~~LECTURE 1~~

QCD : theory of strong interaction.

Elementary quanta of the theory are quarks and gluons. "PARTONS": Quarks & gluons never observed as free particles, but introduced as constituents of hadrons

Bound states : mesons $q\bar{q}$, baryons qqq } hadrons

Quarks: spin $\frac{1}{2}$. There are 6 types of quarks (FLAVOURS)

	charge	mass
u	2/3	2 MeV
d	-1/3	5 MeV
c	2/3	1.3 GeV
s	-1/3	130 MeV
t	2/3	173 GeV
b	-1/3	4.2 GeV

← FRACTIONAL CHARGE

← MASS IS A BIT AMBIGUOUS : WE DON'T MEASURE QUARKS DIRECTLY. ONLY FOR THE TOP QUARK ONE CAN PERFORM A "DIRECT" MEASURE

← for light quarks $m_{\overline{MS}}(\mu)$ $\quad \mu \sim \begin{cases} 2\ \text{GeV} & u,d,s \\ m_b, m_c & b,c \end{cases}$

u, d, s → light quarks

c, b, t → heavy quarks (← at LHC, b and c ~ massless)

↳ they decay because of weak interactions

There is an extra quantum number carried by quarks: COLOUR

$\psi_i^{(f)}(x)$ ← FLAVOUR (f), ← COLOUR (i)

Colour is the quantum number that governs the dynamics.

The gauge symmetry group is $SU(3)_c$ ← QUANTUM CHROMODYNAMICS

QCD = $SU(N_c)$ gauge theory → NON ABELIAN GAUGE THEORY

→ LAGRANGIAN IS FIXED ONCE THE GAUGE GROUP IS FIXED and the matter content is given

Before continuing, let's have a quick look at the historical development, in order to understand where the EXP evidence for QCD comes from

QUARK MODEL ("Eightfold way"): Gell-Mann, Zweig, Neeman

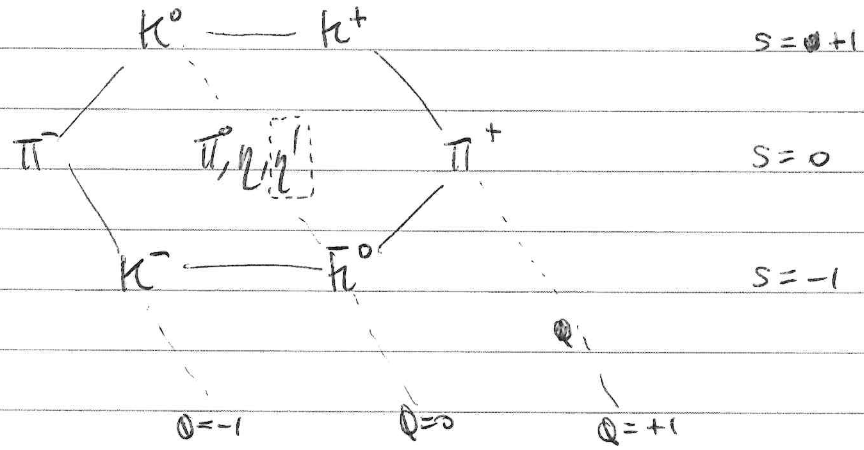
~~SU(3) approximate symmetry ... Not the gauge symmetry~~

Hadronic spectroscopy - organize states according to some symmetry pattern

~~... CHARGE, Q = 1/3~~

Examples:

mesons



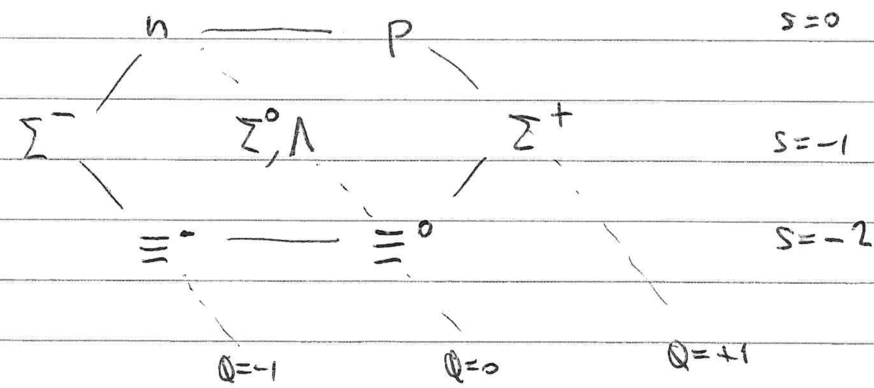
$J^P = 0^-$

- light meson octet
- spin = 0
- pseudoscalar

- η' heavier [sometimes one calls the multiplet "meson nonet"

- There exists a heavier octet, with spin 1

baryons

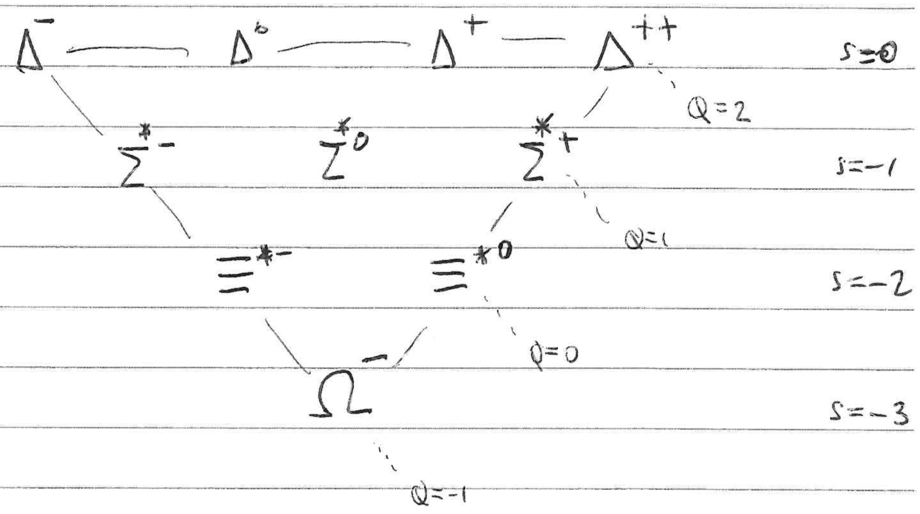


$J^P = \frac{1}{2}^+$

- light baryon octet
- spin $= \frac{1}{2}$

$J^P = \frac{3}{2}^+$

baryon decuplet



~~Ω^-: predicted following the pattern before the EXP discovery~~

Quark model: existence of quarks:
- $SU(3)_F$ approximate symmetry
- spin $\frac{1}{2}$ particles
- fundamental rep. of $SU(3)_F$
- [3 flavours (+ antiparticles)]
- fractional charge

	spin	I_3	S	B	Y	Q
u	1/2	1/2	0	1/3	1/3	2/3
d	1/2	-1/2	0	1/3	1/3	-1/3
s	1/2	0	-1	1/3	-2/3	-1/3

$I_3 = \frac{1}{2}(N_u - N_d)$ ← $SU(2)_F$ isospin [more precisely $I_3 = \frac{1}{2}[(N_u - N_{\bar{u}}) - (N_d - N_{\bar{d}})]$]

$Y = B + S$ ← THIS IS NOT THE U(1) HYPERCHARGE

$$Q = I_3 + \frac{Y}{2}$$

Mesons & baryons: organized according to irreducible representation of products of $SU(3)_F$

Y ↑ → I_3

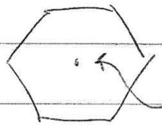
<u>Prediction:</u> Ω^- predicted following the above pattern (and the model) before the EXP discovery (1964

~~Recall of~~

mesons: $q\bar{q} \to \underline{3} \times \underline{\bar{3}} = \underline{1} + \underline{8}$

$\underline{1}$ ↑ singlet $(\eta' = u\bar{u} + d\bar{d} + s\bar{s})$

$\underline{8}$ ↑ Meson octet

 π^0, η, η'

$\eta' = \frac{1}{\sqrt{3}}(u\bar{u} + d\bar{d} + s\bar{s})$

$SU(3)_F$ singlet

Isospin singlet $\quad I = 0$

$\pi^0 = \frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d}) \qquad I = 1$

$\eta = \frac{1}{\sqrt{6}}(u\bar{u} + d\bar{d} - 2s\bar{s}) \qquad I = 0$

($\underline{2} \times \underline{2} = \underline{1} + \underline{3}$

$I = 0 \quad \underline{1} \quad |+-\rangle - |-+\rangle$

$I = 1 \quad \underline{3} \quad |++\rangle \quad |--\rangle \quad |+-\rangle + |-+\rangle$)

- 1965: Additional quantum number: COLOUR, to explain Δ^{++}

Δ^- ddd $\qquad \Delta^0$ udd $\qquad \Delta^+$ uud $\qquad \Delta^{++}$ uuu

Σ^{*-} dds $\qquad \Sigma^{*0}$ uds $\qquad \Sigma^{*+}$ uus

Ξ^{*-} dss $\qquad \Xi^{*0}$ uss

Ω^- sss

$\boxed{\Delta^{++}! \text{ with } S_z = \tfrac{3}{2}}$ spin $\frac{3}{2}$, made of 3 identical particles, with same spin $(|\tfrac{1}{2}\rangle|\tfrac{1}{2}\rangle|\tfrac{1}{2}\rangle)$ and symmetric spatial

~~Being a FERMION + PAULI EXCLUSION → ANTISYMMETRIC~~ ~~Not~~ wrt exchange

⇒ ~~COLOR QUANTUM NUMBER~~

$\Delta^{++}(s_z=\frac{3}{2})$ has fully symmetric wave function $|u^\uparrow, u^\uparrow, u^\uparrow\rangle$
BUT
Fermi-Dirac statistics $\Rightarrow$ total wave function antisymm.

$$|\psi(1,2,3)\rangle = -|\psi(2,1,3)\rangle$$

(in other words, not all constituents can be in the exact same state, because of the Pauli exclusion principle)

'64–'65

$\Rightarrow$ Extra quantum number: <u>Colour</u> $\psi_i^{(f)}$ $\leftarrow$ FRITZSCH/GELL-MANN/LEUTWYLER

Several evidences of $N_c = 3$ (□)
- GREENBERG
- HAN/NAMBU
- BOGOLIUBOV et AL.

For instance $u = \begin{vmatrix} u_1 \\ u_2 \\ u_3 \end{vmatrix} = \begin{vmatrix} u_r \\ u_g \\ u_b \end{vmatrix}$ ("RED" "GREEN" "BLUE")

$$|\Delta^{++}\rangle = \frac{1}{\sqrt{6}} \underbrace{\varepsilon_{ijk} |u_i^\uparrow \, u_j^\uparrow \, u_k^\uparrow\rangle}_{} \qquad i,j,k = 1 \dots N_c$$

antisymmetric for $i \leftrightarrow j$ ✓

Extra postulate:
- quarks are colour triplets
- $SU(3)_c$ is an EXACT SYMMETRY (→ it'll be a gauge symmetry)
- HADRONS ARE COLOUR SINGLET

↳ invariant under the action of $SU(3)_c$

$\boxed{\text{"COLOUR CONFINEMENT"}}$ (← can be understood qualitatively and quantitatively, see later)

(□) $\Delta^{++} \rightarrow N_c \geq 3$

R ratio $= \dfrac{e^+e^- \rightarrow \text{HADRONS}}{e^+e^- \rightarrow \mu^+\mu^-} \propto N_c \cdot \sum_f Q_f^2$

$\pi^0 \rightarrow \gamma\gamma$

Anomaly cancellation

In practice: I need to combine fundam. rep. of $SU(3)_c$ in such a way that singlets can be formed

$\Rightarrow qq \to \underline{3}_c \otimes \underline{3}_c = \underline{6}_s + \bar{\underline{3}}_c$ ×

$q\bar{q} \to \underline{3}_c \otimes \bar{\underline{3}}_c = \boxed{\underline{1}_c} + \underline{8}_c$

↳ ok!

$$|\text{Meson}\rangle \sim \frac{1}{\sqrt{3}} \delta_{ij} \bar{q}_i q'_j \qquad i,j \; SU(3)_c$$

→ invariant tensor

dim: $q : \psi_i \to U_{ij}\psi_j \qquad (U \in SU(3))$

$\bar{q} : \psi^*_i \to (U_{ij}\psi_j)^*$

$$\psi^*_i \psi_i \to \psi^{*\prime}_i \psi'_i = (U_{ij}\psi_j)^* (U_{ik}\psi_k) =$$

$$= \psi^*_j U^*_{ij} U_{ik} \psi_k =$$

$$= \psi^*_j \psi_k (U^\dagger)_{ji} U_{ik}$$

$$= \psi^*_j \psi_k (U^\dagger U)_{jk} = \psi^*_j \psi_j$$

$\Rightarrow qqq \to \underline{3}_c \otimes \underline{3}_c \otimes \underline{3}_c = \underline{10}_c \oplus \underline{8}_c \oplus \underline{8}_c + \boxed{\underline{1}_c}$

↳ ok!

$$|\text{Baryon}\rangle \sim \frac{1}{\sqrt{6}} \varepsilon_{ijk} q_i q'_j q''_k$$

dim:

$$\varepsilon_{ijk}\psi_i\psi_j\psi_k \to \varepsilon_{ijk}\psi'_i\psi'_j\psi'_k =$$

$$= \varepsilon_{ijk} U_{ii'} U_{jj'} U_{kk'} \psi_{i'}\psi_{j'}\psi_{k'}$$

$$= (\det U)\, \varepsilon_{i'j'k'} \psi_{i'}\psi_{j'}\psi_{k'} = \varepsilon_{ijk}\psi_i\psi_j\psi_k$$

⇒ other consequences:
Since $|\psi_{baryon}\rangle$ has to be totally antisymm in colour

and totally symmetric in "orbital" ⊗ "spin" ⊗ "flavour"

⇒ ~~[illegible]~~ In the "proton octet" and in the "Δ decuplet" the "flavour" part CANNOT BE ANTISYMMETRIC
Hence, I explain why I don't have the "flavour singlet" part (that, instead, I have in the meson octet)

~~[illegible]~~
- So far, we've discussed everything as if quarks are mathematical objects, through which we explain hadron prodn successfully

- Are they real?

- Late '60 DIS experiments at SLAC-MIT
$e^- p \to e^- X$

→ can be explained by assuming that electrons scatter elastically on free constituents of the protons "PARTONS"

↳ at high energy, somehow QCD must become weak → ASYMPT FREEDOM

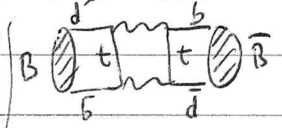
→ Exp evidence that spin of constituents ~~[illegible]~~ is $1/2$ [and evidence also of gluon content]

- charm : - GIM mechanism ~~[illegible]~~ 1970
  - to explain absence of FCNC, ~~[illegible]~~ introduce charm quark
  - Nov 1974, SLAC & BNL : charmonium ($c\bar{c}$ bound state) J/ψ

• bottom : - postulated in '73 by Kobayashi - Maskawa, to explain CP violation (need at least 3 quark generations)

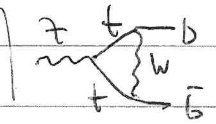
- 1977, at FNAL $\Upsilon = b\bar{b}$ (UPSILON MESON)
(E288)

• top : - expected from B-meson oscillation ~~method~~ and from EW precision measurement



$\Delta m_B \sim G_F^2 m_B |V_{td}|^2 m_t^2$

- 1995 CDF/D0, FNAL $p\bar{p} \to t\bar{t}$



$\sim m_t^2$

• ~~gluon: 3 jet events~~

• gluon 1979 ~~PETRA~~ e^+e^- @ HERA (DESY) : 3 jet events
PDF momentum sum rule
~~3 jet events~~

Other exp tests : quark spins : • DIS (Callan - Gross relation)
• 2 jet events in e^+e^-

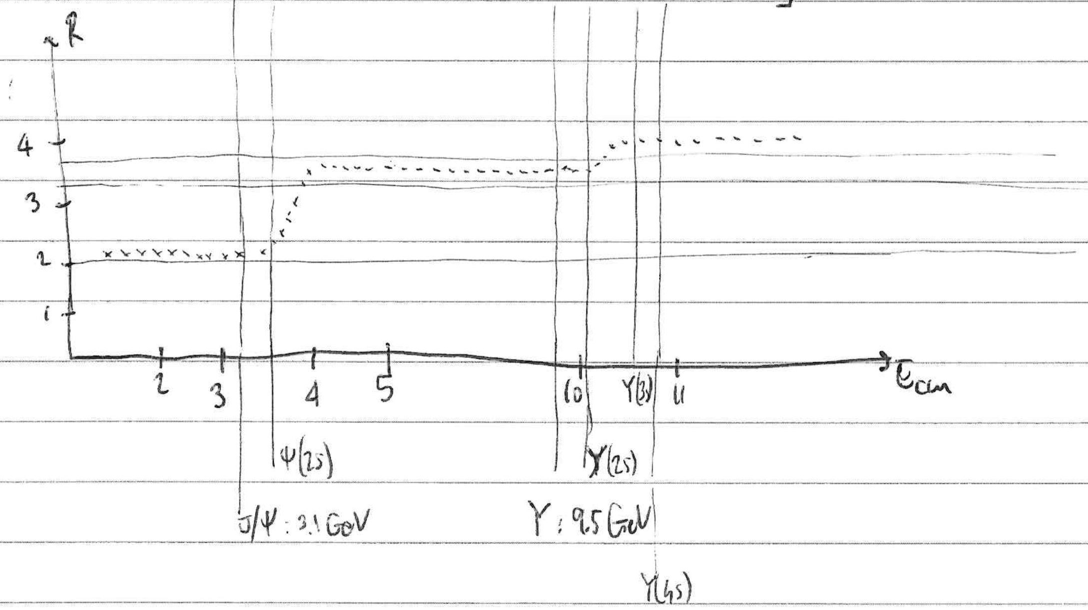
gluon spin : angles and energy distribution in $e^+e^- \to$ jets @ Z peak (DKS chapter 9)

Experimental evidence for colour

① R-ratio $R = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)}$ we are assuming that quarks/gluons → hadrons doesn't change σ ("PARTON-HADRON" DUALITY)

Far from Z peak $\sigma(e^+e^- \to \mu^+\mu^-) = \frac{4}{3}\pi \frac{\alpha_{em}^2}{s}$ $\quad F = \sqrt{1-\frac{4m_f^2}{s}}\left(1+\frac{2m_f^2}{s}\right)$

$$R_{e^+e^-} \sim \frac{N_c \sum_f \left[Q_f^2 \cdot F(s, m_f^2)\right]}{Q_\mu^2} = \begin{cases} \frac{2}{3}N_c & f = u, d, s \\ \frac{10}{9}N_c & f = u, d, s, c \\ \frac{11}{9}N_c & f = u, d, s, c, b \end{cases}$$

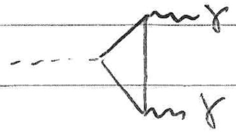


(slides G2)
(1709.04770 CAEVEN)

⇒ $N_c = 3$

⇒ By looking at the details, one also sees that data/TH agreement requires higher order perturbative effects.

② $\pi^0 \to \gamma\gamma$ decay [assuming u & d charges known, from static quark model or from DIS]



$$\Gamma(\pi^0 \to \gamma\gamma) = N_c^2 \left(Q_u^2 - Q_d^2\right)^2 \frac{\alpha_{em}^2}{64\pi^3} m_\pi^3 \frac{1}{f_\pi^2} = 7.63\ \text{eV} \left(\frac{N_c}{3}\right)^2$$

$\Gamma_{Exp} = 7.84$ eV $\qquad$ [Amplitude $\sim N_c$ ⇒ $|M|^2 \sim N_c^2$]

Lagrangian and Feynman rules

$\mathcal{L}_{QCD}$ is built assuming $SU(N_c)$ gauge-theory.
Quarks are in the fundamental represent. of $SU(N_c)$
There are 6 quarks: 3 generations of (u,d)-type quarks
$\begin{vmatrix} u \\ d \end{vmatrix} \quad \begin{vmatrix} c \\ s \end{vmatrix} \quad \begin{vmatrix} t \\ b \end{vmatrix}$

QCD is flavour blind (interaction is the same for $u, d, s, \dots$)

Gauge principle: interactions arise promoting a global symmetry to a local one. Such procedure determines the interaction (including the force carriers).

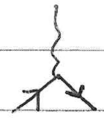
In QED: $\mathcal{L}_{QED} = \bar{\Psi}\left(i\gamma^\mu_\mu \partial_\mu - m\right)\Psi$ ← free Lagrangian for a fermion $\mathcal{L}^{(0)} = \mathcal{L}_{DIRAC}$ invariant under $U(1)$ global

$-\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$ $\quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ ← free E.M. field (Maxwell equation)

$-e\bar{\Psi}\gamma^\mu \Psi A_\mu$ ← interaction $(J^\mu A_\mu)$

• Feynman rules: $\Psi \xrightarrow{p} \bar{\Psi} = \dfrac{i}{p^2 - m^2}(\not{p} + m)$

$A_\mu \overset{p}{\sim\sim} A_\nu = \dfrac{-i g^{\mu\nu}}{p^2}$


$= -ie\gamma^\mu$ $\quad$ [NB $\langle \Psi(x)\bar{\Psi}(y)\rangle = \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\varepsilon} (\not{k} + m) e^{-ik(x-y)}$]

$\mathcal{L}$ can be obtained from $\mathcal{L}^{(0)}$ by requiring gauge invariance: $U(1)_{local}$

$$\begin{cases} \Psi(x) \to e^{i\alpha(x)} \Psi(x) \\ A_\mu \to A_\mu - \frac{1}{e} \partial_\mu \alpha(x) \end{cases}$$

leaves invariant $\mathcal{L} = \bar{\Psi}(i\not{D} - m)\Psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$

D^μ = cov. derivative $= \partial_\mu + ieA_\mu$

← INTERACTIONS FIXED

$\Rightarrow \mathcal{L}_{QED} = \mathcal{L}_{FF} + \mathcal{L}^{(0)}\big|_{cov\ deriv}$

$$\mathcal{L}_{F\tilde{F}} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} \qquad \mathcal{L}^{(0)}_{\partial\to D} = \bar{\psi}(i\not{\partial} - m)\psi$$

Ex: check gauge invariance of $\mathcal{L}_{QED}$

⇝ covariant derivative TRANSFORMS as the field itself, i.e.
$(D_\mu \psi) \to (D_\mu\psi)' = U(x)(D_\mu\psi)$, where $\psi(x) \to \psi'(x) = U(x)\psi(x)$
$U = \exp(i\alpha(x))$

$$\psi'(x) = e^{i\alpha(x)}\psi(x)$$

$$D_\mu \to D'_\mu = \partial_\mu + ieA'_\mu = \partial_\mu + ie\left(A_\mu - \frac{1}{e}\partial_\mu\alpha\right)$$

$$D'\psi' = \left(\partial_\mu + ieA_\mu - i\partial_\mu\alpha\right)\left(e^{i\alpha(x)}\psi(x)\right) =$$

$$= e^{i\alpha(x)}\left(\partial_\mu + ieA_\mu - i\partial_\mu\alpha\right)\psi(x) + (i\partial_\mu\alpha)e^{i\alpha(x)}\psi(x)$$

$$= e^{i\alpha(x)}\left(\partial_\mu + ieA_\mu\right)\psi = U(x)(D_\mu\psi)$$

⇝ $F_{\mu\nu}F$ is gauge invariant

QCD Lagrangian: can be obtained starting from $\mathcal{L}$ which is invariant under SU(N) global and then imposing gauge invariance

$$\mathcal{L}^{(0)} = \sum_f \sum_{ij=1}^{N_c} \bar{q}_i^{(f)}\left[i\gamma^\mu\partial_\mu - m^{(f)}\right]_{ij} q_j^{(f)}$$

← invariant under $SU(N_c)$ global
$U = \exp(i\,t^a\theta^a)$

Focus on 1 flavour only, colour index understood $\mathcal{L}^{(0)} = \bar{q}(i\gamma^\mu\partial_\mu - m)q$

I'll assume you are familiar with basic facts about compact Lie groups (SU(N) is a Lie group)

- $U \in SU(N)$ can be written as $U = \exp\{i\,\theta^a t^a\}$

$a = 1 \dots N_c^2 - 1$

t^a = generators of ~~[illegible]~~ fundam. rep. of SU(N)

θ^a: real parameters

~~I'll assume you are familiar with t^a hermitian traceless matrices $\{t^a\}_{a=1\dots N_c^2-1}$ basis~~

- U : $2N^2$ real numbers
- Imposing $UU^\dagger = U^\dagger U = 1$ and $\det U = 1$
  ~~fixes~~ $\rightarrow N^2+1$ equations

  $\rightarrow N^2-1$ remaining free parameters

- $U = \exp\{i\theta^a t^a\}$

  $UU^\dagger = 1 \rightarrow t^a = (t^a)^\dagger$

  $\det U = 1 \rightarrow \det(e^A) = e^{\mathrm{Tr}(A)}$

  $\rightarrow 1 = \exp\{\mathrm{Tr}(i\theta^a t^a)\} \Rightarrow \mathrm{Tr}(t^a) = 0$

- SU(3) explicit representation: $t^a = \frac{\lambda^a}{2}$ $\quad \lambda^a$ = Gell-Mann matrices

  $\mathrm{Tr}(t^a t^b) = T_R\, \delta^{ab}$ $\quad T_R = \frac{1}{2}$ is the standard normalization

- $[t^a, t^b] = i f^{abc} t^c$

  STRUCTURE CONSTANTS: - they are a fundamental object
  - these commutation rules have a meaning at an abstract level: they define the algebra of a Lie group

- Can get an idea of why f^{abc} algebra and an important looking at commutator

- Since group can be generated by exponentiating, one can expect that properties close to the identity give "complete" information about the group

$$[U(\delta_1), U(\delta_2)] = [1 + i\delta_1^a t^a + \dots, 1 + i\delta_2^b t^b] =$$

THIS IS SLOPPY, BUT WE DON'T NEED MORE

$$= (i\delta_1^a)(i\delta_2^b)[t^a, t^b] + \dots$$

$[t_a, t_b] \neq 0 \Rightarrow$ NON ABELIAN GROUP

Exercise : - ~~...~~ define

$$T^{ab} =: i[t^a, t^b]$$ (ab is the name of the matrix)

show that T^{ab}

has properties • $\mathrm{Tr}(T^{ab}) = 0$

• $(T^{ab})^\dagger = T^{ab}$

$\Rightarrow$ $[t^a, t^b] = i f^{abc} t^c$ ~~where f^{abc}~~

$\Rightarrow$ f totally antisymmetric

$\Rightarrow$ $f \in \mathbb{R}$

• A d-dim representation of the algebra is a set of d×d matrices that satisfy

$$[T^a, T^b] = i f^{abc} T^c$$

• The # of matrices is always N^2-1 : it has to match the dimension of the group. We're interested in 2 rep: FUNDAM and ADJOINT

~~$t^a = \frac{\lambda^a}{2}$, $a=1 \dots N^2-1$ is a N-dim rep.~~

• fund. rep - $\{t^a = \frac{\lambda^a}{2}\}_{a=1 \dots N^2-1}$ is a N-dim rep.

- acts on the space of N-dim vectors (ψ_i, $i=1 \dots N$)

• adjoint rep - $(T^a)_{bc} = i f^{bac}$ is a (N^2-1)-dim rep

- $[T^a, T^b] = i f^{abc} T^c$ can be verified by using Jacobi identity $[T^a,[T^b,T^c]] + \dots + \dots = 0$

For a non-abelian gauge group, the transf. is local:

- $U(x) = \exp\{ i t^a \theta^a(x) \}$

$$\partial_\mu \psi'(x) = \partial_\mu (U(x)\psi(x)) = U \partial_\mu \psi + (\partial_\mu U)\psi$$

- $(D_\mu)_{ij} = \partial_\mu \delta_{ij} + i g t^a_{ij} A^a_\mu$ ← COV. DERIVATIVE

GLUONS (force carriers)

and we want that

$$(D_\mu \psi) \to D'_\mu \psi' = U(x)(D_\mu \psi)$$

- this implies that

$$t^a A^a_\mu \to t^a A'^a_\mu = U(t^a A^a_\mu)U^{-1} + \frac{i}{g}(\partial_\mu U)U^{-1}$$ SHOW THIS

- Field strength tensor:

- By defining $\quad i g t^a F^a_{\mu\nu} = [D_\mu, D_\nu]$

(QED: $ieF^{QED}_{\mu\nu} = [D^{QED}_\mu, D^{QED}_\nu]$

$D_\mu = \partial_\mu + ieA_\mu$

$[D_\mu, D_\nu] = ie(\partial_\mu A_\nu - \partial_\nu A_\mu)$)

$$\Rightarrow \boxed{F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f^{abc} A^b_\mu A^c_\nu}$$ SHOW THIS

- property $\quad t^a F^a_{\mu\nu} \to t^a F'^a_{\mu\nu} = U(t^a F^a_{\mu\nu})U^{-1}$

⇒ kinetic term for gluons

$$-\frac{1}{4} F^{a\mu\nu} F^a_{\mu\nu} = \mathcal{L}_{YM}$$ (CLASSICAL YM Lagrangian)

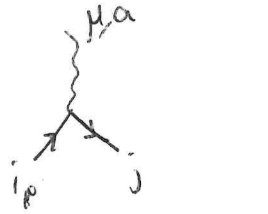
Exercise: to show that $\mathcal{L}_{YM}$ is gauge invariant, show that

$$F^{a\,\mu\nu} F^a_{\mu\nu} = 2\,\mathrm{Tr}(F^{\mu\nu} F_{\mu\nu})$$ where $F^{\mu\nu} = t^a F^a_{\mu\nu}$

$(\mathrm{Tr}(T^a T^b) = \frac{1}{2}\delta^{ab})$

$$\mathcal{L}_{QCD} = \mathcal{L}_{YM} + \mathcal{L}^{(0)}_{\partial \to D} = -\frac{1}{4} F^{a\,\mu\nu} F^a_{\mu\nu} + \sum_{jk} \bar{\psi}_j (i\not{D}_{jk} - m)\psi_k$$

• QCD Vertexes

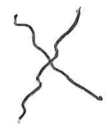


$$-ig\, t^a_{ji}\, \gamma^\mu \quad \longleftrightarrow \quad -g(\bar\psi\, t^a \gamma^\mu \psi) A^a_\mu$$

(spinorial indexes understood)

$$F^{\mu\nu a} F_{\mu\nu}{}^{a} \sim g\, f^{abc} (\partial_\mu A^a_\nu) A^{b\mu} A^{c\nu}$$

$$\sim g^2 f^{abe} \cdot f^{cde}\, A^{\mu a} A^{\nu b} A^c_\mu A^d_\nu$$



SELF INTERACTIONS !

derivation of 3-g vertex (TEXTBOOK WAY)

$$-\tfrac{1}{4} F^{\mu\nu a} F_{\mu\nu}{}^{a} = -\tfrac{1}{4}\left(\partial^\mu A^{\nu a} - \partial^\nu A^{\mu a} - g f^{abc} A^{b\mu} A^{c\nu}\right)\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f^{ade} A^d_\mu A^e_\nu\right)$$

$$= \frac{g}{4}\left[\left(f^{abc} A^{b\mu} A^{c\nu}\right)\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right) + \left(f^{ade} A^d_\mu A^e_\nu\right)\left(\partial^\mu A^{\nu a} - \partial^\nu A^{\mu a}\right)\right]$$

$$= \frac{g}{2} f^{abc}\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right) A^{b\mu} A^{c\nu} = \left\{ f^{abc} = \frac{f^{abc} - f^{acb}}{2} \text{, index relabel} \right\}$$

$$= g f^{abc} (\partial_\mu A^a_\nu)\left(A^{\mu b} A^{\nu c} - A^{\nu b} A^{\mu c}\right)$$

$$= g f^{abc} \left(\partial_\mu A^a_\alpha\right) A^b_\beta A^c_\gamma \left(g^{\alpha\gamma} g^{\mu\beta} - g^{\alpha\beta} g^{\mu\gamma}\right)$$

vertex : $(+i)\left\{ (g f^{abc}) \left(-ip^\mu \text{ of gluon a (incoming momentum)}\right)\left(g^{\alpha\gamma} g^{\mu\beta} - g^{\alpha\beta} g^{\mu\gamma}\right) \times \hat{A}^a_\alpha \hat{A}^b_\beta \hat{A}^c_\gamma \right.$

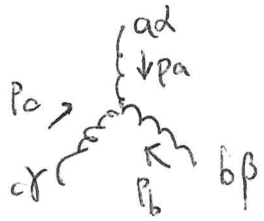
$\left. + \text{ cyclic permutations} \right\}$

factor $(+i)$: from $\exp\{+i\int d^4x\, \mathcal{L}_{int}\}$

$-ip^\mu$: F.T. of ∂_μ, recalling that $\partial_\mu \to i p_\mu$ IF OUT (created)

$\partial_\mu \to -i p_\mu$ IF IN (destroyed)

$\Rightarrow$ vertex



$$= +g f^{abc} \left\{ \left(p_a^\beta g^{\alpha\gamma} - p_a^\gamma g^{\alpha\beta}\right) + \text{permutations} \right\}$$

$$= g f^{abc} \left\{ g^{\alpha\beta}(p_b - p_a)^\gamma + g^{\beta\gamma}(p_c - p_b)^\alpha + g^{\gamma\alpha}(p_a - p_c)^\beta \right\}$$

$$= -g f^{abc} \left[g^{\alpha\beta}(p_a - p_b) + g^{\beta\gamma}(p_b - p_c) + g^{\gamma\alpha}(p_c - p_a)^\beta \right]$$

<u>derivation using more physical arguments (requiring gauge invariance)</u> see QCD notes / "Introduction to QCD" [Mangano]

<u>NB</u> 3g vertex can also be written "counterclockwise"

1st term above :

$$f^{abc} g^{\alpha\beta}(p_a - p_b)^\gamma = -f^{cba} g^{\alpha\beta}(p_a - p_b)^\gamma = +f^{cba} g^{\alpha\beta}(p_b - p_a)^\gamma$$

$$= +f^{bac}(p_b - p_a)^\gamma g^{\beta\alpha}$$

Propagators and gauge fixing

$\mathcal{L}$ is invariant under $SU(3)_c$ local

However, problem in formulating quantum theory:

- large degeneracy between field configurations $(A^a_\mu(x))$ (equivalent because of gauge symmetry)
- With functional quantization approach (Feynman path integral) these degenerate configurations give a divergence

$$Z = \int D[A_\mu] \exp\{ i S[A_\mu] \}$$

- all gauge configurations that are equivalent give same weight
- configurations where $A_\mu(x) = \frac{+i}{g}(\partial_\mu U)U^{-1}$, $U = \exp\{i\theta^a(x) t^a\}$ are equivalent to $A_\mu(x) = 0$

$\hookrightarrow$ $S[A_\mu] = 0$

- Faddeev-Popov '67 treatment: integrate away the physically equivalent configurations

$$\mathcal{L}_{GF} = -\frac{1}{2\lambda}\left[f(A)\right]^2$$

$$\mathcal{L}_{FP} = +\eta^{+} \frac{\delta f(A^\theta)}{\delta\theta}\eta \quad \propto \quad \left(= +\eta^{+a}(x) \underbrace{\frac{\delta f(A^a(\theta(x)))}{\delta\theta^b(y)}}_{M^{ab}} \eta^b(y) \right)$$

$\eta^a(x)$ $a = 1 \ldots N_c^2 - 1$: complex scalar fields obey Fermi statistics $\Rightarrow$ CANNOT BE PHYSICAL

~~2) • covariant gauges~~

~~$f(A) = (\partial_\mu A^{\mu a})^2$; $f(A') = \partial_\mu A'^{\mu a}$~~

~~To extract $f(A')$: $t^a A'^{\mu a} = U(t^a A^{\mu a})U^{-1} + \frac{i}{g}(\partial_\mu U)U^{-1}$~~

~~$t^a \partial_\mu A'^{\mu a} = (\partial_\mu U)(t^a A^{\mu a})U^{-1} + U(t^a \partial_\mu A^{\mu a})U^{-1} + U(t^a A^{\mu a})(\partial_\mu U^{-1})$~~

a) Covariant gauges (Lorentz gauge: $\partial_\mu A^\mu = 0$)

$$\mathcal{L}_{GF} = \frac{-1}{2\lambda}\left(\partial_\mu A^{\mu a}\right)^2$$

$$\mathcal{L}_{FP} = \partial_\mu \eta^{a+} \left[D^\mu\right]^{ab} \eta^b$$

acts on a N^2-1 space (η^b)

$$\left[D^\mu\right]^{ab} = \partial^\mu \delta^{ab} + ig\left[T^c\right]^{ab} A^{c\mu} =$$

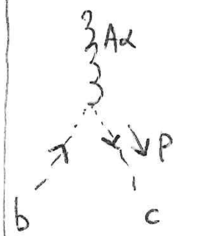
$$= \partial_\mu \delta^{ab} + ig\left(if^{acb}\right) A_\mu^{\mu c}$$

$$= \partial_\mu \delta^{ab} + g f^{abc} A_\mu^c$$

$$\Rightarrow \mathcal{L}_{FP} = \left(\partial_\mu \eta^{a+}\right)\left(\partial^\mu \eta_a\right) + \partial_\mu \eta^{a+} g f^{abc} \eta^b A^\mu_c$$

$\Rightarrow$

$a \dashrightarrow b = \delta^{ab} \dfrac{i}{p^2 + i\varepsilon}$



$= g f^{abc} p^\alpha$

- GHOSTS COUPLED TO GLUONS
- propagator, see next page
- In QED, $f^{abc} = 0$ $\Rightarrow$ no ghosts (fully decoupled)

FACTOR -1 IN LOOPS

b) axial gauge ($n \cdot A = 0$)

$$\mathcal{L}_{GF} = -\frac{1}{2\lambda}\left(n_\mu A^{\mu a}\right)^2 \qquad n \cdot p \neq 0$$

$$\mathcal{L}_{FP} = \eta_a^+ \left[\delta^{ab} n_\mu \partial^\mu + g f^{abc} \underbrace{n_\mu A^{\mu c}}_{=0}\right] \eta_b$$

- propagator, see next page
- $n \cdot A = 0$ $\Rightarrow$ no ghosts (can be reabsorbed)

q and g propagators from $\mathcal{L}_{free}$

$$\mathcal{L}_{q,\bar\Psi\Psi} = \bar\Psi\,(i\not\partial - m)\,\Psi$$

$\downarrow$

F.T. ; $\tilde G(p)$ "propagator = $i \times$ inverse"

$$\tilde G = \frac{i}{\not p - m} = \frac{i}{p^2-m^2}(\not p + m)$$

$$\mathcal{L}_{YM,AA} = \frac{-1}{4}\left(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a\right)\left(\partial^\mu A^{\nu a} - \partial^\nu A^{\mu a}\right)$$

$$= -\frac{1}{2}\left[\left(\partial_\mu A_\nu^a\right)\left(\partial^\mu A^{\nu a}\right) - \left(\partial_\mu A_\nu^a\right)\left(\partial^\nu A^{\mu a}\right)\right]$$

$$= -\frac{1}{2}\left[-A^{\nu a}\,\Box A_\nu^a + A^{\mu a}\,\partial_\mu\partial_\nu A^{\nu a}\right]$$

$$= \frac{1}{2}A^{\mu a}\left[\Box g_{\mu\nu} - \partial_\mu\partial_\nu\right]A^{\nu a}$$

$\Box g_{\mu\nu} - \partial_\mu\partial_\nu$ can't be inverted :

$$\left(-k^2 g_{\mu\nu} + k_\mu k_\nu\right)k^\nu = 0 \quad , \text{ for } k^\nu \neq 0$$

GAUGE FIXING

- Covariant [LORENZ GAUGE]

$$\mathcal{L}_{GF} = -\frac{1}{2\lambda}\left(\partial_\mu A^{a\nu}\right)^2 = -\frac{1}{2\lambda}\left(\partial_\mu A^{a\mu}\right)\left(\partial_\nu A^{a\nu}\right) = +\frac{1}{2\lambda}A^{a\mu}\partial_\mu\partial_\nu A^{a\nu}$$

$$\Rightarrow \mathcal{L}_{YM,AA} + \mathcal{L}_{GF} = \frac{1}{2}A^{\mu a}\left[\Box g_{\mu\nu} - \left(1-\frac{1}{\lambda}\right)\partial_\mu\partial_\nu\right]A^{\nu a}$$

$$\left(-k^2 g_{\mu\nu} + \left(1-\frac{1}{\lambda}\right)k_\mu k_\nu\right)k^\nu \neq 0 \quad , \text{ for } k^\nu \neq 0$$

$$\tilde G_{\mu\nu} = \frac{+i}{k^2}\left(-g_{\mu\nu} + (1-\lambda)\frac{k_\mu k_\nu}{k^2}\right)$$

$\lambda = 1$ Feynman gauge

$\lambda = 0$ Landau gauge

- axial (also called physical gauge) $\mathcal{L}_{GF} = -\frac{1}{2\lambda}\left(n_\mu A^{a\mu}\right)^2$

$$\Rightarrow \mathcal{L}_{YM} + \mathcal{L}_{GF} = \frac{1}{2} A^{\mu a}\left[\Box g_{\mu\nu} - \partial_\mu \partial_\nu - \frac{1}{\lambda} n_\mu n_\nu\right] A^{\nu a}$$

$$\left(-k^2 g_{\mu\nu} + k_\mu k_\nu - \frac{1}{\lambda} n_\mu n_\nu\right) k^\nu \neq 0 \quad \text{for } k^\nu \neq 0 \qquad (n \cdot k \neq 0)$$

$$\tilde{G}_{\mu\nu} = \frac{+i}{k^2}\left[-g_{\mu\nu} + \frac{(nk+kn)_{\mu\nu}}{(nk)} - \frac{(n^2 + \lambda k^2) k_\mu k_\nu}{(nk)^2}\right]$$

- $\{n^2 = 0, \lambda = 0\} \to$ light-cone gauge

Comments:

- Gauge fixing breaks GAUGE INVARIANCE in $\mathcal{L}$
  However physical result are indep. of gauge choice

- cov. gauges $\to$ Lorentz invariance
  axial gauge $\to$ explicit arbitrary direction $\boxed{n^\mu}$

- # of propagating D.O.F.

  cov. gauges $\tilde{G}_{\mu\nu} = \frac{i}{k^2} d_{\mu\nu}$ $\qquad d_{\mu\nu} \neq \sum_{PHYS} \epsilon_\mu(k)\, \epsilon^*_\nu(k)$

  axial gauges $\tilde{G}_{\mu\nu} = \frac{i}{k^2} d_{\mu\nu}$ $\qquad d_{\mu\nu} = \sum_{PHYS} \epsilon_\mu(k)\, \epsilon^*_\nu(k)$

  - AXIAL GAUGE also called physical gauges : $d_{\mu\nu} k^\nu = 0$, $d_{\mu\nu} n^\nu = 0 \Rightarrow$ 2 DOF propag
    $\Rightarrow$ USEFUL IF PHYSICAL ARGUMENT NEEDED
  - COV. GAUGE $(\lambda = 1)$ $d_{\mu\nu} = -g_{\mu\nu} \Rightarrow 4$ DOF propagating
    $\Rightarrow$ GHOSTS CANCEL THE UNPHYSICAL DOF

Exercise: show that, in light-cone gauge, $d_{\mu\nu} = \begin{pmatrix} 0 & & & \\ & 1 & & \\ & & 1 & \\ & & & 0 \end{pmatrix}$ if $k^\mu = (E\,0\,0\,E)$

(just need to choose $n^\mu = (E\,0\,0\,-E)$): $nk = 2E^2$

$$d_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & +1 & & \\ & & +1 & \\ & & & +1 \end{pmatrix} + \frac{1}{2E^2}\begin{pmatrix} 2E^2 & & & \\ & 0 & & \\ & & 0 & \\ & & & -2E^2 \end{pmatrix} = \begin{pmatrix} 0 & & & \\ & 1 & & \\ & & 1 & \\ & & & 0 \end{pmatrix}$$