Lecture 3

Feynman rules of QED

	particle annihilation	antiparticle creation

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\boldsymbol{p}}}} \sum_s \left[a_{\boldsymbol{p}}^s u^s(p) e^{-ip\cdot x} + b_{\boldsymbol{p}}^{s\dagger} v^s(p) e^{ip\cdot x} \right]$$

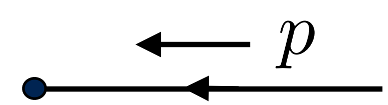
$$\bar{\psi}(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\boldsymbol{p}}}} \sum_s \left[b_{\boldsymbol{p}}^s \bar{v}^s(p) e^{-ip\cdot x} + a_{\boldsymbol{p}}^{s\dagger} \bar{u}^s(p) e^{ip\cdot x} \right]$$

	antiparticle annihilation	particle creation

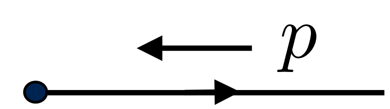
$$A_\mu(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\boldsymbol{p}}}} \sum_r \left[a_{\boldsymbol{p}}^r \epsilon_\mu^r(p) e^{-ip\cdot x} + a_{\boldsymbol{p}}^{r\dagger} \epsilon_\mu^{r*}(p) e^{ip\cdot x} \right]$$

	annihilation	creation

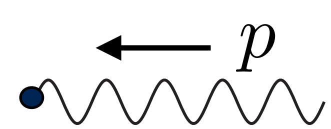
External line (incoming)



$= u^s(p)$ particle annihilation

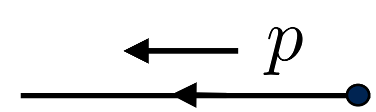


$= \bar{v}^s(p)$ antiparticle annihilation

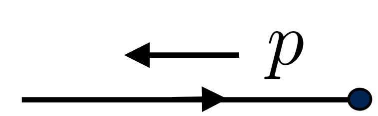


$= \epsilon_\mu(p)$ photon annihilation

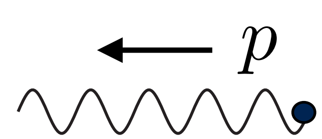
External line (outgoing)



$= \bar{u}^s(p)$ particle creation

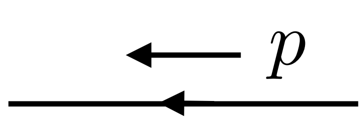


$= v^s(p)$ antiparticle creation

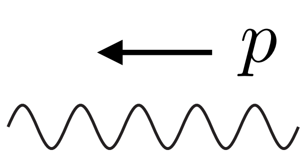


$= \epsilon_\mu^*(p)$ photon creation

Propagator



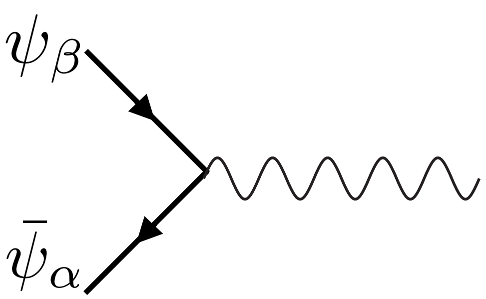
$$= \frac{i(\not{p} + m_f)_{\alpha\beta}}{p^2 - m_f^2 + i\epsilon}$$



$$= \frac{-ig_{\mu\nu}}{p^2 + i\epsilon}$$

Feynman gauge
ξ =1

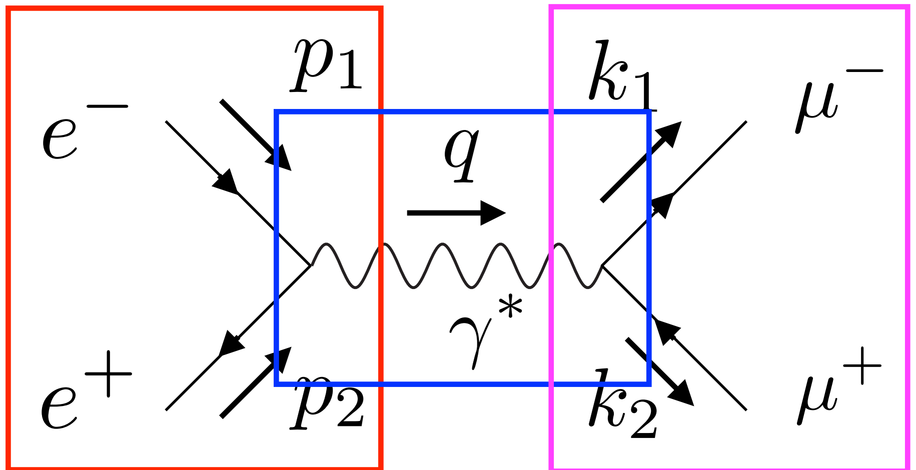
Vertex



$= ie(\gamma^\mu)_{\alpha\beta}$

$e^+e^- \rightarrow \mu^+\mu^-$

Let's consider a simple QED process.



$$q = p_1 + p_2 = k_1 + k_2$$

$q^2 \neq 0$ photon cannot be on-shell (virtual)

Amplitude

$$i\mathcal{M} = \left[\bar{u}^r_{\mu^-}(k_1)(+ie\gamma^\mu)v^{r'}_{\mu^+}(k_2)\right]\frac{-ig_{\mu\nu}}{q^2 + i\epsilon}\left[\bar{v}^{s'}_{e^+}(p_2)(+ie\gamma^\nu)u^s_{e^-}(p_1)\right]$$

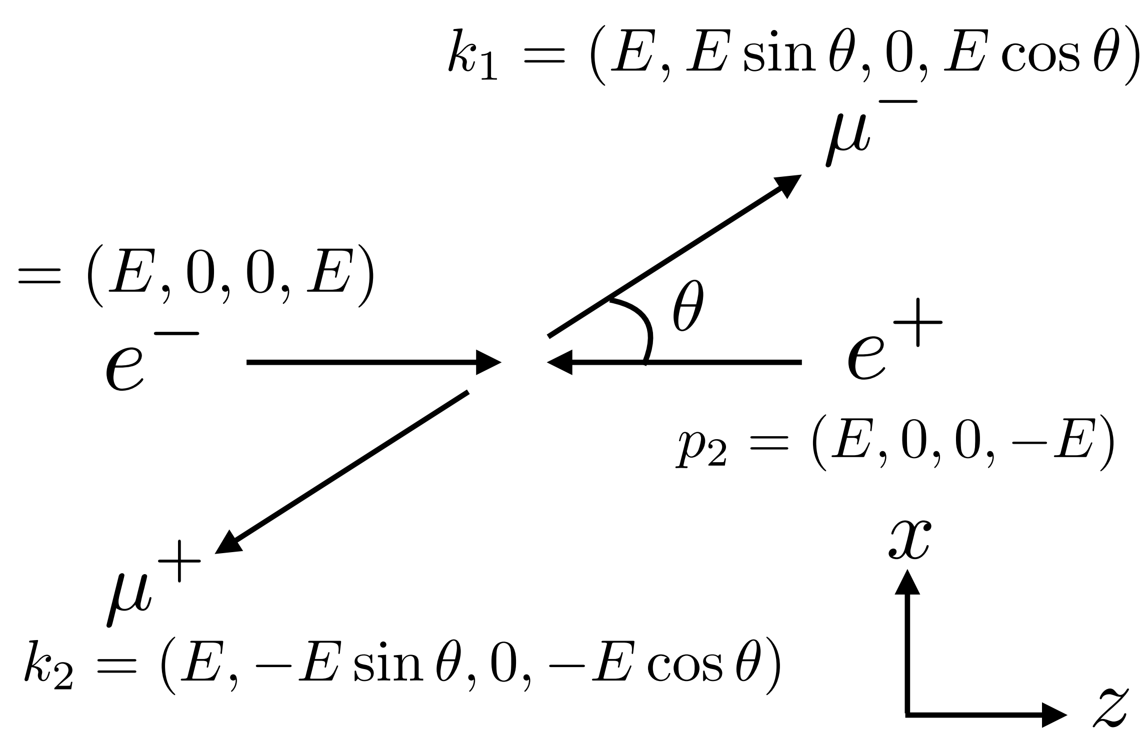
$$\overline{\sum_{\text{spins}}}|\mathcal{M}|^2 = \frac{1}{2}\cdot\frac{1}{2}\sum_{s,s',r,r'}|\mathcal{M}|^2 \qquad \sum_s u^s(p)\bar{u}^s(p) = \not{p} + m, \quad \sum_s v^s(p)\bar{v}^s(p) = \not{p} - m$$

Differential cross section

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s}\overline{\sum_{\text{spins}}}|\mathcal{M}|^2 \underset{E\gg m_\mu}{=} \frac{\alpha^2}{4s}(1+\cos^2\theta)$$

$$s = E^2_{\text{CM}} = 4E^2$$

$$\sigma = \frac{\alpha^2}{4s}\int_0^{2\pi}d\phi\int_{-1}^{1}d\cos\theta\ (1+\cos^2\theta) = \frac{4\pi\alpha^2}{3s}$$

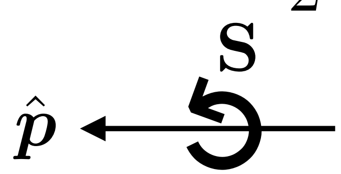


Chirality & Helicity

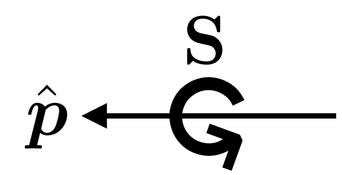
4 component spinor can be expressed by chirality and helicity.

<u>Chirality</u> eigenvalues of γ_5 $\gamma_5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ $\gamma_5 \psi_R = +\psi_R, \quad \gamma_5 \psi_L = -\psi_L$

<u>Helicity</u> $h \equiv \hat{\boldsymbol{p}} \cdot \mathbf{S} = \frac{1}{2} \hat{p}_i \begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}$ $\hat{\boldsymbol{p}}$: unit momentum vector, $\mathbf{S}$: spin

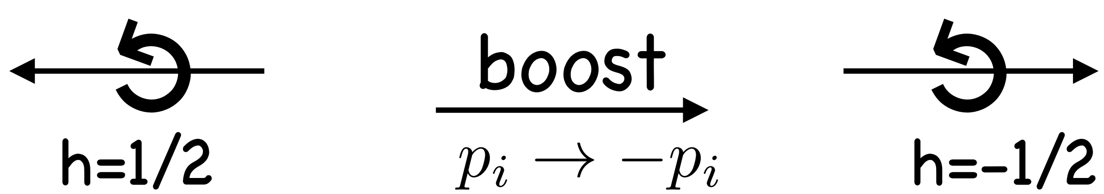


h=+1/2 right-handed (RH)



h=-1/2 left-handed (LH)

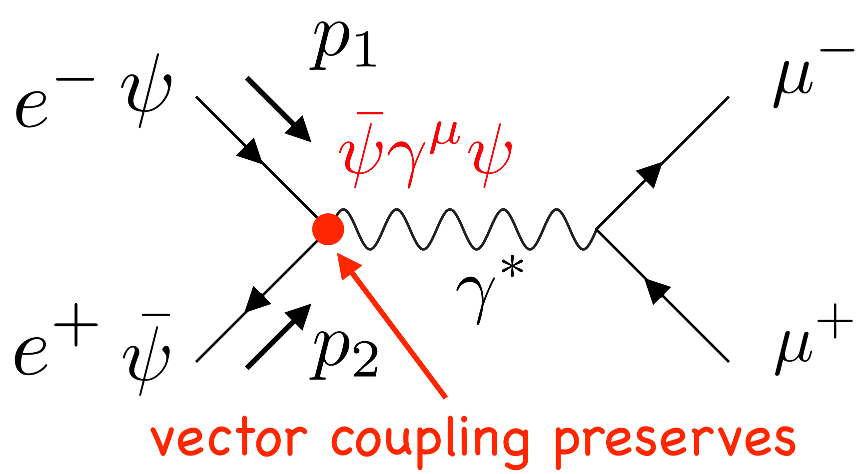
- For massive particles, helicity is frame dependent!



- In the high-energy limit (E>>m) or massless limit (m=0),

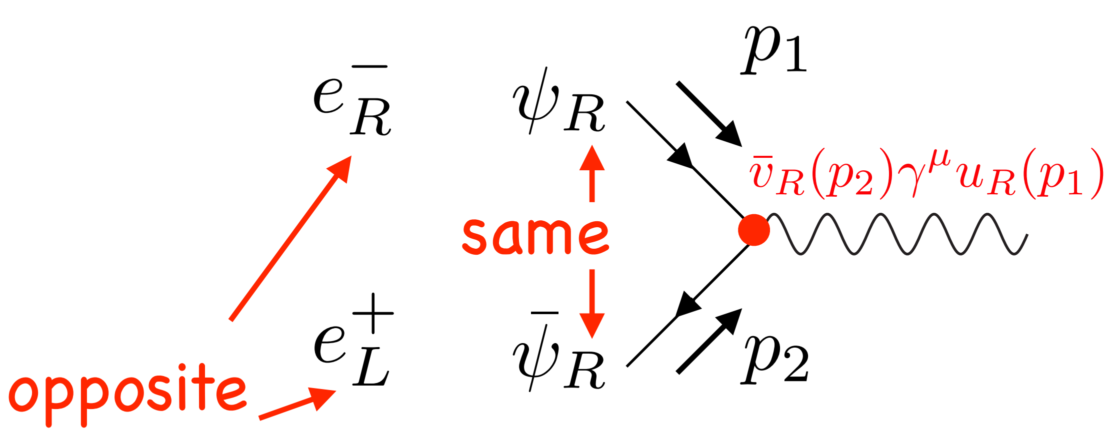
helicity (x2) = chirality particles

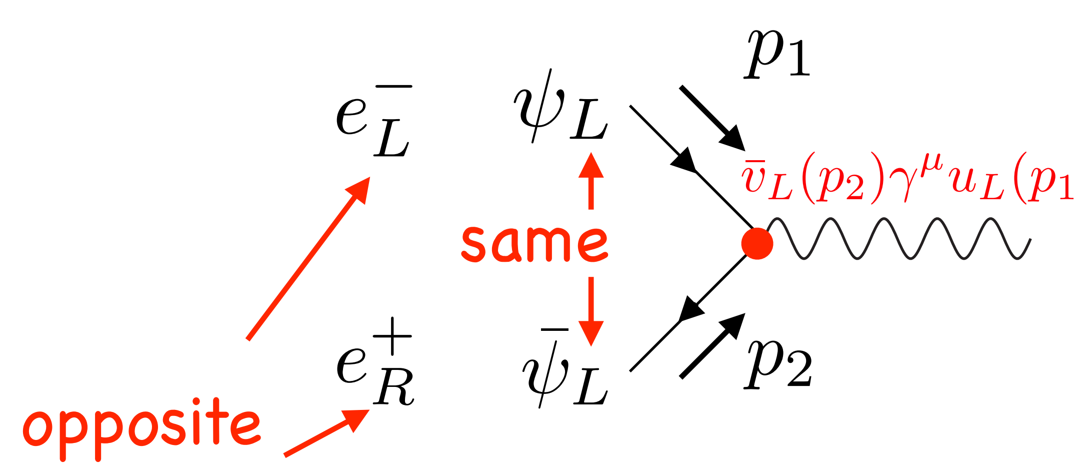
helicity (x2) = -chirality antiparticles e.g., $e^+e^- \to \gamma^* \to \mu^+\mu^-$ Ref. Sec.5.2 in Peskin-Schroeder's book

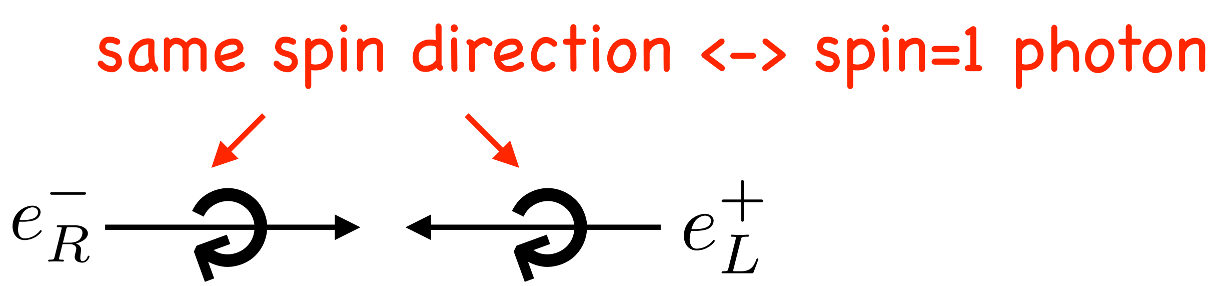


$\bar{\psi}\gamma^\mu\psi = \bar{\psi}_R\gamma^\mu\psi_R + \bar{\psi}_L\gamma^\mu\psi_L$

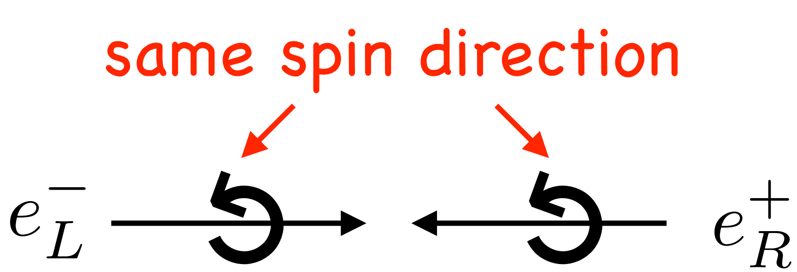
In the high-energy limit (massless limit),





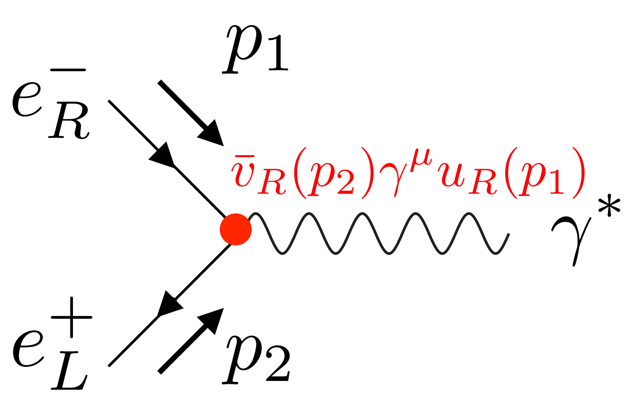


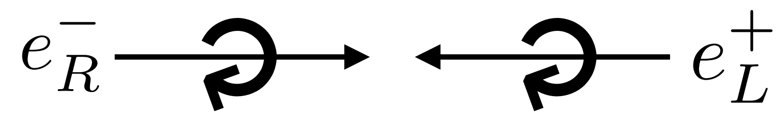
$\bar{v}^{(-)}_{e^+R}(p_2)\gamma^\mu u^{(+)}_{e^-R}(p_1)$

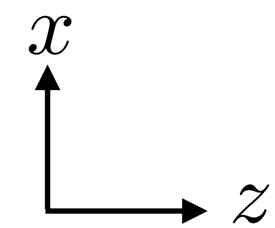


$\bar{v}^{(+)}_{e^+L}(p_2)\gamma^\mu u^{(-)}_{e^-L}(p_1)$

$e^-_R e^+_L \to \gamma^*$







$$\bar{v}^{(-)}_{e^+R}(p_2)\gamma^\mu u^{(+)}_{e^-R}(p_1) = -2E\begin{pmatrix}0\\1\\i\\0\end{pmatrix}$$

what is this??

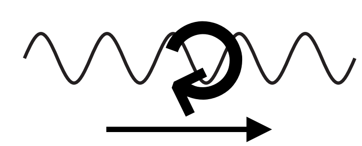
Recall that photon polarizations in the z direction (helicity eigenstates).

$$J^3\epsilon^\mu_+ = \epsilon^\mu_+, \quad J^3\epsilon^\mu_- = -\epsilon^\mu_-$$

$$J^3(= M^{12}) = \begin{pmatrix}1 & 0 & 0 & 0\\0 & 0 & -i & 0\\0 & i & 0 & 0\\0 & 0 & 0 & 0\end{pmatrix}$$

$$\epsilon^\mu_+ = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\i\\0\end{pmatrix} \qquad \epsilon^\mu_- = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\-i\\0\end{pmatrix}$$

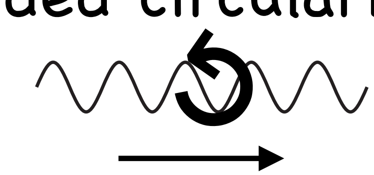
$$\therefore\ \bar{v}^{(-)}_{e^+R}(p_2)\gamma^\mu u^{(+)}_{e^-R}(p_1) = -2E\sqrt{2}\epsilon^\mu_+$$ (right-handed circularly polarized)



$e^-_L e^+_R \to \gamma^*$

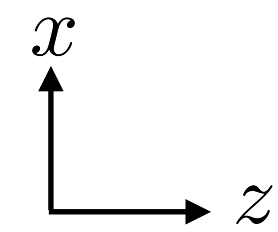
$$\bar{v}^{(+)}_{e^+L}(p_2)\gamma^\mu u^{(-)}_{e^-L}(p_1) = -2E\begin{pmatrix}0\\1\\-i\\0\end{pmatrix} = -2E\sqrt{2}\epsilon^\mu_-$$

(left-handed circularly polarized)



$\theta = \theta,\ \phi = 0$

$$\xi^{(+)} = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix},\ \xi^{(-)} = \begin{pmatrix} -\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \end{pmatrix}$$

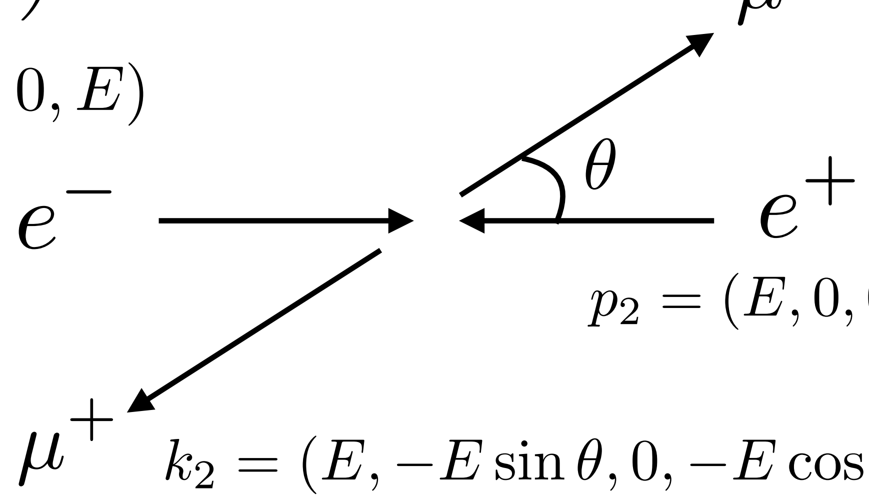


$\theta = 0,\ \phi = 0$

$$\xi^{(+)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ \xi^{(-)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$k_1 = (E, E\sin\theta, 0, E\cos\theta)$

$p_1 = (E, 0, 0, E)$



$\theta = \pi,\ \phi = 0$

$$\xi^{(+)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},\ \xi^{(-)} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

$p_2 = (E, 0, 0, -E)$

$\underline{\theta = \pi - \theta,\ \phi = \pi}$

$$\xi^{(+)} = \begin{pmatrix} \sin\frac{\theta}{2} \\ -\cos\frac{\theta}{2} \end{pmatrix},\ \xi^{(-)} = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix}$$

$k_2 = (E, -E\sin\theta, 0, -E\cos\theta)$

$\mu^-_R \mu^+_L$

$$\bar{u}^{(+)}_{\mu^- R}(k_1)\gamma^\mu v^{(-)}_{\mu^+ R}(k_2) = 2E \begin{pmatrix} 0 \\ \cos\theta \\ -i \\ -\sin\theta \end{pmatrix}$$

$\mu^-_L \mu^+_R$

$$\bar{u}^{(-)}_{\mu^- L}(k_1)\gamma^\mu v^{(+)}_{\mu^+ L}(k_2) = 2E \begin{pmatrix} 0 \\ \cos\theta \\ i \\ -\sin\theta \end{pmatrix}$$

parity invariance (L ↔ R)

$$\mathcal{M}(e^-_R e^+_L \to \mu^-_R \mu^+_L) = \mathcal{M}(e^-_L e^+_R \to \mu^-_L \mu^+_R) = e^2(1 + \cos\theta)$$

$$\mathcal{M}(e^-_R e^+_L \to \mu^-_L \mu^+_R) = \mathcal{M}(e^-_L e^+_R \to \mu^-_R \mu^+_L) = e^2(1 - \cos\theta)$$

parity invariance (L ↔ R)

$$\longrightarrow \boxed{\frac{d\sigma}{d\Omega} \underset{E \gg m_\mu}{=} \frac{\alpha^2}{4s}(1 + \cos\theta^2)}$$

(same as before)

Cancellation of ξ

Amplitude

$$\mathcal{M} = e^2 \left[\bar{u}^r_{\mu^-}(k_1) \gamma^\mu v^{r'}_{\mu^+}(k_2) \right] \frac{1}{q^2 + i\epsilon} \left(g_{\mu\nu} - (1 - \boxed{\xi}) \frac{q_\mu q_\nu}{q^2} \right) \left[\bar{v}^{s'}_{e^+}(p_2) \gamma^\nu u^s_{e^-}(p_1) \right]$$

$$\bar{v}^{s'}_{e^+}(p_2) \not{q} u^s_{e^-}(p_1) = \bar{v}^{s'}_{e^+}(p_2)(\not{p}_1 + \not{p}_2) u^s_{e^-}(p_1) = \bar{v}^{s'}_{e^+}(p_2)(m - m) u^s_{e^-}(p_1) = 0$$

EOMs

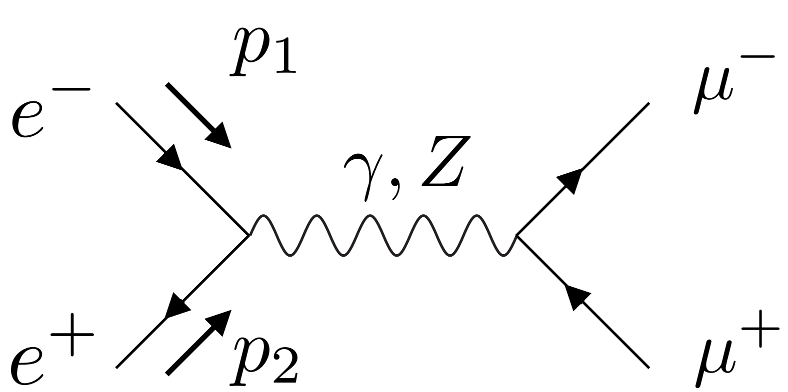
$$(\not{p} - m) u^s(p) = \bar{u}^s(p)(\not{p} - m) = 0,$$
$$(\not{p} + m) v^s(p) = \bar{v}^s(p)(\not{p} + m) = 0$$

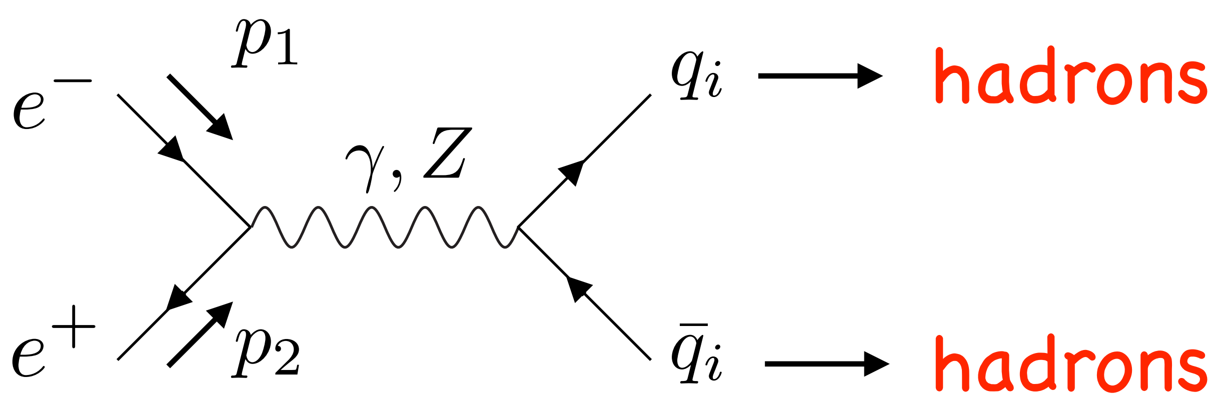
Therefore, the ξ dependent term drops.

R-ratio

R-ratio (Drell ratio)

$$R = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)}$$



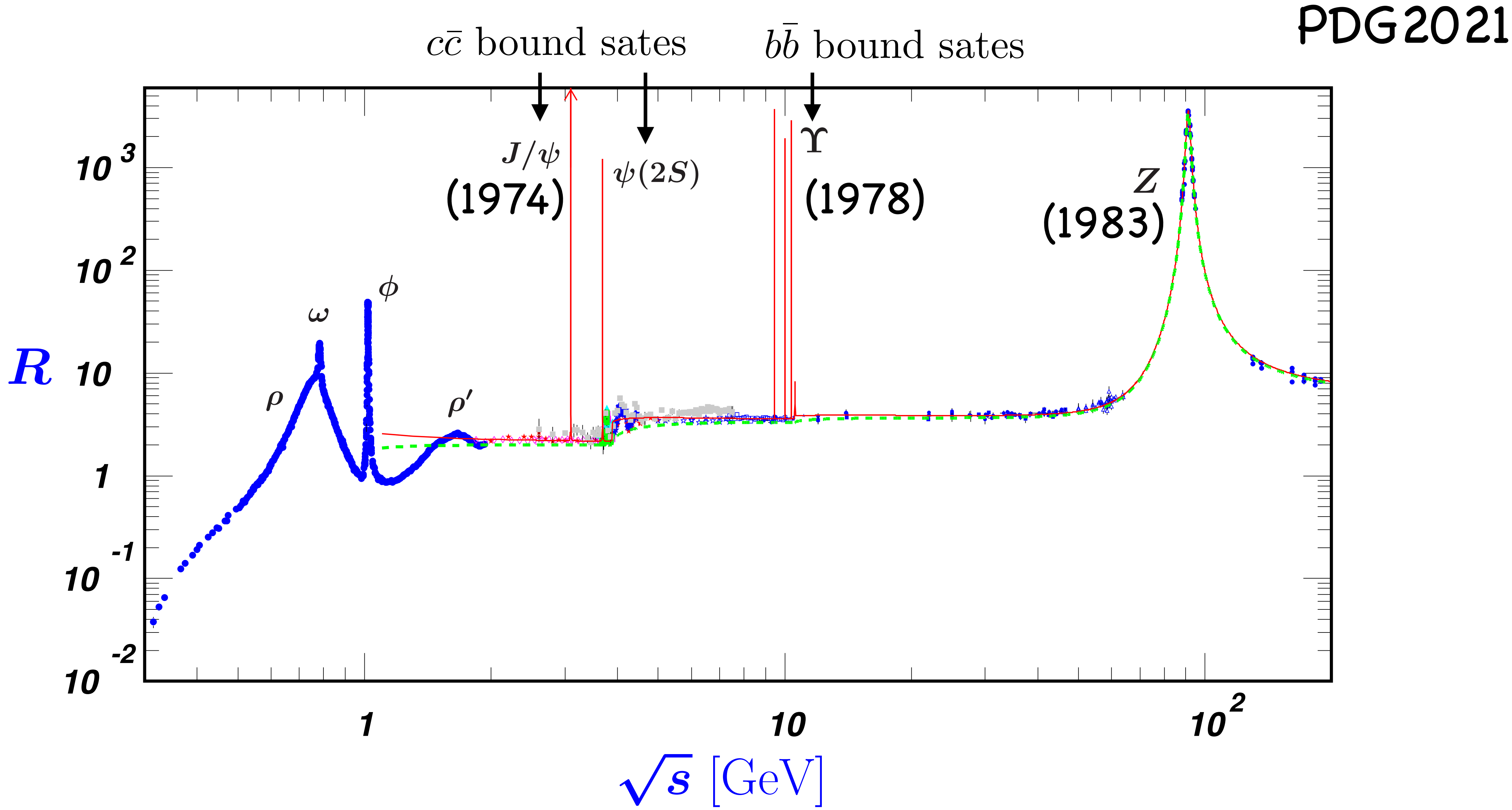


electric charge (Q_u=2/3, Q_d=-1/3)

color

$$R(s) = \frac{\sum_i \sigma(e^+e^- \to q_i\bar{q}_i, s)}{\sigma(e^+e^- \to \mu^+\mu^-, s)} \simeq \frac{\sum_i \frac{4\pi\alpha^2}{3s} Q_i^2 N_C}{\frac{4\pi\alpha^2}{3s}} = \sum_i Q_i^2 N_C \qquad s = (p_1 + p_2)^2$$

R-ratio



Just below the J/ψ ($c\bar{c}, m_{J/\psi} \sim 3$ GeV) threshold

$$R(s) = \sum_{i=u,d,s} Q_i^2 N_C = N_C \left[\left(\frac{2}{3}\right)^2 + \left(-\frac{1}{3}\right)^2 + \left(-\frac{1}{3}\right)^2 \right] = \frac{2N_C}{3} \underset{N_C=3}{=} 2$$

$$R_{\text{EXP}} \simeq 2$$

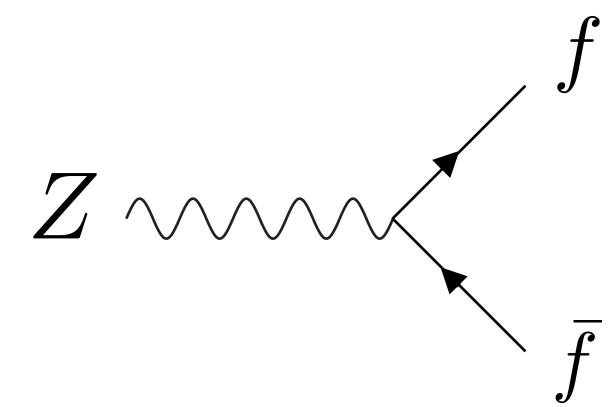
Z boson decays

$$\mathcal{L} = -g_Z Z_\mu \bar{f}\gamma^\mu \left[g^L_{Z\bar{f}f} P_L + g^R_{Z\bar{f}f} P_R \right] f, \quad g_Z = g_2/c_W$$

$$g^L_{Z\bar{f}f} = T^3_f - Q_f s^2_W, \quad g^R_{Z\bar{f}f} = -Q_f s^2_W$$

Partial Z decay width $\quad m_f \ll m_Z$

$$\Gamma(Z \to f\bar{f}) = \frac{N_C m_Z}{24\pi} g^2_Z \left[|g^L_{Z\bar{f}f}|^2 + |g^R_{Z\bar{f}f}|^2 \right]$$



$N_C = 3$ for quarks, $N_C = 1$ for leptons

$$\frac{\Gamma_\ell}{\Gamma_Z} \simeq \frac{|g^L_{Z\bar{\ell}\ell}|^2 + |g^R_{Z\bar{\ell}\ell}|^2}{\sum_{f\neq t} \left[|g^L_{Z\bar{f}f}|^2 + |g^R_{Z\bar{f}f}|^2 \right]} \simeq 0.034 \qquad \Gamma_\ell = \Gamma_e = \Gamma_\mu = \Gamma_\tau$$

One of the most important predictions of the SM is lepton flavor universality (governed by gauge interaction).

Experimental values

PDG2021

$$\mathcal{B}(Z \to e^+e^-) = (3.3632 \pm 0.0042)\%,$$

$$\mathcal{B}(Z \to \mu^+\mu^-) = (3.3662 \pm 0.0066)\%,$$

$$\mathcal{B}(Z \to \tau^+\tau^-) = (3.3696 \pm 0.0083)\%$$

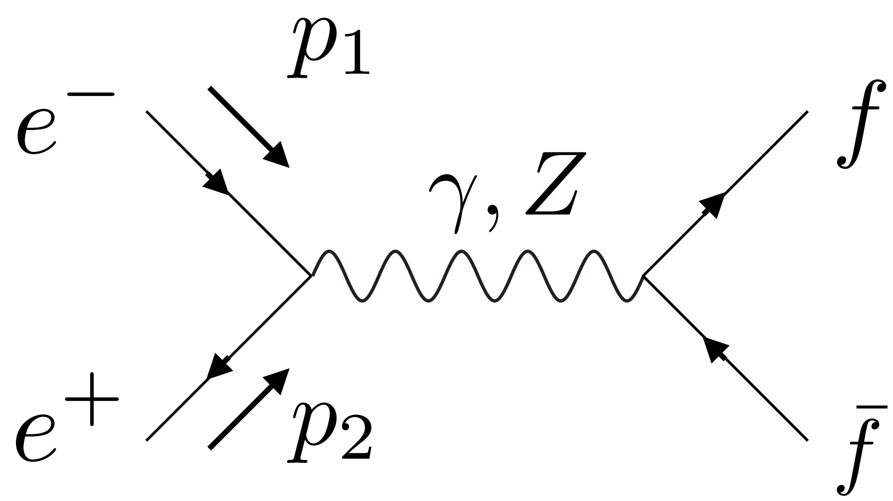
Excercise $\quad$ Estimate $\Gamma_{\rm had}/\Gamma_Z$, where $\Gamma_{\rm had} = \sum_{q\neq t} \Gamma_q$.

Number of neutrinos

Ref. Phys.Rept. 427 (2006) 257

@LEP (Large Electron Positron Collider)

$$\sigma^{f\bar{f}}_{\rm peak} \equiv \sigma(e^+e^- \to Z \to f\bar{f})|_{\sqrt{s}=m_Z} = \frac{12\pi}{m_Z^2}\frac{\Gamma_e\Gamma_f}{\Gamma_Z^2}$$



$$\Gamma_e = \Gamma(Z \to e^+e^-) \qquad \Gamma_f = \Gamma(Z \to f\bar{f})$$

$$\Gamma_Z = \sum_{\ell=e,\mu,\tau} \Gamma_\ell + \Gamma_{\rm had} + \Gamma_{\rm inv} = 3\Gamma_\ell + \Gamma_{\rm had} + N_\nu\Gamma_\nu \qquad \Gamma_{\rm had} = \sum_{q\neq t}\Gamma_q$$

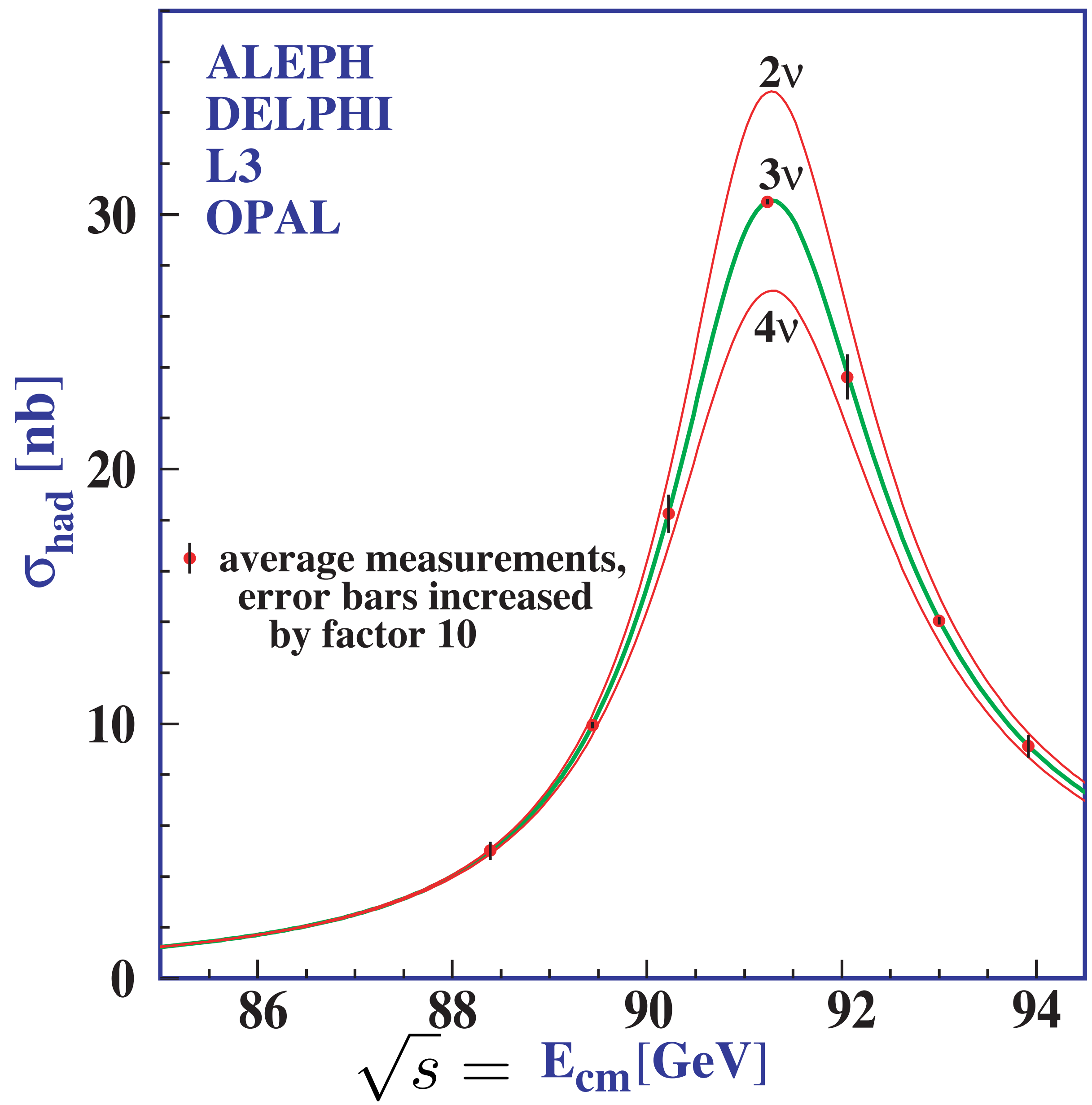
$$\Gamma_\ell = \Gamma_e = \Gamma_\mu = \Gamma_\tau$$
$$\Gamma_{\rm inv} = N_\nu\Gamma_\nu$$

Number of neutrinos

$$N_\nu = \frac{\Gamma_\ell}{\Gamma_\nu}\left[\sqrt{\frac{12\pi R_\ell}{\sigma^{\rm had}_{\rm peak}}} - R_\ell - 3\right] \qquad R_\ell = \frac{\Gamma_{\rm had}}{\Gamma_\ell}$$

Number of neutrinos

PDG2021



The number of neutrinos with masses less than $m_Z/2$ is 3.

Z-pole observables

PDG2021

Quantity	Value	Standard Model	Pull
M_Z [GeV]	91.1876 ± 0.0021	91.1882 ± 0.0020	-0.3
Γ_Z [GeV]	2.4955 ± 0.0023	2.4942 ± 0.0009	0.6
$\sigma_{\rm had}$ [nb]	41.481 ± 0.033	41.482 ± 0.008	0.0
R_e	20.804 ± 0.050	20.736 ± 0.010	1.4
R_μ	20.784 ± 0.034	20.735 ± 0.010	1.4
R_τ	20.764 ± 0.045	20.781 ± 0.010	-0.4
R_b	0.21629 ± 0.00066	0.21581 ± 0.00002	0.7
R_c	0.1721 ± 0.0030	0.17221 ± 0.00003	0.0
$A_{FB}^{(0,e)}$	0.0145 ± 0.0025	0.01619 ± 0.00007	-0.7
$A_{FB}^{(0,\mu)}$	0.0169 ± 0.0013		0.5
$A_{FB}^{(0,\tau)}$	0.0188 ± 0.0017		1.5
$A_{FB}^{(0,b)}$	0.0996 ± 0.0016	0.1030 ± 0.0002	-2.1
$A_{FB}^{(0,c)}$	0.0707 ± 0.0035	0.0736 ± 0.0002	-0.8
$A_{FB}^{(0,s)}$	0.0976 ± 0.0114	0.1031 ± 0.0002	-0.5
$\bar{s}_\ell^2$	0.2324 ± 0.0012	0.23153 ± 0.00004	0.7
	0.23148 ± 0.00033		-0.2
	0.23129 ± 0.00033		-0.7
A_e	0.15138 ± 0.00216	0.1469 ± 0.0003	2.1
	0.1544 ± 0.0060		1.2
	0.1498 ± 0.0049		0.6
A_μ	0.142 ± 0.015		-0.3
A_τ	0.136 ± 0.015		-0.7
	0.1439 ± 0.0043		-0.7
A_b	0.923 ± 0.020	0.9347	-0.6
A_c	0.670 ± 0.027	0.6677 ± 0.0001	0.1
A_s	0.895 ± 0.091	0.9356	-0.4

Pull: standard deviation

deviation from SM

$$\sim \frac{\mathcal{O}^{\rm EXP} - \mathcal{O}^{\rm SM}}{\sqrt{(\Delta\mathcal{O}^{\rm EXP})^2 + (\Delta\mathcal{O}^{\rm SM})^2}}$$

No significan deviation from SM!

Must-read-paper

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Precision electroweak measurements on the Z resonance☆, ☆☆

The ALEPH Collaboration
The DELPHI Collaboration
The L3 Collaboration
The OPAL Collaboration
The SLD Collaboration
The LEP Electroweak Working Group
The SLD Electroweak and Heavy Flavour Groups

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To know details of EW physics, you must read this paper.

Giga/Tera-Z (future)

Production of 10^9-10^{12} Z bosons to study its properties in great detail (discovery from precision measurements).

Some new physics models predict lepton-flavor-changing Z decays:

$$Z \to e^\pm \mu^\mp,\ e^\pm \tau^\mp, \mu^\pm \tau^\mp$$

current upper bounds

$$\mathcal{B}(Z \to e^\pm \mu^\mp) < 7.5 \times 10^{-7},$$
$$\mathcal{B}(Z \to e^\pm \tau^\mp) < 9.8 \times 10^{-6},$$
$$\mathcal{B}(Z \to \mu^\pm \tau^\mp) < 1.2 \times 10^{-5}$$

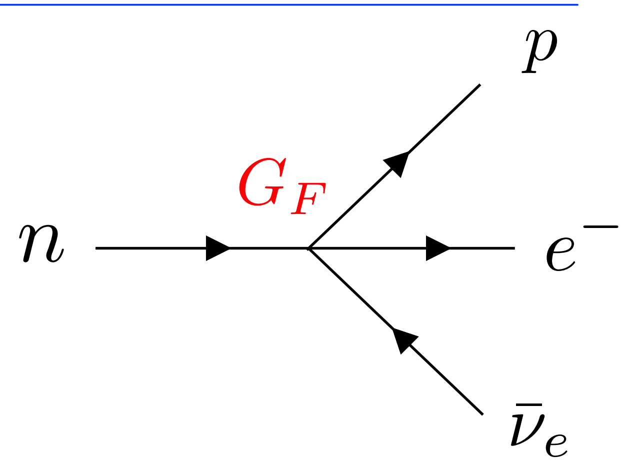
- ILC-GigaZ 1905.00220
- FCCee-TeraZ 1809.01830
- CEPC CEPC-SPPC Preliminary Conceptual Design Report. 1. Physics and Detector

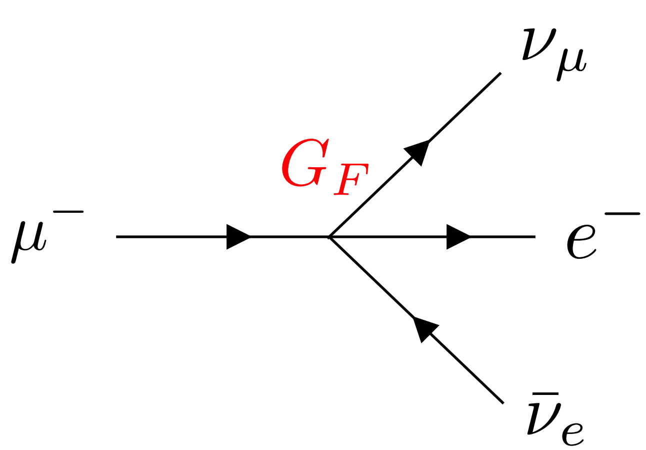
Weak interacting processes

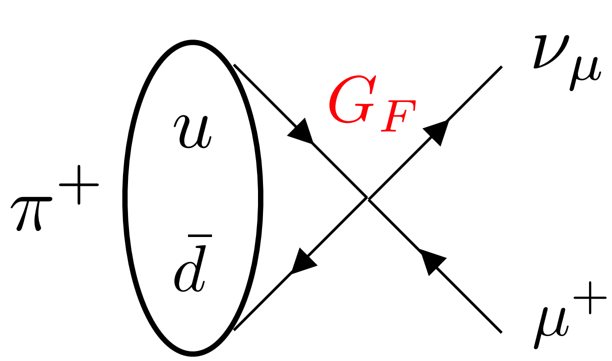
Various weak interacting processes

- Hadron only $K^+ \to \pi^+\pi^+\pi^-, \quad K^+ \to \pi^+\pi^0$, etc
- Lepton only $\mu^- \to e^-\bar{\nu}_e\nu_\mu, \quad \nu_e e^- \to \nu_e e^-, \quad$ etc
- Hadron and lepton $n \to p e^- \nu_e, \quad \pi^+ \to \mu^+\nu_\mu, \quad$ etc

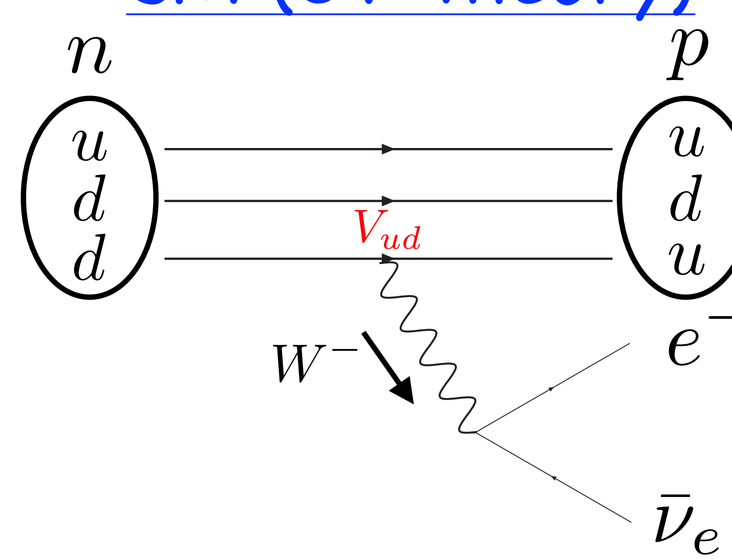
4 fermion interaction

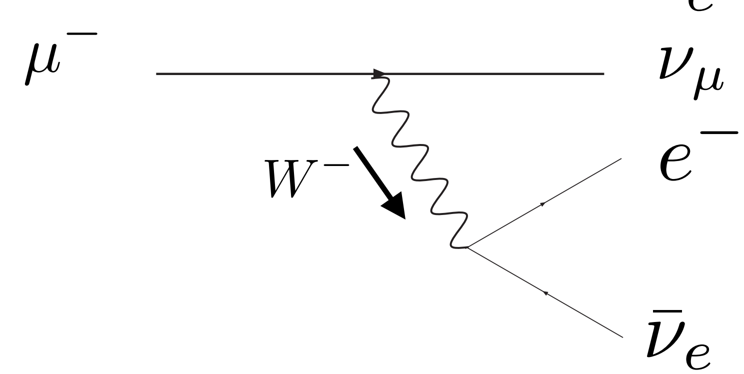


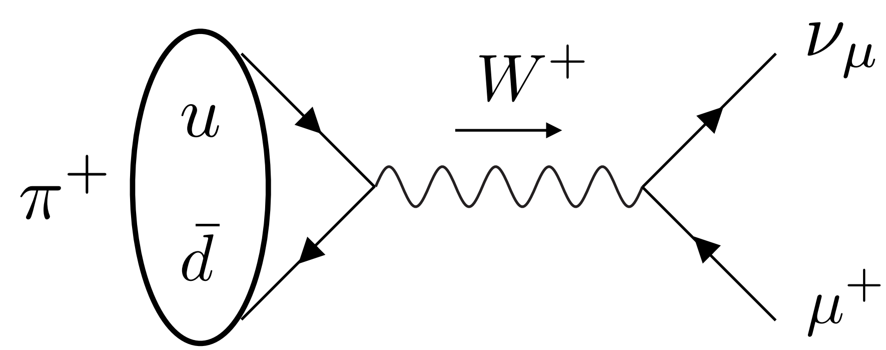




SM (UV theory)

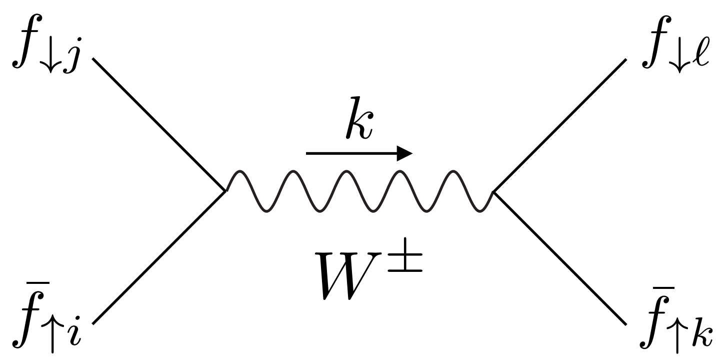






Effective Lagrangian

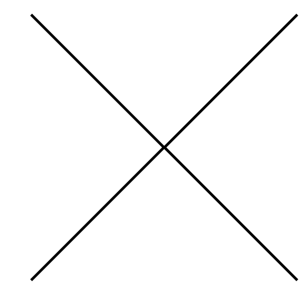
Effective Lagrangian is more convenient for low energy physics.



$$\frac{-g_{\mu\nu}}{k^2 - m_W^2} \underset{k^2 \ll m_W^2}{\simeq} \frac{g_{\mu\nu}}{m_W^2}$$

"integrating out W bosons"

4 fermion interactions



Integration of W bosons

G. Buchalla, A. Buras, M. Lautenbacher, hep-ph/9512380 (RMP)

We literally integrate out W boson fields.

$$Z_W = N \int [dW^+][dW^-] \exp\left[i \int d^4x \, \mathcal{L}_W \right]$$

$$\mathcal{L}_W = W_\mu^+ (\Box g^{\mu\nu} - \partial^\mu \partial^\nu) W_\nu^- + m_W^2 W_\mu^+ W^{-\mu} - \frac{g_2}{2\sqrt{2}} \left(J^{+\mu} W_\mu^+ + J^{-\mu} W_\mu^- \right),$$

$$J^{+\mu} = \bar{u}_i \gamma^\mu (V_{\rm CKM})_{ij} (1-\gamma_5) d_j + \bar{\nu}_i \gamma^\mu (1-\gamma_5) e_i,$$
$$J^{-\mu} = \bar{d}_i \gamma^\mu (V_{\rm CKM}^\dagger)_{ij} (1-\gamma_5) u_j + \bar{e}_i \gamma^\mu (1-\gamma_5) \nu_i.$$

$$Z_W = N \int [dW^+][dW^-] \exp\left[i \int d^4x d^4y\ W_\mu^+ K^{\mu\nu} W_\nu^- - \frac{ig_2}{2\sqrt{2}} \int d^4x \left(J^{+\mu} W_\mu^+ + J^{-\mu} W_\mu^- \right) \right]$$

$$K^{\mu\nu}(x,y) = \delta^{(4)}(x-y) \Big[g^{\mu\nu}(\Box + m_W^2) - \partial_y^\mu \partial_y^\nu \Big]$$ inverse of propagator

$$\Delta_{\mu\nu}(x,y) = \int \frac{d^4k}{(2\pi)^4}\ e^{-ik(x-y)} \Delta_{\mu\nu}(k), \quad \Delta_{\mu\nu}(k) = \frac{-1}{k^2 - m_W^2} \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{m_W^2} \right)$$ propagator

$$\longrightarrow \quad \int d^4z\ K^{\mu\lambda}(x,z) \Delta_{\lambda\nu}(z,y) = \delta^\mu{}_\nu \delta^{(4)}(x-y)$$

Shifting the W fields to complete the square and integrating them out,

$$Z_W = N \int [dW'^+][dW'^-] \exp\left[i \int d^4x d^4y\ W_\mu'^+(x) K^{\mu\nu}(x,y) W_\nu'^-(y) \right]$$

$$\times \exp\left[- \frac{ig_2^2}{8} \int d^4x d^4y\ J^{+\mu}(x) \Delta_{\mu\nu}(x,y) J^{-\nu}(y) \right]$$

$$= C \exp\left[- \frac{ig_2^2}{8} \int d^4x d^4y\ J^{+\mu}(x) \Delta_{\mu\nu}(x,y) J^{-\nu}(y) \right] \underset{k^2 \ll m_W^2}{\simeq} C \exp\left[i \int d^4x\ \mathcal{L}_W^{\rm eff} \right],$$

where $\mathcal{L}_W^{\rm eff} = -\frac{g_2^2}{8m_W^2} J^{+\mu}(x) J_\mu^-(x) = -\frac{G_F}{\sqrt{2}} J^{+\mu}(x) J_\mu^-(x), \quad G_F = \frac{1}{\sqrt{2}v^2}$ Fermi constant

$8m_W^2 = g_2^2 v^2/4$

4 fermion interactions

determined by μ decay (see later)

Equivalent method

When W bosons are sufficiently heavy, we may drop kinetic term in the Lagrangian.

$$\mathcal{L}_W \to m_W^2 W_\mu^+ W^{-\mu} - \frac{g_2}{2\sqrt{2}}\left(J^{+\mu}W_\mu^+ + J^{-\mu}W_\mu^-\right)$$

We erase W fields by their equation of motion (EOM).

EOM $$\frac{\partial \mathcal{L}_W}{\partial W^{-\mu}} = m_W^2 W_\mu^+ - \frac{g_2}{2\sqrt{2}} J_\mu^- = 0 \longrightarrow W_\mu^+ = \frac{g_2}{2\sqrt{2}}\frac{J_\mu^-}{m_W^2}$$

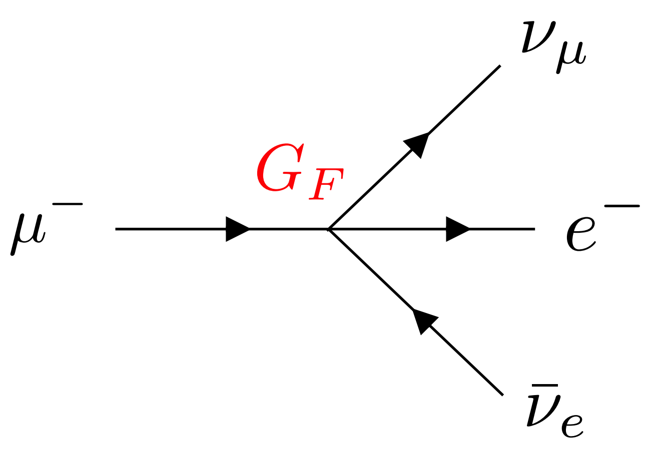
Plugging W+ field back to the above Lagrangian, one gets

$$\mathcal{L}_W^{\text{eff}} = m_W^2\left(\frac{g_2}{2\sqrt{2}m_W^2}J_\mu^-\right)W^{-\mu} - \frac{g_2}{2\sqrt{2}}\left[J^{+\mu}\left(\frac{g_2}{2\sqrt{2}m_W^2}J_\mu^-\right) + J^{-\mu}W_\mu^-\right]$$

$$= -\frac{g_2^2}{8m_W^2}J^{+\mu}J_\mu^- = -\frac{G_F}{\sqrt{2}}J^{+\mu}J_\mu^-$$

Higgs VEV from μ decay

Let us estimate Higgs VEV from the muon decay.



$$\tau_\mu^{-1} \simeq \Gamma(\mu^- \to \nu_\mu e^- \bar{\nu}_e) = \frac{1}{192\pi^3} G_F^2 m_\mu^5 \ [\text{GeV}]$$

PDG 2022

$$m_\mu = (105.6583755 \pm 0.0000023) \ \text{MeV}$$

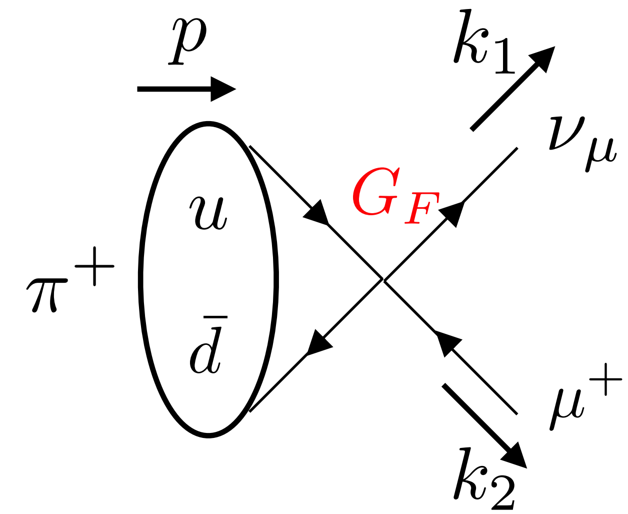
$$\tau_\mu^{\text{EXP}} = (2.1969811 \pm 0.0000022) \times 10^{-6} \ \text{s}$$

$$\hbar = 6.5821 \times 10^{-25} \ \text{GeV s}$$

$$\longrightarrow \quad G_F = \sqrt{\frac{192\pi^3}{m_\mu^5 (\tau_\mu^{\text{EXP}} / 6.5821 \times 10^{-25})}} \simeq 1.164 \times 10^{-5} \ \text{GeV}^{-2}$$

$$\longrightarrow \quad v_{\text{phys}} = \frac{1}{2^{1/4}\sqrt{G_F}} \simeq 246 \ \text{GeV}$$

$\pi^+ \to \mu^+ \nu_\mu$



Effective Lagrangian

$$\mathcal{L}_W^{\text{eff}} = -\frac{G_F}{\sqrt{2}} V_{ud} \Big[(\bar{u}d)_{V-A} (\bar{\mu}\nu_\mu)_{V-A} \Big] \qquad (\bar{f}f')_{V-A} \equiv \bar{f}\gamma^\mu (1-\gamma_5) f'$$

$$\begin{aligned}
\langle \nu_\mu(k_1)\mu^+(k_2)|iT|\pi^+(p)\rangle &= \langle \nu_\mu(k_1)\mu^+(k_2)|T\left[i\int d^4x\, \mathcal{L}_W^{\text{eff}}(x)\right]|\pi^+(p)\rangle \\
&= -i\frac{G_F}{\sqrt{2}} V_{ud} \int d^4x\, \langle \mu^+\nu_\mu|(\bar{u}d)_{V-A}(\bar{\mu}\nu_\mu)_{V-A}|\pi^+\rangle \\
&= -i\frac{G_F}{\sqrt{2}} V_{ud} \int d^4x\, \langle \mu^+\nu_\mu|\bar{\mu}\gamma_\mu(1-\gamma_5)\nu_\mu|0\rangle\langle 0|\bar{u}\gamma^\mu(1-\gamma_5)d|\pi^+\rangle
\end{aligned}$$

vacuum saturation approximation

$$\langle \nu_\mu(k_1)\mu^+(k_2)|\bar{\mu}\gamma_\mu(1-\gamma_5)\nu_\mu|0\rangle = \bar{u}^r_{\nu_\mu}(k_1)\gamma_\mu(1-\gamma_5)v^s_\mu(k_2)e^{i(k_1+k_2)x}$$

$$\langle 0|\bar{u}\gamma^\mu(1-\gamma_5)d|\pi^+(p)\rangle = -\langle 0|\bar{u}\gamma^\mu\gamma_5 d|\pi^+(p)\rangle = -if_\pi p^\mu e^{-ipx}$$

pion decay constant

Amplitude

$$i\mathcal{M} = -\frac{G_F}{\sqrt{2}} V_{ud} f_\pi m_\mu \left[\bar{u}^r_{\nu_\mu}(k_1)\gamma_\mu(1-\gamma_5)v^s_\mu(k_2) \right]$$

Decay rate

$$\Gamma(\pi^+ \to \mu^+\nu_\mu) = \frac{G_F^2}{8\pi}|V_{ud}|^2 f_\pi^2 m_\pi m_\mu^2 \left(1 - \frac{m_\mu^2}{m_\pi^2}\right)^2$$

Electron mode is also calculated in the same way.

$$\frac{\Gamma(\pi^+ \to e^+\nu_e)}{\Gamma(\pi^+ \to \mu^+\nu_\mu)} = \frac{m_e^2(m_\pi^2 - m_e^2)^2}{m_\mu^2(m_\pi^2 - m_\mu^2)^2} \simeq 1.28 \times 10^{-4}$$

Experimental data PDG2021

$$\mathrm{BR}(\pi^+ \to \mu^+\nu_\mu) = (99.98770 \pm 0.00004)\%$$

$\gg$ Why??

$$\mathrm{BR}(\pi^+ \to e^+\nu_e) = (1.230 \pm 0.004) \times 10^{-4}$$

consistent

You may think the latter should be dominant because of the larger phase space $\propto \left(1 - \frac{m_{\ell^+}^2}{m_{\pi^+}^2}\right)$ $\quad m_e \ll m_\mu$

Exercises

(1) π^+ mostly decays to μ^+ rather than e^+. Why?

(2) $G_F = 1.16 \times 10^{-5}\ \mathrm{GeV}^{-2},\ |V_{ud}| = 0.973,\ m_{\pi^+} = 139.6\ \mathrm{MeV},\ \tau_{\pi^+} = 2.6 \times 10^{-8}\ \mathrm{s},$
$m_\mu = 105.7\ \mathrm{MeV}$. Using those values, determine f_π.

(3) Since π^+ is spin 0 and W^+ is spin 1, it seems to violate the angular momentum conservation at the rest frame of π^+. How can we explain this?

Lecture 4

CP transformation

- Dirac fermions -

See, e.g., Bjorken-Drell's book

C trf. $\psi(x) \to \psi^C(x) \equiv \mathcal{C}\psi(x)\mathcal{C}^{-1} = C(\bar{\psi}(x))^T, \quad \bar{\psi} = \psi^\dagger\gamma^0, \quad C^{-1}\gamma^\mu C = -(\gamma^\mu)^T$

$C = i\gamma^2\gamma^0$

$\mathcal{C}$ is unitary operator that acts on creation/annihilation operators.

P trf. $\psi(t, \boldsymbol{x}) \to \psi^P(x) = \mathcal{P}\psi(t, \boldsymbol{x})\mathcal{P}^{-1} = \gamma^0\psi(t, -\boldsymbol{x})$

e.,g. QED $\quad \Big[i\gamma^\mu(\partial_\mu - ieA_\mu) - m\Big]\psi(x) = 0,$ EOM of electron

$\Big[i\gamma^\mu(\partial_\mu + ieA_\mu) - m\Big]\psi^C(x) = 0,$ EOM of positron

In theories with P conservation, particle-anti-particle symmetry is described by C transformation.

N.B.

$$\psi = \begin{pmatrix} \chi_\alpha \\ \bar{\eta}^{\dot{\alpha}} \end{pmatrix} = \begin{pmatrix} \varphi_L \\ \varphi_R \end{pmatrix} \qquad \psi^C = C(\bar{\psi})^T = \begin{pmatrix} \eta_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix} = \begin{pmatrix} i\sigma^2\varphi_R^* \\ -i\sigma^2\varphi_L^* \end{pmatrix}$$

$\chi_\alpha \in (1/2, 0), \quad \bar{\eta}^{\dot{\alpha}} \in (0, 1/2)$ SL(2,C) spinors

CP transformation

- Chiral fermions -

P trf. $\psi_{L,R}(t,\boldsymbol{x}) \to \psi^P_{L,R} = \mathcal{P}\psi_{L,R}(t,\boldsymbol{x})\mathcal{P}^{-1} = \gamma^0\psi_{R,L}(t,-\boldsymbol{x})$

CP trf. $\psi_{L,R}(t,\boldsymbol{x}) \to \mathcal{CP}\psi_{L,R}(t,\boldsymbol{x})\mathcal{P}^{-1}\mathcal{C}^{-1} = \mathcal{C}\gamma^0\psi_{R,L}(t,-\boldsymbol{x})\mathcal{C}^{-1} = \gamma^0 C\left(\bar{\psi}_{L,R}(t,-\boldsymbol{x})\right)^T$

Yukawa interaction

$$\mathcal{L}_Y = y_{ij}\bar{\psi}_{Li}(t,\boldsymbol{x})\chi_{Rj}(t,\boldsymbol{x})\phi(t,\boldsymbol{x}) + y^*_{ij}\bar{\chi}_{Rj}(t,\boldsymbol{x})\psi_{Li}(t,\boldsymbol{x})\phi^*(t,\boldsymbol{x})$$

$$\bar{\psi}_L(t,\boldsymbol{x})\chi_R(t,\boldsymbol{x}) \stackrel{CP}{\to} \bar{\chi}_R(t,-\boldsymbol{x})\psi_L(t,-\boldsymbol{x}), \qquad \phi(t,\boldsymbol{x}) \stackrel{CP}{\to} \phi^*(t,-\boldsymbol{x})$$ for scalars

$$\bar{\chi}_R(t,\boldsymbol{x})\psi_L(t,\boldsymbol{x}) \stackrel{CP}{\to} \bar{\psi}_L(t,-\boldsymbol{x})\chi_R(t,-\boldsymbol{x})$$

$$\begin{aligned}
S_Y = \int d^4x\,\mathcal{L}_Y \to S^{CP}_Y &= \int d^4x\,\mathcal{CP}\mathcal{L}_Y(\mathcal{CP})^{-1} \\
&= \int d^4x\left[y_{ij}\bar{\chi}_{jR}(t,-\boldsymbol{x})\psi_{Li}(t,-\boldsymbol{x})\phi^*(t,-\boldsymbol{x}) + y^*_{ij}\bar{\psi}_{Li}(t,-\boldsymbol{x})\chi_{Rj}(t,-\boldsymbol{x})\phi(t,-\boldsymbol{x})\right] \\
&\underset{\boldsymbol{x}\to-\boldsymbol{x}}{=} \int d^4x\left[y^*_{ij}\bar{\psi}_{Li}(t,\boldsymbol{x})\chi_{Rj}(t,\boldsymbol{x})\phi(t,\boldsymbol{x}) + y_{ij}\bar{\chi}_{Rj}(t,\boldsymbol{x})\psi_{Li}(t,\boldsymbol{x})\phi^*(t,\boldsymbol{x})\right].
\end{aligned}$$

Therefore, $S_Y = S^{CP}_Y$ if $y_{ij} \in \mathbb{R}$. However, $y_{ij} \in \mathbb{C}$ does not always break CP.

CP violation

E.g., 1 generation

$$\mathcal{L}_Y = y\bar{\psi}_L \chi_R \phi + y^* \bar{\chi}_R \psi_L \phi^*$$

For $y = |y|e^{i\theta}$, we can remove the phase by the field definition $\chi'_R = e^{i\theta}\chi_R$.

Condition for CPV

"To have CP violation, at least 3 generations (6 quarks) are needed."

(* At that time, only 3 quarks were known.)

M. Kobayashi

T. Maskawa

Progress of Theoretical Physics, Vol. 49, No. 2, February 1973

CP-Violation in the Renormalizable Theory of Weak Interaction

Makoto KOBAYASHI and Toshihide MASKAWA

Department of Physics, Kyoto University, Kyoto

(Received September 1, 1972)

In a framework of the renormalizable theory of weak interaction, problems of CP-violation are studied. It is concluded that no realistic models of CP-violation exist in the quartet scheme without introducing any other new fields. Some possible models of CP-violation are also discussed.

Let us consider N generation fermions and 1 Higgs doublet.

$$-\mathcal{L}_Y = Y^d_{ij}\bar{q}_{Li}d_{Rj}\Phi + Y^u_{ij}\bar{q}_{Li}u_{Rj}\tilde{\Phi} + Y^e_{ij}\bar{\ell}_{Li}e_{Rj}\Phi + \text{h.c.} \qquad i,j = 1,2,\ldots,N$$

$$\Phi = \begin{pmatrix}\phi^+ \\ \phi^0\end{pmatrix} \underset{\text{unitary gauge}}{\rightarrow} \begin{pmatrix}0 \\ \frac{1}{\sqrt{2}}(v+h(x))\end{pmatrix}, \quad \tilde{\Phi} = i\tau^2\Phi^* = \begin{pmatrix}\frac{1}{\sqrt{2}}(v+h(x)) \\ 0\end{pmatrix}$$

Rotate the fields as

$$u_L = V^u_L u'_L, \quad u_R = V^u_R u'_R; \quad d_L = V^d_L d'_L, \quad d_R = V^d_R d'_R; \quad e_L = V^e_L e'_L, \quad e_R = V^e_R e'_R$$

Mass eigenstate

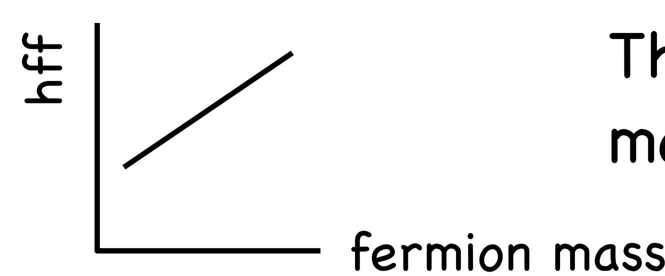
$(Y^{u,d,e})_{ij}$ can be diagonalized by bi-unitary transformations.

$$V_L^{u,d,e\dagger}\left(\frac{Y^{u,d,e}_{ij}v}{\sqrt{2}}\right)V_R^{u,d,e} = \begin{pmatrix} m_1^{u,d,e} & & & \\ & m_2^{u,d,e} & & \\ & & \ddots & \\ & & & m_N^{u,d,e}\end{pmatrix}$$

Masses and hff couplings

$$-\mathcal{L}_Y \ni m_i^f\bar{f}_i f_i + \frac{m_i^f}{v}h\bar{f}_i f_i$$

same flavor



This relation can be modified in NP models

Flavor mixing matrix

$$\mathcal{L}_W = -\frac{g_2}{\sqrt{2}}\left[\bar{u}_{Li}\gamma^\mu d_{Li}W_\mu^+ + \bar{d}_{Li}\gamma^\mu u_{Li}W_\mu^- + \bar{\nu}_{Li}\gamma^\mu e_{Li}W_\mu^+ + \bar{e}_{Li}\gamma^\mu \nu_{Li}W_\mu^-\right]$$
$$= -\frac{g_2}{\sqrt{2}}\left[\bar{u}'_{Li}\gamma^\mu (V)_{ij} d'_{Lj}W_\mu^+ + \bar{d}'_{Li}\gamma^\mu (V^\dagger)_{ij} u'_{Lj}W_\mu^- + \bar{\nu}_{Li}\gamma^\mu (V_L^e)_{ij} e'_{Lj}W_\mu^+ + \bar{e}'_{Li}\gamma^\mu (V_L^{e\dagger})_{ij}\nu_{Lj}W_\mu^-\right],$$

where $V = V_L^{u\dagger}V_L^d$ (flavor mixing matrix, called CKM matrix for N=3)

Cabibbo-Kobayashi-Maskawa

We can simply redefine $\nu'_L = V_L^{e\dagger}\nu_L.$

-> no flavor mixing matrix in the lepton sector

* if neutrinos have masses, a similar flavor mixing matrix (U_{PMNS}) would appear.

Pontecorvo-Maki-Nakagawa-Sakata

Condition for CPV

In general, # of degrees of freedom (dof) in the NxN unitary matrix is given by

$V^\dagger V = VV^\dagger = 1$

$$(N^2 \times 2) - \left(N + \frac{N(N-1)}{2} \times 2\right) = \boxed{N^2}$$

Since we have 2N quarks, we naively think that we can remove 2N complex phases by redefinitions of LH quark fields.

$$q_{iL} \to e^{i\theta_i} q_{iL}$$

However, # of usable phases = 2N-1. Why is there "-1"?

$\because$

$$V_L^u \to V_L^u P_\alpha, \quad V_L^d \to V_L^d P_\beta, \quad P_\alpha = \mathrm{diag}(e^{i\alpha_1}, e^{i\alpha_2}, \cdots, e^{i\alpha_N}), \quad P_\beta = \mathrm{diag}(e^{i\beta_1}, e^{i\beta_2}, \cdots, e^{i\beta_N})$$

$$V = V_L^{u\dagger} V_L^d \to P_\alpha^\dagger (V_L^{u\dagger} V_L^d) P_\beta = P_\alpha^\dagger V P_\beta \longrightarrow e^{-i(\alpha_i-\beta_j)} V_{ij} = e^{-i(\alpha_i-\beta_j-\theta_{ij})} |V_{ij}|$$

matrix elements

Only "relative phases" can be used to eliminate phases of V.

This is because Lagrangian has $U(1)_B$ symmetry ($\alpha_i = \beta_j$) -> "-1"

Conditions we can use are $\alpha_i - \beta_j - \theta_{ij} = 0, \quad i, j = 1, 2, \cdots, N$

N=2 case for illustration

$$\begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} \theta_{11} \\ \theta_{12} \\ \theta_{21} \\ \theta_{22} \end{pmatrix}$$

=A

rank(A)=3, det(A) = 0

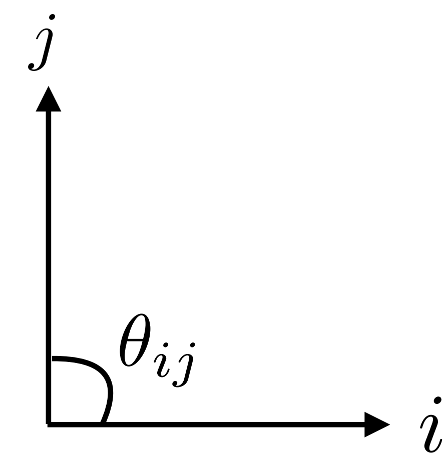
only 3θ_{ij} can be removed.

We must subtract (2N-1) from N².

$$\text{total dof} = N^2 - (2N-1) = \boxed{(N-1)^2}$$

# of angles in the N dimensional space:

angles are defined by choosing any 2 axises.



$$_N C_2 = \frac{N(N-1)}{2}$$

# of complex phases = total dof - (# of angles)

$$\boxed{(N-1)^2 - \frac{N(N-1)}{2} = \frac{1}{2}(N-1)(N-2)}$$

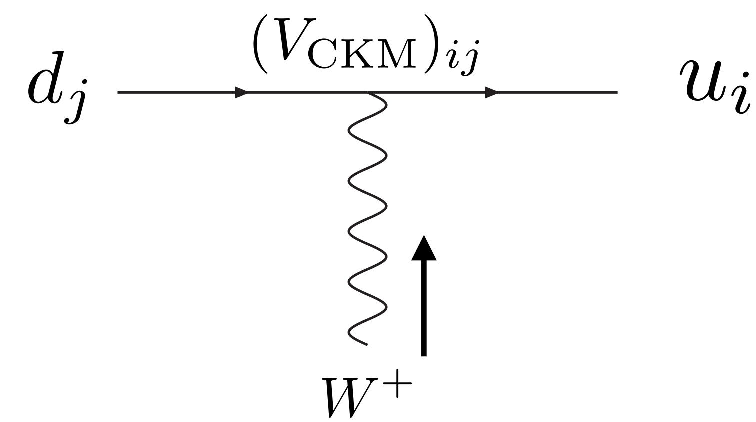
E.g., N=3 in the SM: # of angles = 3; # of phases = 1.

To realize CP violation, at least 3 generations (6 quarks) are needed!!

Quark mixing

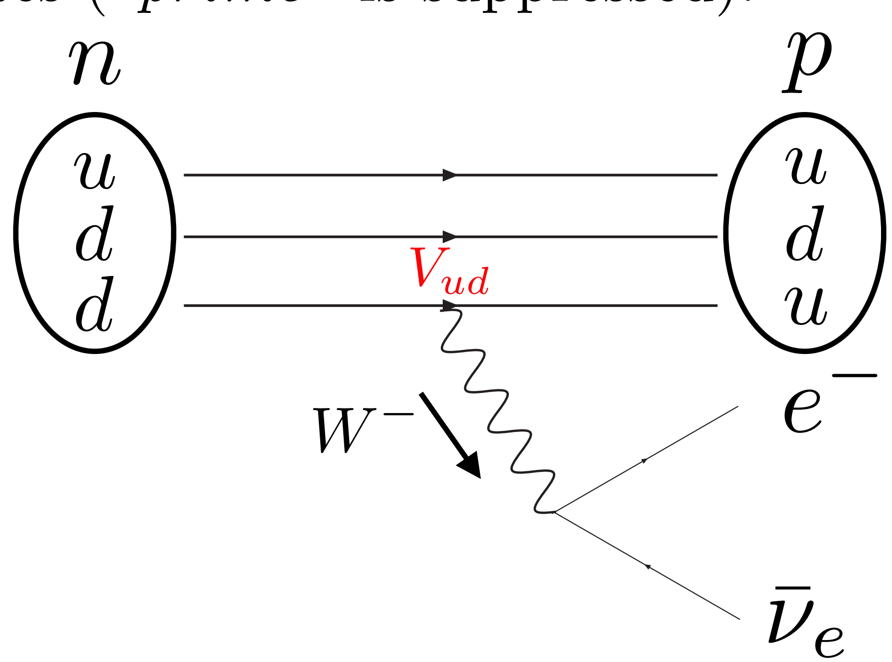
Quark flavor can change via the CKM matrix.

$$\mathcal{L}_W \ni -\frac{g_2}{\sqrt{2}} W^+_\mu \begin{pmatrix} \bar{u}_L & \bar{c}_L & \bar{t}_L \end{pmatrix} \gamma^\mu \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \begin{pmatrix} d_L \\ s_L \\ b_L \end{pmatrix}$$

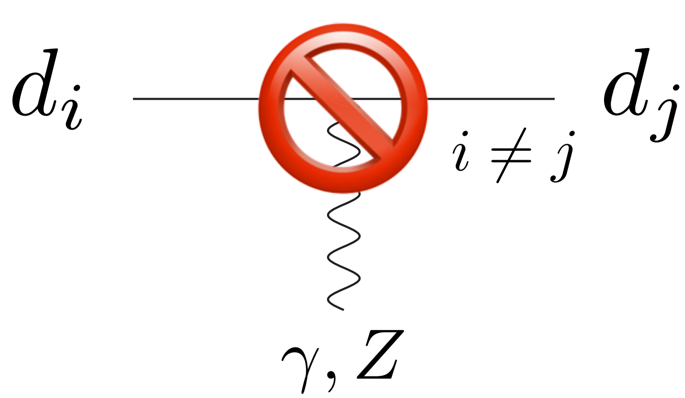


f_L are the mass eigenstates ("*prime*" is suppressed).

e.g., β-decay



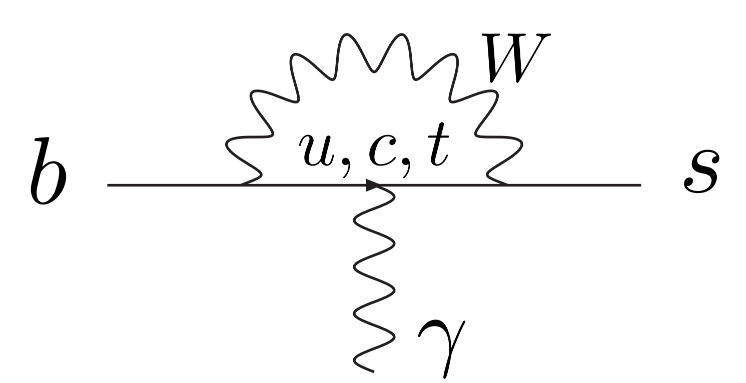
No tree-level Flavor-Changing Neutral Current (FCNC)



However, FCNC at loop levels exists.

e.g. b->s+ γ

penguin diagram



Wolfenstein parametrization

V_{CKM} matrix is parametrized by 3 angles and 1 phase

$$V_{\rm CKM} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{32} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$s_{ij} = \sin\theta_{ij} \geq 0, \;\; c_{ij} = \cos\theta_{ij} \geq 0$$

Experimentally known -> $s_{13} \ll s_{23} \ll s_{12} \ll 1$

$$(\theta_{12}, \theta_{13}, \theta_{23}, \delta) \to (\lambda, A, \rho, \eta) \qquad s_{12} = \lambda, \quad s_{23} = A\lambda^2, \quad s_{13}e^{i\delta} = A\lambda^3(\rho + i\eta)$$

V_{CKM} is approximated to

$$V_{\rm CKM} = \begin{pmatrix} 1-\lambda^2/2 & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1-\lambda^2/2 & A\lambda^2 \\ A\lambda^3(1-\rho-i\eta) & -A\lambda^2 & 1 \end{pmatrix} + \mathcal{O}(\lambda^4)$$

Unitarity triangle

Unitarity conditions

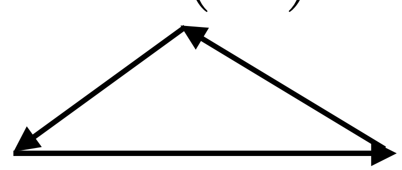
$$\sum_{k=u,c,t} V_{ki}V^*_{kj} = \delta_{ij} \ (i,j=d,s,b), \qquad \sum_{k=d,s,b} V_{ik}V^*_{jk} = \delta_{ij} \ (i,j=u,c,t)$$ 6 conditions

$$V_{ud}V^*_{us} + V_{cd}V^*_{cs} + V_{td}V^*_{ts} = 0 \qquad s \to d \qquad K^0 \sim d\bar{s}, \ \overline{K^0} \sim s\bar{d}$$

$\mathcal{O}(\lambda)$ $\mathcal{O}(\lambda)$ $\mathcal{O}(\lambda^5)$

$$V_{ud}V^*_{ub} + V_{cd}V^*_{cb} + V_{td}V^*_{tb} = 0 \qquad b \to d \qquad B^0 \sim d\bar{b}, \ \overline{B^0} \sim b\bar{d}$$

$\mathcal{O}(\lambda^3)$ $\mathcal{O}(\lambda^3)$ $\mathcal{O}(\lambda^3)$



3 sides are the same order in magnitude.

$$V_{us}V^*_{ub} + V_{cs}V^*_{cb} + V_{ts}V^*_{tb} = 0 \qquad b \to s \qquad B^0_s \sim s\bar{b}, \ \overline{B^0_s} \sim b\bar{s}$$

$\mathcal{O}(\lambda^4)$ $\mathcal{O}(\lambda^2)$ $\mathcal{O}(\lambda^2)$

Areas of all the triangles are the same (see later)

Angles are determined by CP asymmetry

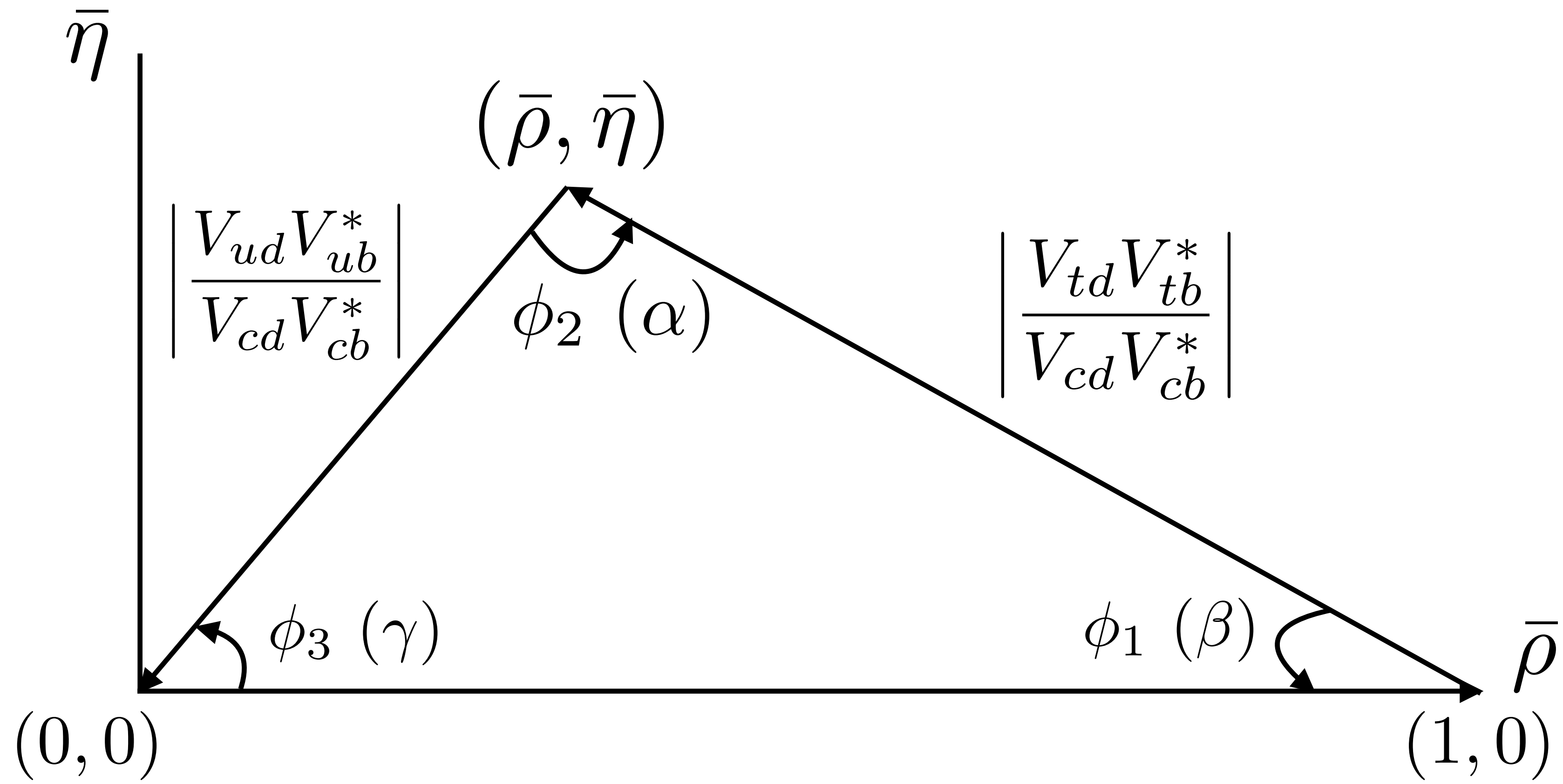
Unitarity triangle

Unitarity condition

$$V_{ud}V_{ub}^* + V_{cd}V_{cb}^* + V_{td}V_{tb}^* = 0$$

↓

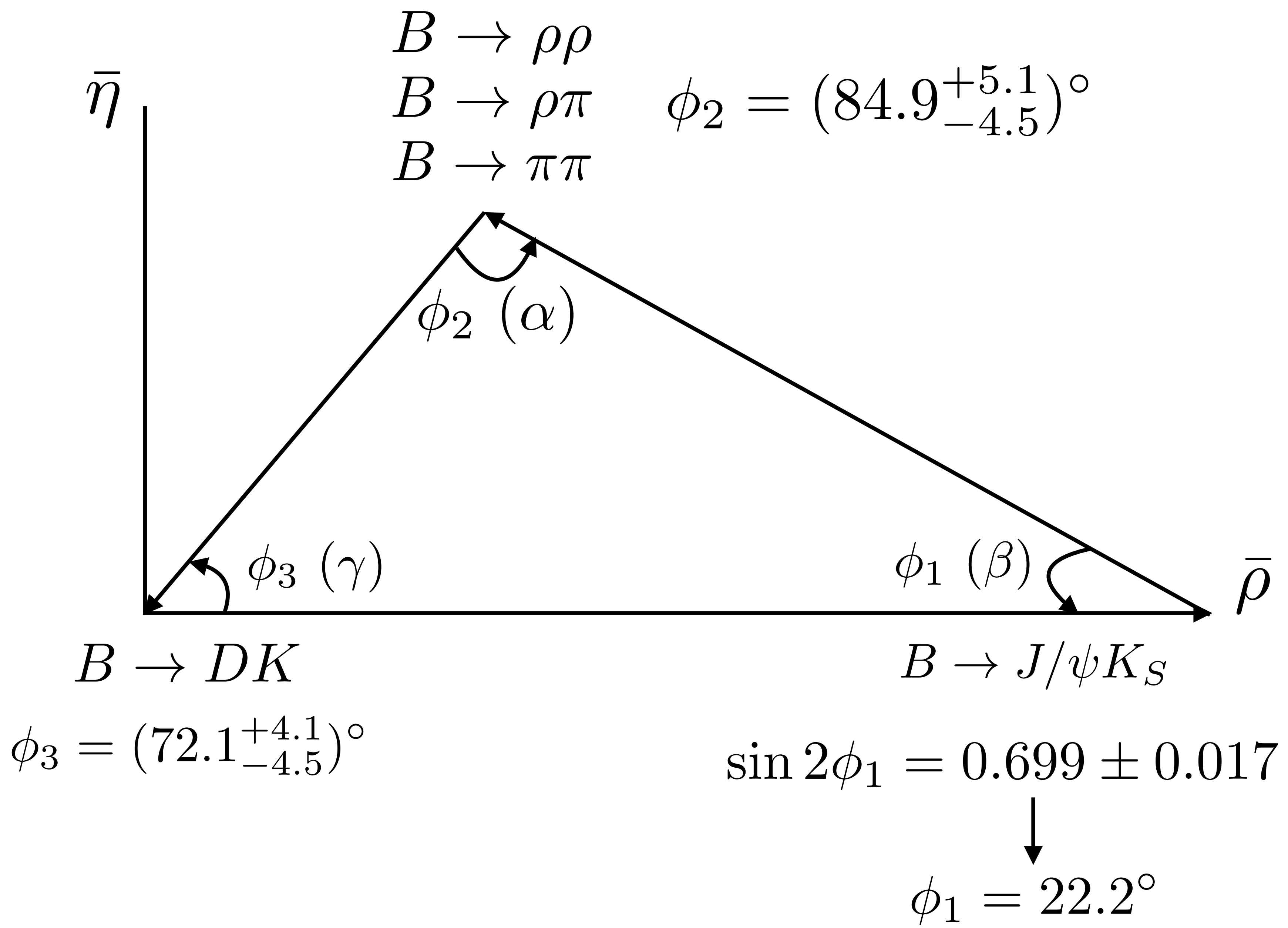
$$\frac{V_{ud}V_{ub}^*}{V_{cd}V_{cb}^*} + 1 + \frac{V_{td}V_{tb}^*}{V_{cd}V_{cb}^*} = 0$$



$$\phi_1 = \beta = \mathrm{Arg}\left(-\frac{V_{cd}V_{cb}^*}{V_{td}V_{tb}^*}\right), \quad \phi_2 = \alpha = \mathrm{Arg}\left(-\frac{V_{td}V_{tb}^*}{V_{ud}V_{ub}^*}\right), \quad \phi_3 = \gamma = \mathrm{Arg}\left(-\frac{V_{ud}V_{ub}^*}{V_{cd}V_{cb}^*}\right)$$

$$\bar{\rho} + i\bar{\eta} = -\frac{V_{ud}V_{ub}^*}{V_{cd}V_{cb}^*} \simeq (\rho + i\eta)\left(1 - \frac{\lambda^2}{2} + \ldots\right)$$ re-phasing invariant

Unitarity triangle



* Experimental values are quoted from PDG 2021.

$\phi_1 + \phi_2 + \phi_3 = 179.2^\circ$ no indication new physics...

CP asymmetry

- general discussion -

Given amplitudes such as

$$\mathcal{A}(i \to f) = \mathcal{A}_1 + \mathcal{A}_2 = |\mathcal{A}_1| e^{i\phi_1} e^{i\delta_1} + |\mathcal{A}_2| e^{i\phi_2} e^{i\delta_2},$$

$$\overline{\mathcal{A}}(\bar{i} \to \bar{f}) = \overline{\mathcal{A}}_1 + \overline{\mathcal{A}}_2 = |\mathcal{A}_1| e^{-i\phi_1} e^{i\delta_1} + |\mathcal{A}_2| e^{-i\phi_2} e^{i\delta_2}$$ (CP-conjugate)

where ϕ: CP phases; δ: phases from cuts (absorptive parts)
CP-violating CP-conserving

In B physics, ϕ is often called weak phase and δ strong phase.

CP asymmetry

$$A_{CP} = \frac{\Gamma(i \to f) - \Gamma(\bar{i} \to \bar{f})}{\Gamma(i \to f) + \Gamma(\bar{i} \to \bar{f})}$$

$$= \frac{-2|\mathcal{A}_1 \mathcal{A}_2| \sin(\phi_1 - \phi_2) \sin(\delta_1 - \delta_2)}{|\mathcal{A}_1|^2 + |\mathcal{A}_2|^2 + 2|\mathcal{A}_1 \mathcal{A}_2| \cos(\phi_1 - \phi_2) \cos(\delta_1 - \delta_2)}$$

To have CP asymmetry, both CP-conserving and -violating phases are needed!!

Jarlskog invariant

C. Jarlskog, PRL55, 1039 (1985); Z.Phys.C 29 491 (1985)

Let us define

$$[\mathcal{M}_u^\dagger \mathcal{M}_u, \mathcal{M}_d^\dagger \mathcal{M}_d] = iC, \quad \mathcal{M}_{u,d} = \frac{Y^{u,d} v}{\sqrt{2}}, \quad C^\dagger = C, \quad \mathrm{tr} C = 0.$$

$$\det C = -2(m_c^2 - m_u^2)(m_t^2 - m_u^2)(m_t^2 - m_c^2)(m_s^2 - m_d^2)(m_b^2 - m_d^2)(m_b^2 - m_s^2) J$$

- If any of the 2 masses are degenerated in up/down sectors, detC=0.
- $J = (-1)^{r+s} \mathrm{Im}(V_{ij} V_{k\ell} V_{i\ell}^* V_{jk}^*)$

e.g.

$r = s = 3$

$$\begin{array}{c} \\ i \\ k \\ \end{array} \begin{array}{c} \begin{array}{ccc} j & \ell & \phantom{V_{13}} \end{array} \\ \begin{pmatrix} V_{11} & V_{12} & V_{13} \\ V_{21} & V_{22} & V_{23} \\ V_{31} & V_{32} & V_{33} \end{pmatrix} \end{array} \longrightarrow \begin{pmatrix} V_{11} & V_{12}^* \\ V_{21}^* & V_{22} \end{pmatrix} \longrightarrow J = \mathrm{Im}(V_{11} V_{22} V_{12}^* V_{21}^*)$$

$r = 1, s = 3$

$$\begin{array}{c} \\ \\ i \\ k \end{array} \begin{array}{c} \begin{array}{ccc} j & \ell & \phantom{V_{13}} \end{array} \\ \begin{pmatrix} V_{11} & V_{12} & V_{13} \\ V_{21} & V_{22} & V_{23} \\ V_{31} & V_{32} & V_{33} \end{pmatrix} \end{array} \longrightarrow \begin{pmatrix} V_{21} & V_{22}^* \\ V_{31}^* & V_{32} \end{pmatrix} \longrightarrow J = \mathrm{Im}(V_{21} V_{32} V_{22}^* V_{31}^*)$$

9 different ways to express J.

Jarlskog invariant

J is rephasing invariant

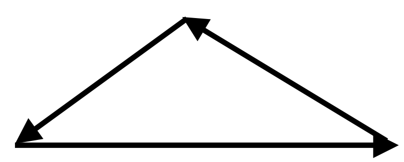
$$V_L^u \to V_L^u P_\alpha, \quad V_L^d \to V_L^d P_\beta, \quad P_\alpha = \mathrm{diag}(e^{i\alpha_1}, e^{i\alpha_2}, e^{i\alpha_3}), \quad P_\beta = \mathrm{diag}(e^{i\beta_1}, e^{i\beta_2}, e^{i\beta_3})$$

$$V_{\mathrm{CKM}} = V_L^{u\dagger} V_L^d \to P_\alpha^\dagger (V_L^{u\dagger} V_L^d) P_\beta = P_\alpha^\dagger V_{\mathrm{CKM}} P_\beta \longrightarrow e^{-i(\alpha_i - \beta_j)} (V_{\mathrm{CKM}})_{ij}$$

matrix elements

$$V_{11} V_{22} V_{12}^* V_{21}^* \to e^{-i(\alpha_1 - \beta_1)} V_{11} e^{-i(\alpha_2 - \beta_2)} V_{22} e^{i(\alpha_1 - \beta_2)} V_{12}^* e^{i(\alpha_2 - \beta_1)} V_{21}^* = V_{11} V_{22} V_{12}^* V_{21}^*$$

-> J is rephasing invariant.

- |J|= (are of unitarity triangle )x2