the self energy has to bee proportional to the mass itself, forcing again the divergences of the one loop self energy to be logarithmical and not linear, as power counting would tell.

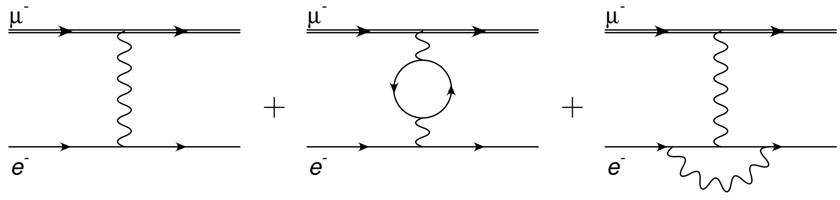
4.4 The Lehmann–Symanzik–Zimmermann (LSZ) formula

4.5 The running of the coupling constant

We begin analyzing the physical meaning of previous calculations, by recalling the definition of renormalization constants Z_i (see (4.12), (4.16) and (4.25))

$$Z_1 \equiv 1 + \delta_1 \tag{4.27}$$
$$Z_2 \equiv 1 + \delta_2 \tag{4.28}$$
$$Z_3 \equiv 1 + \delta_3 \tag{4.29}$$

If we consider $e^- \mu^- \to e^- \mu^-$ scattering and compute higher order virtual corrections to the amplitude, at order e_B^4, we have to sum the following graphs



(4.30)

Box corrections (that are not UV divergent) and the vertex correction at the muon vertex are not shown. This last correction will play a role when considering the renormalization of the muon charge. We then considered only the radiative corrections on the lower half of the diagrams (on the electron part of the amplitude) that means that in the following we will think only to the prediction of the theory for the physical measurable electron charge e_P. From this it follows also that the second graph will contribute with a one half factor (or square root factor). We thus have that the sum of the graphs goes like

$$\begin{aligned} &\sim \; e_B \left(1 + \frac{1}{2}\,\delta_3 - \delta_1 + \delta_2 + \delta_2\right) \\ &\sim \; e_B\, Z_3^{1/2}\, Z_2^2\, Z_1^{-1} \end{aligned} \tag{4.31}$$

In addition, a factor $\left(Z_2^{1/2}\right)^2$ has been added, to comply with the LSZ formula[4]. In this case, we're dealing with external fermions (electrons), so we must multiply twice by $Z_2^{-1/2}$. We note that this gives exactly the same result that one would obtain adding only connected diagrams shorn of self energy corrections on external legs and multiplying this result with a $Z_i^{1/2}$ factor for every external leg of type "i", as in the standard LSZ formula. In both cases the correct answer for the amplitude is

$$M \sim e_B\, Z_3^{1/2}\, Z_2\, Z_1^{-1} = e_B\, Z_3^{1/2} \tag{4.32}$$

where we have used the fact that at the first order our calculation gives $Z_1 = Z_2$ [5]. Equation (4.32) also suggests us that in some sense the renormalization of QED is related only to the correction of the photon self energy (Z_3): we will come back on this at the end of this section.

From all these considerations, we are now left with something proportional to $e_B\sqrt{Z_3}$ and this will be our definition for the physical electron charge e_P, since the cross section we would obtain from the amplitude contains a $(e_B\sqrt{Z_3})^2$ factor and the cross section is the link between theory and measurable quantities. Thus we define[6]

$$e_P = \sqrt{Z_3}\, e_B \tag{4.33}$$

[4]The usual conventions are $i = 2$ for fermions and $i = 3$ for gauge bosons.

[5]As we shall see later on, this equality holds at all orders, by virtue of Ward identities.

[6]Instead of e_P, usually one calls this quantity the *renormalized* charge e_R but in this part we will continue to use e_P in order to remind that this is the value that in the theory has the meaning of measurable, physical electron charge.

From the previous equation it is now easy to see how the physical charge varies with the energy scale μ: at the scale μ^2 its value is

$$e_P(\mu^2) = e_B \left\{1 - \frac{2}{3}\frac{\alpha_B}{2\pi}\log\left(\frac{\Lambda^2}{\mu^2}\right)\right\}^{1/2} \simeq e_B \left\{1 - \frac{1}{3}\frac{\alpha_B}{2\pi}\log\left(\frac{\Lambda^2}{\mu^2}\right)\right\} \tag{4.34}$$

while at the scale μ_0^2

$$e_P(\mu_0^2) = e_B \left\{1 - \frac{1}{3}\frac{\alpha_B}{2\pi}\log\left(\frac{\Lambda^2}{\mu_0^2}\right)\right\}. \tag{4.35}$$

The difference is thus no more dependent on the cutoff scale Λ

$$e_P(\mu^2) - e_P(\mu_0^2) = e_B \frac{1}{3}\frac{\alpha_B}{2\pi}\left\{\log\left(\frac{\Lambda^2}{\mu_0^2}\right) - \log\left(\frac{\Lambda^2}{\mu^2}\right)\right\} = \frac{e_B^3}{24\pi^2}\log\left(\frac{\mu^2}{\mu_0^2}\right). \tag{4.36}$$

Now we are free to replace the bare charge e_B with the physical (renormalized) one e_P in the right hand side of the previous equation, up to terms of higher order, finding

$$e_P(\mu^2) - e_P(\mu_0^2) = \frac{e_P^3}{24\pi^2}\log\left(\frac{\mu^2}{\mu_0^2}\right) + \mathcal{O}\left(e_P^4\right). \tag{4.37}$$

The running of the coupling constant is thus

$$e_P(\mu^2) = e_P(\mu_0^2) + \frac{e_P^3}{24\pi^2}\log\left(\frac{\mu^2}{\mu_0^2}\right) + \mathcal{O}\left(e_P^4\right). \tag{4.38}$$

The previous formula is very important since, given the value of the physical charge at one fixed scale μ_0, one can extrapolate the new e_P value at any other scale μ, keeping in mind that one have to remain in the perturbative regime.

Before going on, a remark on the way we introduced the scale μ is due: μ^2 was a scale of the order of the external momenta of the legs of which we calculated the radiative corrections. In particular, looking at the graphs we added (eq. (4.30)), for the photon self energy and the vertex corrections we can think at μ^2 as the off–shellness of the virtual exchanged photon, i.e. the typical scale of the process.

Keeping in mind all these observations, eq. (4.38) tells us a fundamental unexpected thing: if we make two measurements for a process involving the electron charge at different energies and we want to predict the correct result, we have to use different values for the electron charge itself. In this sense we can also say that the constant e_P is no longer a constant but it runs in a way predicted by the theory. It is also clear that for the theory to be predictive it is needed to fix the value of the constant at one scale[7] and then use (4.38) to extract the corresponding value at another scale and use it in the computation.

To obtain the running of the renormalized coupling α_R one can proceed in a slightly different way: it is useful to see how it works because we will use the following argument to

[7]Typically in QED one fix the fine-structure constant α to be equal to the low energy measured value $\approx 1/137$ at the scale $\mu^2 = m_e^2$.

Appendix A

Useful mathematical functions

A.1 The Γ and B functions

The Gamma function is defined by

$$\Gamma(z) \equiv \begin{cases} \displaystyle\int_0^\infty dx\, \mathrm{e}^{-x} x^{z-1} & \mathrm{Re}\, z > 0 \\ \displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{1}{k+z} + \int_1^\infty dx\, \mathrm{e}^{-x} x^{z-1} & \mathrm{Re}\, z < 0,\ z \neq -n,\ n \in N_0 \end{cases} \tag{A.1}$$

With a simple change of variables $x \to x^2$

$$\Gamma(z) = \int_0^\infty dx\, \mathrm{e}^{-x} x^{z-1} = 2 \int_0^\infty dx\, \mathrm{e}^{-x^2} x^{2z-1} \tag{A.2}$$

It can be easily shown that

$$\Gamma(1) = 1 \tag{A.3}$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \tag{A.4}$$

$$\Gamma(z+1) = z\,\Gamma(z) \tag{A.5}$$

Using eq. (A.2) and changing to polar coordinates ($x = r\cos\theta, y = r\sin\theta$) we can write

$$\begin{aligned} \Gamma(a)\,\Gamma(b) &= 4 \int_0^\infty dx\, dy\, \mathrm{e}^{-x^2-y^2} y^{2a-1} x^{2b-1} \\ &= 4 \int_0^\infty dr\, r \int_0^{\pi/2} d\theta\, \mathrm{e}^{-r^2} r^{2a+2b-2} (\sin\theta)^{2a-1} (\cos\theta)^{2b-1} \\ &= 2 \int_0^{\pi/2} d\theta\, (\sin\theta)^{2a-1} (\cos\theta)^{2b-1}\, 2 \int_0^\infty dr\, \mathrm{e}^{-r^2} r^{2(a+b)-1} \\ &= 2 \int_0^{\pi/2} d\theta\, (\sin\theta)^{2a-1} (\cos\theta)^{2b-1}\, \Gamma(a+b) \end{aligned} \tag{A.6}$$

Since $d\sin^2\theta = 2\sin\theta\cos\theta\, d\theta$ we can write

$$\frac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)} = 2\int_0^{\pi/2} d\theta\,\left(\sin\theta\right)^{2a-1}\left(\cos\theta\right)^{2b-1} = \int_0^1 d\sin^2\theta\left(\sin^2\theta\right)^{a-1}\left(\cos^2\theta\right)^{b-1} \tag{A.7}$$

Calling $x = \sin^2\theta$ we have

$$\frac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)} = \int_0^1 dx\, x^{a-1}(1-x)^{b-1} \tag{A.8}$$

We can define the "beta" function as

$$B(a,b) \equiv \int_0^1 dx\, x^{a-1}(1-x)^{b-1} = 2\int_0^{\frac{\pi}{2}} d\theta\,\left(\sin\theta\right)^{2a-1}\left(\cos\theta\right)^{2b-1} = \frac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)}\,. \tag{A.9}$$

A useful expansion is given by

$$\Gamma(1+\epsilon) = 1 - \gamma_E\,\epsilon + \frac{6\,\gamma_E^2+\pi^2}{12}\,\epsilon^2 + \mathcal{O}\left(\epsilon^3\right)\;, \tag{A.10}$$

where $\gamma_E = 0.5772157\ldots$ is the Euler-Mascheroni constant.

A.2 The angular volume Ω_d in d dimensions

In order to compute the total angular volume in d dimensions we proceed as follows. We consider the integral I

$$I \equiv \left(\int_{-\infty}^{\infty} dx\, \mathrm{e}^{-x^2}\right)^d = \left(\sqrt{\pi}\right)^d \tag{A.11}$$

and we rewite the lhs of the equation as

$$I = \int_{-\infty}^{\infty} dx_1\, dx_2 \ldots dx_d\, \mathrm{e}^{-(x_1^2+x_2^2+\ldots+x_d^2)} = \int d\Omega_d \int_0^{\infty} dr\, r^{d-1}\mathrm{e}^{-r^2} \tag{A.12}$$

The r integration can be performed using eq. (A.2)

$$I = \Omega_d\,\frac{\Gamma\left(\frac{d}{2}\right)}{2} \tag{A.13}$$

so that

$$\Omega_d = \frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)} \tag{A.14}$$