COLOUR ALGEBRA AND COLOUR FACTORS

- The QCD vertices contain

$$t^a$$

$$f^{abc}$$

- When computing amplitudes/amplitude squared, we need to sum over colours
- No need to use explicit forms of t^a and f^{abc}

1) we have 2 representations

$$\mathbb{T} = t \qquad (\mathbb{T}^a)_{ij} = t^a_{ij}$$

$$\mathbb{T} = T \qquad (\mathbb{T}^a)_{bc} = i f^{bac}$$

$$\mathbb{T}^2 =: \mathbb{T}^a \mathbb{T}^a$$

One can show that $[\mathbb{T}^2, \mathbb{T}^b] = 0 \quad \forall b$

$\Rightarrow$ there is a theorem that states that $\mathbb{T}^2 \sim \mathrm{Id}$

2) $\mathbb{T}^2$ is also called a CASIMIR of the group

- fund: $(\mathbb{T}^2)_{ik} = t^a_{ij} t^a_{jk} \overset{!}{=} C_F \mathbb{1}_{ik}$

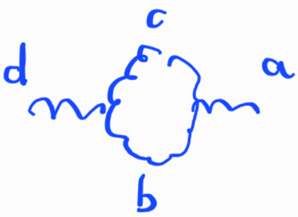
- adj: $(\mathbb{T}^2)_{ad} = T^b_{ac} T^b_{cd} = i f^{abc} \, i f^{cbd} \overset{!}{=} C_A \mathbb{1}_{ad}$

3) CASIMIRS appear in 2 diagrams (and also in the squared amplitudes)

$t^a_{ij}\, t^a_{jk} = C_F\, \delta_{ik}$

C_F vs. C_A

q vs g radiating



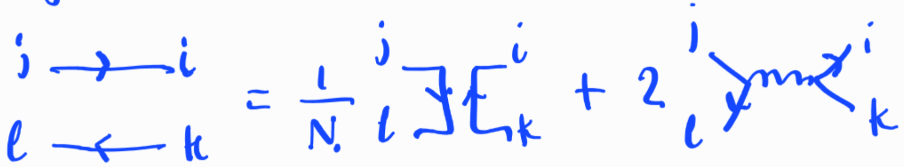
$(-) f^{abc} f^{bdc} = i f^{abc}\, i f^{cbd} = C_A\, \delta_{ad}$

incoming/outgoing momenta

4) Fierz identity

$$\delta_{ij}\, \delta_{\ell k} = \frac{1}{N}\, \delta_{ik}\, \delta_{\ell j} + 2\, t^a_{\ell j}\, t^a_{ik}$$

graphically



5) $\mathrm{Tr}(t^a) = 0$

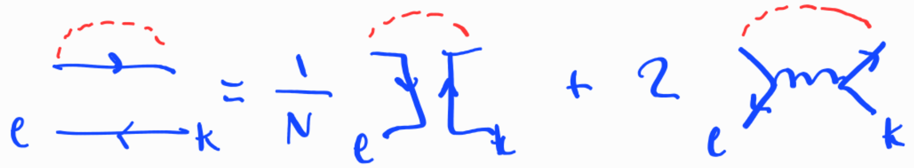
graphically

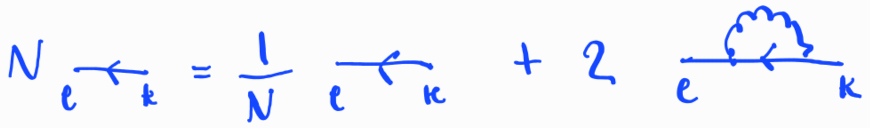
 $= \sum_i \delta_{ii} = N$

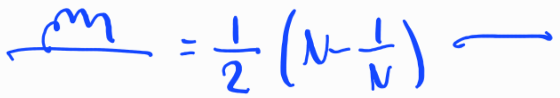
 $= \sum_a \delta_{aa} = N^2 - 1$

With the above rules, we can do many computations just graphically and/or using basic commutator rules

ⓐ C_F



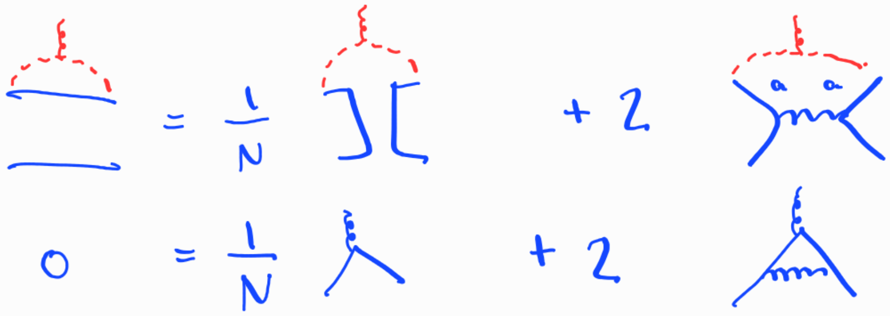




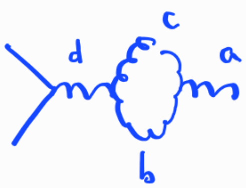
$\Rightarrow C_F = \frac{N^2-1}{2N}$

ⓑ

(last equality from next point)



ⓒ Using ⓐ and ⓑ prove that $C_A = N$

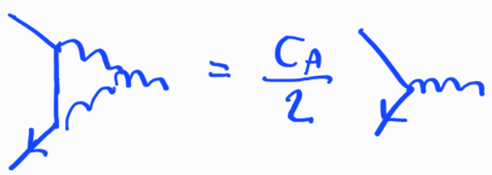


$$= i f^{abc}\, i f^{cbd}\, t^d \overset{!}{=} C_A\, \delta^{ad}\, t^d$$

$$i f^{cbd} t^d = [t^c, t^b]$$

$\vdots$

ⓓ Show that $t^a t^c\, i f^{abc} = \frac{C_A}{2} t^b$ and hence that



$$t^a t^c (i f^{abc}) = \left[\frac{1}{2}[t^a, t^c] + \frac{1}{2}\{t^a, t^c\}\right] i f^{abc}$$