

COLOUR ALGEBRA AND COLOUR FACTORS

The \mathbb{O} s vertices contain

$$t^a$$

$$f^{abc}$$

• When computing amplitudes/amplitude squared, we need to sum over colours

• No need to use explicit forms of t^a and f^{abc}

1) we have 2 representations

$$\Gamma^a = t \quad (\Gamma^a)_{ij} = t^a_{ij}$$

$$\Gamma^a = T \quad (\Gamma^a)_{bc} = if^{bac}$$

$$\Gamma^2 =: \Gamma^a \Gamma^a$$

one can show that $[\Gamma^a, \Gamma^b] = 0 \quad \forall a, b$

\Rightarrow there is a theorem that states that $\Gamma^2 \sim \text{Id}$

2) Γ^2 is also called a Casimir of the group

$$\bullet \text{fund: } (\Gamma^2)_{ik} = t^a_{ij} t^a_{jk} \stackrel{!}{=} C_F \mathbb{1}_{ik}$$

$$\bullet \text{adj: } (\Gamma^2)_{ad} = T^b_{ac} T^b_{cd} = if^{abc} if^{cbd} \stackrel{!}{=} C_A \mathbb{1}_{ad}$$

3) CASIMIRS appear in 2 diagrams (and also in the squared amplitudes)

$$k \xrightarrow{\text{blob}} i \quad t_{ij}^a t_{jk}^a = C_F \delta_{ik}$$

C_F vs. C_A
q. vs g radiating

$$d \xrightarrow{\text{blob}} a \quad (-) \int_{abc} \int_{bdc} = \int_{abc} \int_{cbd} = C_A \delta_{ad}$$

↑
incoming/outgoing momenta

4) Fierz identity

$$\delta_{ij} \delta_{lk} = \frac{1}{N} \delta_{ik} \delta_{lj} + 2 t_{lj}^a t_{ik}^a$$

graphically

$$\begin{array}{c} j \rightarrow i \\ l \leftarrow k \end{array} = \frac{1}{N} \begin{array}{c} j \\ \downarrow \\ l \end{array} \begin{array}{c} i \\ \uparrow \\ k \end{array} + 2 \begin{array}{c} j \\ \searrow \\ l \end{array} \begin{array}{c} i \\ \swarrow \\ k \end{array}$$

5) $\text{Tr}(t^a) = 0$

graphically $\begin{array}{c} \updownarrow \\ i \end{array} \rightarrow \bigcirc = 0$

$$\bigcirc = \sum_i \delta_{ii} = N$$

$$\text{blob} = \sum_a \delta_{aa} = N^2 - 1$$

With the above rules, we can do many computations just graphically and/or using basic combinatorics rules

① C_F

$$e \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} k = \frac{1}{N} \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} k + 2 \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} k$$

$$N \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} k = \frac{1}{N} \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} k + 2 \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} k$$

$$\begin{array}{c} \text{---} \\ \text{---} \end{array} = \frac{1}{2} \left(N - \frac{1}{N} \right) \longrightarrow \Rightarrow C_F = \frac{N-1}{2N}$$

② $\begin{array}{c} \text{---} \\ \text{---} \end{array} = \frac{-1}{2N} \begin{array}{c} \text{---} \\ \text{---} \end{array} = \left(C_F - \frac{C_A}{2} \right) \begin{array}{c} \text{---} \\ \text{---} \end{array}$ (last equality from next point)

$$\begin{array}{c} \text{---} \\ \text{---} \end{array} = \frac{1}{N} \begin{array}{c} \text{---} \\ \text{---} \end{array} + 2 \begin{array}{c} \text{---} \\ \text{---} \end{array}$$

$$0 = \frac{1}{N} \begin{array}{c} \text{---} \\ \text{---} \end{array} + 2 \begin{array}{c} \text{---} \\ \text{---} \end{array}$$

③ Using ① and ② prove that $C_A = N$

$$\begin{array}{c} \text{---} \\ \text{---} \end{array} = \text{if}^{abc} \text{if}^{cbd} t^d = C_A \delta^{ad} t^d$$

$$\text{if}^{cbd} t^d = [t^c, t^b]$$

⋮

④ Show that $t^a t^c \text{if}^{abc} = \frac{C_A}{2} t^b$ and hence that

$$\begin{array}{c} \text{---} \\ \text{---} \end{array} = \frac{C_A}{2} \begin{array}{c} \text{---} \\ \text{---} \end{array}$$

$$t^a t^c (\text{if}^{abc}) = \left[\frac{1}{2} [t^a, t^c] + \frac{1}{2} \{t^a, t^c\} \right] \text{if}^{abc}$$