Introduction to Standard Model

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Plan

1. Symmetries, properties of scalars, spinors and vectors

2. Gauge theories, SM Lagrangian, Higgs mechanism

3. QED process (e^+e^- -> $\mu^+\mu^-$), Z boson decays, weak interacting processes (μ decays, π^+ decays)

4. CP transformations, CKM matrix, Jarlskog invariant

5. Higgs properties (decays, productions), rho-parameter

6. Vacuum structure (effective potential, renormalization group equation)

7. Exercises, discussions

Lecture 1

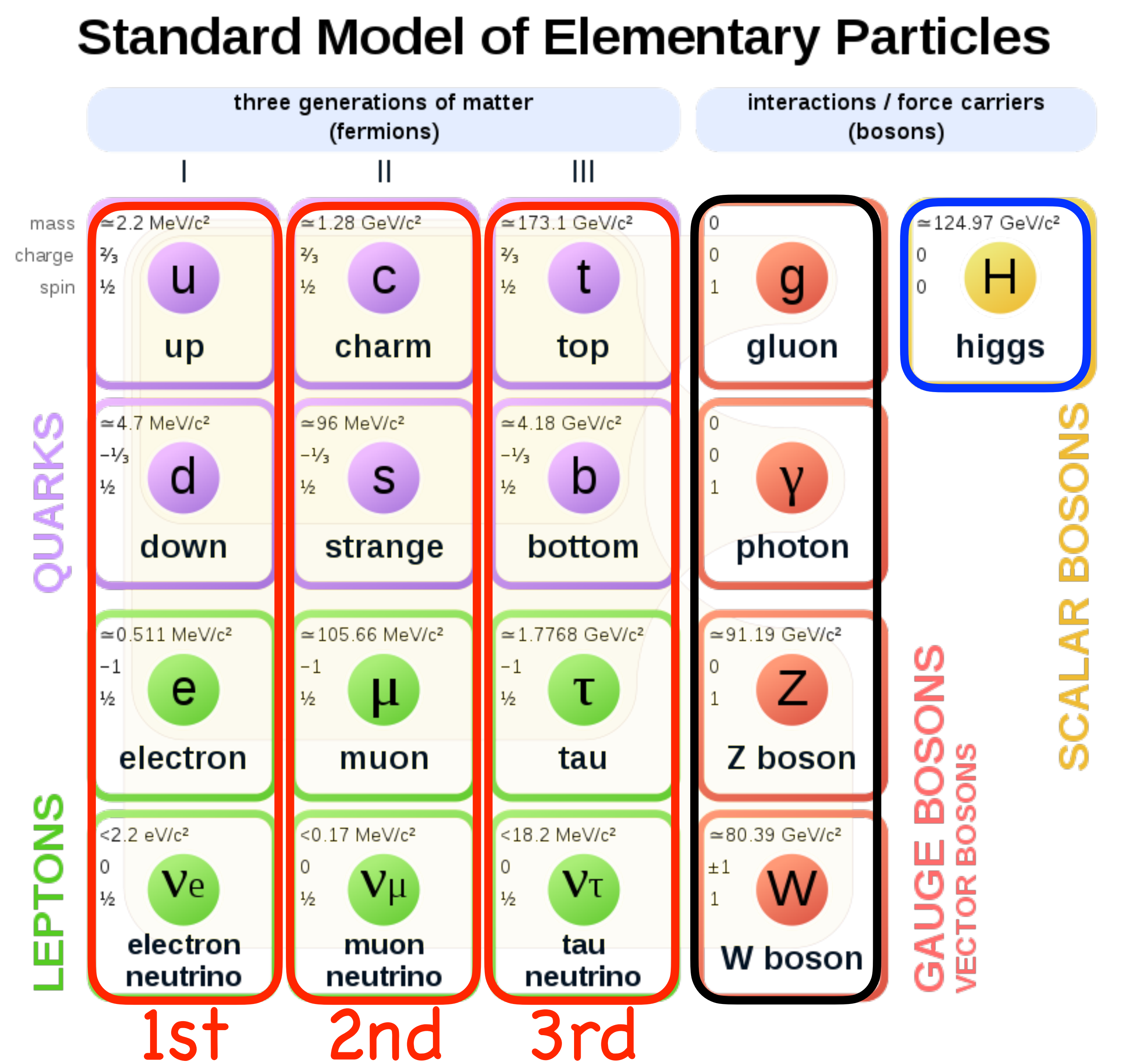
Standard Model (SM)

- Any particle is made of elementary particles, e.g., proton=(uud), neutron=(udd)

  - 3 generations of quarks and leptons

  - 3 forces (strong, weak, electromagnetic forces)

  - Higgs is symmetry breaker and mass giver.



[Standard Model of Elementary Particles.svg]

- SM is the theory of elementary particles.

Standard Model (SM)

SM was established by Glashow, Slam and Weinberg.

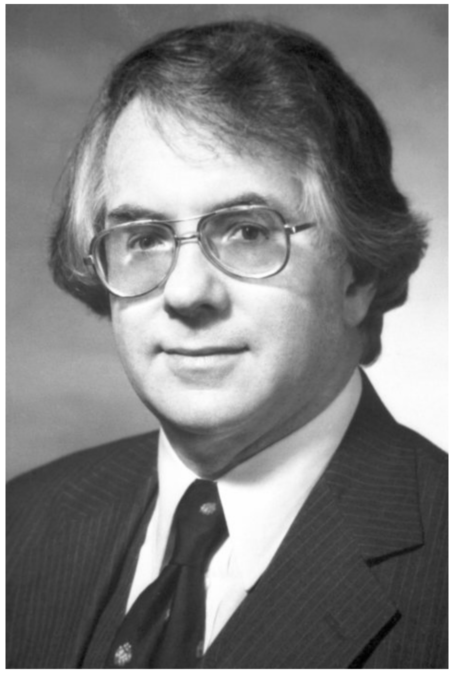


Photo from the Nobel Foundation archive.
Sheldon Lee Glashow
Prize share: 1/3

Photo from the Nobel Foundation archive.
Abdus Salam
Prize share: 1/3

Photo from the Nobel Foundation archive.
Steven Weinberg
Prize share: 1/3

The Nobel Prize in Physics 1979 was awarded jointly to Sheldon Lee Glashow, Abdus Salam and Steven Weinberg "for their contributions to the theory of the unified weak and electromagnetic interaction between elementary particles, including, inter alia, the prediction of the weak neutral current."

2 pillars of SM

(1) Gauge principle

SM is constructed based on gauge symmetry $SU(3)_C \times SU(2)_L \times U(1)_Y$.

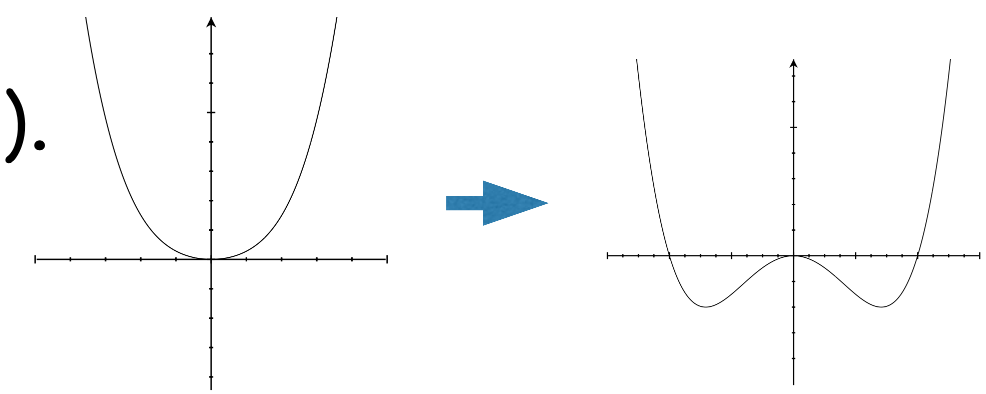
- Lagrangian respects the symmetry.

-> All elementary particles are massless.

(2) Higgs mechanism

- symmetry is broken by vacuum (Spontaneous Symmetry Breaking (SSB)).
- All elementary particles become massive.

Higgs potential



symmetry is spontaneously broken.

from wikipedia

Yoichiro Nambu (Nobel prize 2008):
The basic laws are very simple, yet this world is not boring.

There are a bunch of particles.
how do we classify their
properties and interactions?

-> symmetries!

Symmetries

(Bosonic) symmetries of the S-matrix in relativistic field theory

= Poiancare symmetry $\oplus$ internal symmetries   Coleman, Mandula, PRD159, 1251 ('67)

Poiancare symmetry

$\hbar = c = 1$ (natural unit)

$\gamma = \frac{1}{\sqrt{1-\beta^2}}$ $\beta = v$

Translation

$x^\mu \to x'^\mu = x^\mu + a^\mu$

Lorentz trf.

$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu$

boost

e.g. z direction $\begin{pmatrix} \gamma & 0 & 0 & \gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma\beta & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} m \\ 0 \\ 0 \\ 0 \end{pmatrix}$ p^μ rest frame

rotation

about y axis $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & 0 & \sin\theta \\ 0 & 0 & 1 & 0 \\ 0 & -\sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} \gamma m \\ 0 \\ 0 \\ \gamma\beta m \end{pmatrix}$

Internal symmetries

Abelian groups $g_1, g_2 \in G,\ \ g_1 g_2 = g_2 g_1$ U(1), SO(2), Z_2 etc

Non-abelian groups $g_1, g_2 \in G,\ \ g_1 g_2 \neq g_2 g_1$ SU(2), SO(3), SU(3), etc

Lagrangian is constructed to respect some symmetries.

Poincare algebra

P_μ : generator of translation, $M_{\mu\nu}$: generator of Lorentz trf.

$$\begin{aligned}[P_\mu, P_\nu] &= 0, \\ [P_\mu, M_{\rho\sigma}] &= i(g_{\mu\rho}P_\sigma - g_{\mu\sigma}P_\rho), \\ [M_{\mu\nu}, M_{\rho\sigma}] &= i(g_{\nu\rho}M_{\mu\sigma} - g_{\nu\sigma}M_{\mu\rho} - g_{\mu\rho}M_{\nu\sigma} + g_{\mu\sigma}M_{\nu\rho})\end{aligned}$$

Massive particles

$$P^2 = P_\mu P^\mu = m^2, \quad W^2 = W_\mu W^\mu \quad \left(W^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}P_\nu M_{\rho\sigma}\right)$$

-> characterized by mass and spin (i.e., J=0, 1/2, 1,...)

rest frame

$$P^\mu = \begin{pmatrix} m \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$W^2 = -m^2 \underbrace{\boldsymbol{J}^2}_{\text{spin}} \qquad \boldsymbol{J} = \begin{pmatrix} M_{23} \\ M_{31} \\ M_{12} \end{pmatrix} \qquad [J_i, J_j] = i\epsilon_{ijk}J_k, \quad (i, j, k = 1, 2, 3)$$

-> (2J+1) states (m=-J, -J+1 ,..., J-1, J)

e.g, J=1/2 $\quad |\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \; |\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Massless particles

$$P^2 = W^2 = 0, \;\; W^\mu = \lambda P^\mu$$

e.g.

$$P^\mu = \begin{pmatrix} E \\ 0 \\ 0 \\ E \end{pmatrix}$$

-> characterized by helicity (-λ, λ)

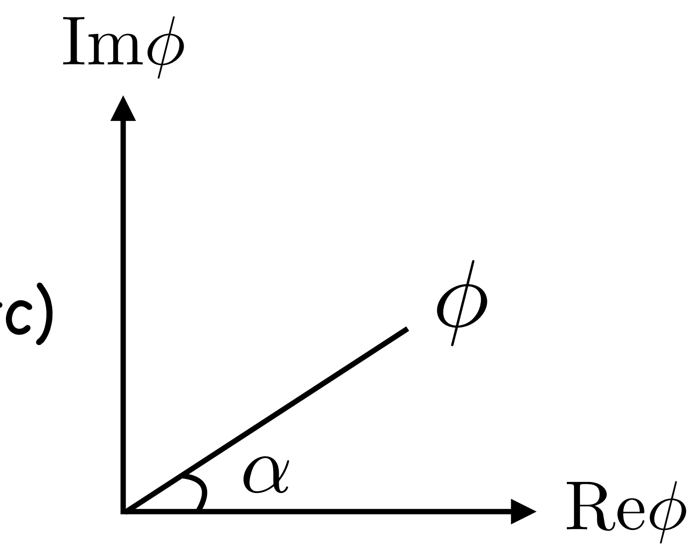
Lorentz invariant

* |helicity| is also customarily called spin.

U(1)

e.g., symmetry of electromagnetism

phase transformation



$$\phi \to \phi' = e^{i\alpha}\phi$$

α is constant -> global U(1) (baryon/lepton number, etc)

α depends on x -> local U(1) (QED, etc.)

Let us rewrite ϕ as $\phi = \phi_R + i\phi_I$

$$\phi'_R + i\phi'_I = (c_\alpha + is_\alpha)(\phi_R + i\phi_I) = c_\alpha\phi_R - s_\alpha\phi_I + i(s_\alpha\phi_R + c_\alpha\phi_I)$$

$$\begin{pmatrix}\phi'_R \\ \phi'_I\end{pmatrix} = \underbrace{\begin{pmatrix} c_\alpha & -s_\alpha \\ s_\alpha & c_\alpha\end{pmatrix}}_{\equiv R}\begin{pmatrix}\phi_R \\ \phi_I\end{pmatrix}$$

orthogonality special

$$R^T R = RR^T = 1, \quad \det R = 1$$

$$R \in SO(2) \qquad U(1) \simeq SO(2)$$

1-to-1 correspondence

[N.B.] Determinant = 1 is important.

$$A = \begin{pmatrix}\cos\alpha & -\sin\alpha \\ -\sin\alpha & -\cos\alpha\end{pmatrix}, \quad A^T A = AA^T = 1, \ \det A = -1 \qquad A \in O(2)$$

$$A(\alpha = 0) \neq 1_{2\times 2}$$

SU(2)

e.g., symmetry of weak interaction

Algebra

dimSU(2)=2²−1=3

$$[T^a, T^b] = i\epsilon^{abc}T^c, \quad a,b,c = 1,2,3 \qquad \mathrm{Tr}(T^aT^b) = \frac{1}{2}\delta^{ab}$$

$$U = e^{iT^a\theta^a}$$

$$U^\dagger U = UU^\dagger = 1 \longrightarrow T^{a\dagger} = T^a$$ Hermitian

$$\det U = 1 \longrightarrow \mathrm{Tr}(T^a) = 0$$ Traceless

Representation of SU(2)

$$[J^a, J^b] = i\epsilon^{abc}J^c$$

$$\boldsymbol{J}^2|j,m\rangle = j(j+1)|j,m\rangle, \quad j = 0,\ \frac{1}{2},\ 1,\ \frac{3}{2},\cdots$$

$$J^3|j,m\rangle = m|j,m\rangle \qquad m = -j, -j+1, \cdots, j-1, j$$

(2j+1)-plet	singlet (j=0)	doublet (j=1/2)	triplet (j=1)
$\begin{pmatrix} \phi_j \\ \phi_{j-1} \\ \vdots \\ \phi_{-(j-1)} \\ \phi_{-j} \end{pmatrix}$	ϕ_0	$\begin{pmatrix} \phi_{1/2} \\ \phi_{-1/2} \end{pmatrix}$	$\begin{pmatrix} \phi_1 \\ \phi_0 \\ \phi_{-1} \end{pmatrix}$
	right-handed fermions	left-handed fermions, Higgs	weak gauge bosons

no higher (j>1) SU(2) rep. states in SM particles.

SU(2)

Doublet (j=1/2)

$$\Phi = \begin{pmatrix} \phi_{1/2} \\ \phi_{-1/2} \end{pmatrix} \qquad \Phi \to \Phi' = U\Phi, \quad U = e^{i\frac{\tau^a}{2}\theta^a} \qquad \left[\frac{\tau^a}{2}, \frac{\tau^b}{2}\right] = i\epsilon^{abc}\frac{\tau^c}{2}$$

$$\tau^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

*Pauli matrices τ^i are also denoted as σ^i.

Triplet (j=1)

$$\Sigma = \begin{pmatrix} \phi_1 \\ \phi_0 \\ \phi_{-1} \end{pmatrix} \quad \Sigma \to \Sigma' = U\Sigma, \quad U = e^{it^a\theta^a}, \quad (t^a)_{bc} = -i\epsilon^{abc}$$

$$[t^a, t^b] = i\epsilon^{abc}t^c$$

$$t^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad t^2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad t^3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

pure imaginary

it^a are real anti-symmetric. $U^* = U,\ U^TU = UU^T = 1,\ \det U = 1. \longrightarrow \mathrm{SO}(3)$

3 dim. rotational group

Exercises

(1) Let Φ be a SU(2) doublet. Show that $\tilde{\Phi} \to \tilde{\Phi}' = U\tilde{\Phi}$, where $\tilde{\Phi} = i\tau^2\Phi^*$.

(2) Let Φ_u and Φ_d are SU(2) doublets. Show that $\epsilon_{ij}\Phi_u^i\Phi_d^j (= \Phi_u^1\Phi_d^2 - \Phi_u^2\Phi_d^1)$ is SU(2) invariant, where $\Phi_{u,d} = \begin{pmatrix} \Phi_{u,d}^1 \\ \Phi_{u,d}^2 \end{pmatrix}$.

(3) Let X and Y operators. Show that

$$e^X Y e^{-Y} = Y + [X, Y] + \frac{1}{2!}[X, [X, Y]] + \frac{1}{3!}\Big[X, [X, [X, Y]]\Big] + \cdots$$

(4) Let Φ and Σ^a be SU(2) doublet and triplet, respectively. Show that $\Phi^\dagger\tau^a\Phi\Sigma^a$ is SU(2) invariant.

SU(3)

e.g., symmetry of strong interaction

Algebra

dimSU(3)=3²-1=8

$$[T^a, T^b] = if^{abc}T^c, \quad a,b,c = 1,\cdots,8 \qquad \mathrm{Tr}(T^aT^b) = \frac{1}{2}\delta^{ab}$$

Defining representation is expressed by Gell-Mann matrices (λs).

$$\left[\frac{\lambda^a}{2}, \frac{\lambda^b}{2}\right] = if^{abc}\frac{\lambda^c}{2}, \quad a,b,c = 1,\cdots,8$$

where

$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda^8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

Hermitian and traceless

$SU(3)_C \times SU(2)_L \times U(1)_Y$

symmetry of Standard Model

Direct product -> each transformation is independent.

e.g., $q_L(x) : (\mathbf{3}, \mathbf{2}, 1/6)$

SU(3) SU(2) U(1)

$$q_{L,I,i}(x) \to q'_{L,I,i}(x) = (e^{i\frac{\lambda^s}{2}\theta_3^s(x)})_{IJ}(e^{i\frac{\tau^a}{2}\theta_2^a(x)})_{ij}(e^{i\theta_1(x)/6})q_{L,J,j}(x)$$

$s = 1, ..., 8$: SU(3) generator indices, $I, J = 1, 2, 3$: SU(3) triplet indicies (color);
$a = 1, 2, 3$: SU(2) generator indices, $i, j = 1, 2$: SU(2) doublet indices;
Y_q: U(1) hypercharge of q

Likewise $u_R(x) : (\mathbf{3}, \mathbf{1}, 2/3)$

$$u_{R,I}(x) \to u'_{R,I}(x) = (e^{i\frac{\lambda^s}{2}\theta_3^s(x)})_{IJ}(e^{i2\theta_1(x)/3})q_{L,J}(x)$$

Mass and spin in multiplet

- All members of an (irreducible) multiplet of internal symmetry group have the same spin and mass.

$$\begin{pmatrix} \phi_j \\ \phi_{j-1} \\ \vdots \\ \phi_{-(j-1)} \\ \phi_{-j} \end{pmatrix}$$

- If not, symmetry should be broken.

* The above statement does not hold in supersymmetric theories. Super-multiplet contains fields with different spins.

Lagrangian

Lagrangian (up to total derivatives) must satisfy

1. Hermiticity $\mathcal{L} = \mathcal{L}^\dagger \longrightarrow$ Hamiltonian is Hermitian (time evolution is unitary)

2. Symmetry $\mathcal{L} \to \mathcal{L}' = \mathcal{L}$

   Lorentz and gauge symmetries, and could be something more.

3. Renormalizability

   UV divergences are removed by a finite number of counterterms

   mass dimension of operators <= 4 $[\mathcal{L}] = M^4$

$$\mathcal{L} = \mathcal{O}_4 + \frac{1}{\Lambda}\mathcal{O}_5 + \frac{1}{\Lambda^2}\mathcal{O}_6 + \cdots \quad \Lambda \text{ is mass dimensionful parameter}$$

* Effective Field Theory (EFT) allows higher-dimensional operators.

scalars $\phi(x)$

$$\mathcal{O} = \{\phi^n,\ (\partial_\mu\phi)^n;\ |\phi|^n,\ |\partial_\mu\phi|^n\}$$ Lorentz inv.

Renormalizability $\longrightarrow$ $\mathcal{O} = \{\phi^{n\le 4},\ (\partial_\mu\phi)^2;\ |\phi|^{n\le 4},\ |\partial_\mu\phi|^2\}$

$$[\partial_\mu] = M,\ [\phi] = M$$

The possible operators are further reduced if a symmetry exists.

E.g., global U(1) $\phi \to \phi' = e^{i\alpha}\phi$ $\longrightarrow$ $\mathcal{O} = \{|\phi|^{n\le 4},\ |\partial_\mu\phi|^2\}$

spinors $\psi(x)$

Lorentz, renormalizability, hermitian $\longrightarrow$ $\mathcal{O} = \{\bar\psi\psi, i\bar\psi\gamma^\mu(\overrightarrow{\partial}_\mu - \overleftarrow{\partial}_\mu)\psi\}$

$$[\psi] = M^{3/2} \quad \cancel{\psi^\dagger\psi}$$

vectors $A_\mu(x)$

Lorentz, renormalizability, hermitian $\longrightarrow$ $\mathcal{O} = \{F_{\mu\nu}F^{\mu\nu}\}$ $[A_\mu] = M$

The form of $F_{\mu\nu}$ depends on gauge groups. e.g., U(1) $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

$$\cancel{A_\mu A^\mu}$$

Free Lagrangian

no interaction

Spin 0 (scalars)

real scalar

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 \qquad [\mathcal{L}] = M^4, \quad [\partial_\mu] = M, \quad [\phi] = M$$

Equation of Motion (EOM)

Klein-Gorden equation

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right) - \frac{\partial\mathcal{L}}{\partial\phi} = 0 \longrightarrow (\partial_\mu\partial^\mu + m^2)\phi = 0 \qquad p^2 = m^2$$

complex scalar

$$\mathcal{L} = \partial_\mu\phi^*\partial^\mu\phi - m^2\phi^*\phi = \sum_{i=1,2}\frac{1}{2}\left[(\partial_\mu\phi_i)^2 - \boxed{m^2}\phi_i^2\right] \qquad \phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$$

common mass

2 real fields

invariant under $\phi \to \phi' = e^{i\alpha}\phi$ global U(1)

cf. 2 real scalars with different masses

$$\mathcal{L} = \sum_{i=1,2}\frac{1}{2}\left[(\partial_\mu\phi_i)^2 - \boxed{m_i^2}\phi_i^2\right] = \partial_\mu\phi^*\partial^\mu\phi - \frac{1}{2}(m_1^2 + m_2^2)\phi^*\phi - \frac{1}{4}(m_1^2 - m_2^2)(\phi^2 + \phi^{*2})$$

different masses

global U(~~1~~)

common mass ↔ symmetry

Solutions

$$(\partial_\mu \partial^\mu + m^2)\phi = 0$$

$$p^0 = E = \pm E_{\boldsymbol{p}}$$

$$E_{\boldsymbol{p}} = \sqrt{\boldsymbol{p}^2 + m^2}$$

positive and negative frequency solutions

$$\phi_+(x) = e^{-ip\cdot x} = e^{-i(E_{\boldsymbol{p}}t - \boldsymbol{p}\cdot\boldsymbol{x})}$$

$$\phi_-(x) = e^{+ip\cdot x} = e^{+i(E_{\boldsymbol{p}}t - \boldsymbol{p}\cdot\boldsymbol{x})}$$

Complex scalar case

particle annihilation

antiparticle creation

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\boldsymbol{p}}}} \left[a_{\boldsymbol{p}} e^{-ip\cdot x} + b^\dagger_{\boldsymbol{p}} e^{ip\cdot x} \right]$$

$$a_{\boldsymbol{p}}|0\rangle = b_{\boldsymbol{p}}|0\rangle = 0$$

$$\phi^\dagger(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\boldsymbol{p}}}} \left[b_{\boldsymbol{p}} e^{-ip\cdot x} + a^\dagger_{\boldsymbol{p}} e^{ip\cdot x} \right]$$

antiparticle annihilation

particle creation

Exercise

Why does the positive frequency mode has the annihilation operator, namely, $a_{\boldsymbol{p}} e^{-ip\cdot x}$? What happens if $a^\dagger_{\boldsymbol{p}} e^{-ip\cdot x}$?

Symmetry and conservation law

Noether's theorem see, e.g., Peskin's book.

If a symmetry exists, corresponding conservation law exists.

$$\phi(x) \to \phi'(x) = \phi(x) + \alpha\Delta\phi(x) \qquad \mathcal{L}(x) \to \mathcal{L}' = \mathcal{L}(x) + \alpha\partial_\mu \mathcal{J}^\mu(x)$$

One obtains

$$\partial_\mu j^\mu(x) = 0, \quad j^\mu(x) = \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi - \mathcal{J}^\mu \qquad Q = \int d^3x\, j^0(x) \quad \frac{\partial Q}{\partial t} = 0$$

charge conservation

Complex scalar case

$$\phi \to \phi' = e^{-i\alpha}\phi$$ global U(1)

$$N_\phi = \int d^3\boldsymbol{x}\; j^0(x) = i\int d^3\boldsymbol{x}\Big[\phi^\dagger(\partial^0\phi) - (\partial^0\phi^\dagger)\phi\Big] = \int \frac{d^3\boldsymbol{p}}{(2\pi)}(a^\dagger_{\boldsymbol{p}}a_{\boldsymbol{p}} - b^\dagger_{\boldsymbol{p}}b_{\boldsymbol{p}})$$

$$= \int \frac{d^3\boldsymbol{p}}{(2\pi)}(n - \bar{n}) = N - \bar{N}$$ = # of particle - # of anti-particle

-> particle-antiparticle "pair" annihilation/creation

global U(1) (phase transformation) invariance
= particle number conservation (= charge conservation)

Spin 1/2 (spinors)

Dirac field

$$\mathcal{L} = \frac{1}{2}\bar{\psi}\Big[i\gamma^\mu(\overrightarrow{\partial}_\mu - \overleftarrow{\partial}_\mu) - m\Big]\psi \to \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi \qquad [\psi] = M^{2/3}$$

hermitian

integration by parts

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}, \quad \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3, \quad \{\gamma^\mu, \gamma_5\} = 0$$

Dirac representation convenient for non-relativistic limit

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Chiral representation convenient for relativistic limit (or massless limit)

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$P_{L,R} = \frac{1 \mp \gamma_5}{2}, \quad \psi_{L,R} = P_{L,R}\psi$$ 4-component $$\psi = \psi_L + \psi_R$$

* Some people's notation $\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$ 2-component (Weyl fermion)

I don't use this notation in my lecture.

Chirality eigenvalues of γ_5 $\gamma_5\psi_L = -\psi_L, \ \gamma_5\psi_R = +\psi_R$

Equation of Motion (EOM)

$$(i\partial\!\!\!/ - m)\psi(x) = 0, \quad \partial\!\!\!/ \equiv \gamma^\mu \partial_\mu$$

In momentum space

helicity*2 (±1)

$$(p\!\!\!/ - m)u^s(p) = 0, \quad u^s(p) = \begin{pmatrix} u^s(p)_{\boxed{-}} \\ u^s(p)_{\boxed{+}} \end{pmatrix}$$

particle

chirality

$$(p^0 - \boldsymbol{p}\cdot\boldsymbol{\sigma})u^s(p)_+ = m u^s(p)_-,$$
$$(p^0 + \boldsymbol{p}\cdot\boldsymbol{\sigma})u^s(p)_- = m u^s(p)_+$$
$$p^0 = E_{\boldsymbol{p}} = \sqrt{\boldsymbol{p}^2 + m^2}$$

$$(p\!\!\!/ + m)v^s(p) = 0, \quad v^s(p) = C(\bar{u}^s(p))^T, \quad C = i\gamma^2\gamma^0$$

charge conjugation

$C = -i\gamma^2\gamma^0$ (Peskin's book)

antiparticle

particle annihilation

antiparticle creation

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\boldsymbol{p}}}} \sum_s \left[a^s_{\boldsymbol{p}} u^s(p) e^{-ip\cdot x} + b^{s\dagger}_{\boldsymbol{p}} v^s(p) e^{ip\cdot x} \right]$$

$$\bar{\psi}(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\boldsymbol{p}}}} \sum_s \left[b^s_{\boldsymbol{p}} \bar{v}^s(p) e^{-ip\cdot x} + a^{s\dagger}_{\boldsymbol{p}} \bar{u}^s(p) e^{ip\cdot x} \right]$$

antiparticle annihilation

particle creation

u spinor (particle)

$$u^s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi^s \\ \sqrt{p\cdot\bar{\sigma}}\,\xi^s \end{pmatrix} \begin{matrix} \longleftarrow \text{ chirality = L(-1)} \\ \longleftarrow \text{ chirality = R(+1)} \end{matrix}$$

$$(p\cdot\sigma)\xi^s = (p^0 - \boldsymbol{p}\cdot\boldsymbol{\sigma})\xi^s = (E - s|\boldsymbol{p}|)\xi^s \qquad \frac{\boldsymbol{p}\cdot\boldsymbol{\sigma}}{|\boldsymbol{p}|}\xi^s = s\xi^s \quad s = \pm 1$$

(helicity*2)

$$\frac{\boldsymbol{p}\cdot\boldsymbol{\sigma}}{|\boldsymbol{p}|} = \begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{pmatrix}, \quad \xi^{(+)} = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}e^{i\phi} \end{pmatrix}, \quad \xi^{(-)} = \begin{pmatrix} -\sin\frac{\theta}{2}e^{-i\phi} \\ \cos\frac{\theta}{2} \end{pmatrix}, \quad \xi^{(\pm)} = \pm i\sigma^2\xi^{(\mp)*}$$

$$\theta \in [0,\pi], \;\; \phi \in [0, 2\pi]$$

$$u^s(p) = \begin{pmatrix} \sqrt{E - s|\boldsymbol{p}|}\,\xi^s \\ \sqrt{E + s|\boldsymbol{p}|}\,\xi^s \end{pmatrix} \underset{E\gg m}{\rightarrow} \begin{pmatrix} \sqrt{E(1-s)}\,\xi^s \\ \sqrt{E(1+s)}\,\xi^s \end{pmatrix} \quad u_R^{(+)}(p) = \begin{pmatrix} 0 \\ \sqrt{2E}\,\xi^{(+)} \end{pmatrix}, \quad u_L^{(-)}(p) = \begin{pmatrix} \sqrt{2E}\,\xi^{(-)} \\ 0 \end{pmatrix}$$

For E>>m or m=0, helicity = chirality!!

v spinor (antiparticle)

$$v^s(p) = C(\bar{u}^s(p))^T = (i\gamma^2\gamma^0)(u^{s\dagger}(p)\gamma^0)^T = \begin{pmatrix} 0 & i\sigma^2 \\ -i\sigma^2 & 0 \end{pmatrix} u^{s*}(p) = \begin{pmatrix} \sqrt{E + s|\boldsymbol{p}|}(i\sigma^2)\xi^{s*} \\ \sqrt{E - s|\boldsymbol{p}|}(-i\sigma^2)\xi^{s*} \end{pmatrix}$$

$$\underset{E\gg m}{\rightarrow} \begin{pmatrix} \sqrt{E(1+s)}(i\sigma^2)\xi^{s*} \\ \sqrt{E(1-s)}(-i\sigma^2)\xi^{s*} \end{pmatrix}, \quad v_L^{(+)}(p) = \begin{pmatrix} -\sqrt{2E}\,\xi^{(-)} \\ 0 \end{pmatrix}, \quad v_R^{(-)}(p) = \begin{pmatrix} 0 \\ -\sqrt{2E}\,\xi^{(+)} \end{pmatrix}$$

For E>>m or m=0, helicity = -chirality

Vector current

U(1) vector current

$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi$ is invariant under $\psi \to \psi' = e^{-i\alpha}\psi$

Corresponding Noether current is

$$j^\mu(x) = \bar{\psi}(x)\gamma^\mu\psi(x), \quad \partial_\mu j^\mu(x) = 0.$$

$$j^\mu(x) = \big(n(x), \boldsymbol{j}(x)\big) \qquad \frac{\partial n}{\partial t} + \nabla \cdot \boldsymbol{j} = 0 \qquad \text{continuity eq.}$$

Let us decompose j^μ into right- and left-handed currents:

$$j^\mu(x) = \bar{\psi}(x)\gamma^\mu\psi(x) = \underbrace{\bar{\psi}_R(x)\gamma^\mu\psi_R(x)}_{j^\mu_R(x)} + \underbrace{\bar{\psi}_L(x)\gamma^\mu\psi_L(x)}_{j^\mu_L(x)}$$

$$\partial_\mu j^\mu_R(x) = im\bar{\psi}(x)\gamma_5\psi(x), \quad \partial_\mu j^\mu_L(x) = -im\bar{\psi}(x)\gamma_5\psi(x)$$

Each current is not conserved if $m \neq 0$.

Axial-vector current

U(1) axial-vector current

chiral transformation

$$\mathcal{L} = \bar{\psi} i \gamma^\mu \partial_\mu \psi \quad \text{is invariant under} \quad \psi \to \psi' = e^{-i\alpha\gamma_5}\psi$$

$$\psi_R \to \psi'_R = e^{-i\alpha}\psi_R, \ \psi_L \to \psi'_L = e^{+i\alpha}\psi_L$$

Corresponding Noether current is

$$j^{\mu 5}(x) = \bar{\psi}(x)\gamma^\mu\gamma_5\psi(x), \quad \partial_\mu j^{\mu 5}(x) = 0.$$

However, the chiral symmetry is broken by the mass term.

$$-\mathcal{L} \ni m\bar{\psi}\psi = m(\bar{\psi}_R\psi_L + \bar{\psi}_L\psi_R)$$

In this case

$$j^{\mu 5}(x) = \bar{\psi}(x)\gamma^\mu\gamma_5\psi(x) = j^\mu_R(x) - j^\mu_L(x)$$

$$\partial_\mu j^{\mu 5}(x) = 2im\bar{\psi}(x)\gamma_5\psi(x)$$

The axial-vector current is not conserved.

Spin 1 (vectors)

Massive vector field   2J+1 = 3 polarization states

transverse   longitudinal

rest frame

$$p^\mu = \begin{pmatrix} m \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad \boxed{\epsilon_1^\mu = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad \epsilon_2^\mu = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}} \qquad \boxed{\epsilon_3^\mu = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}}$$

boost in z direction $\downarrow$

$$E^2 - p_z^2 = m^2$$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}$$

$$\beta = v$$

$$\begin{pmatrix} E = \gamma m \\ 0 \\ 0 \\ p_z = \beta\gamma m \end{pmatrix} \qquad \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} \gamma\beta = \frac{p_z}{m} \\ 0 \\ 0 \\ \gamma = \frac{E}{m} \end{pmatrix}$$

grows with energy

$$p_\mu \epsilon_i^\mu = 0, \;\; \epsilon_{i\mu}\epsilon_j^{\mu *} = -\delta_{ij} \;\; (i,j = 1,2,3)$$

-> equivalence theorem

Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m^2 V_\mu V^\mu, \quad F_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu$$

V_μ: Proca field

EOM

$$\partial_\mu F^{\mu\nu} + m^2 V^\nu = 0 \qquad m^2 \neq 0 \longrightarrow \partial_\nu V^\nu = 0$$

$$\partial_\mu F^{\mu\nu} = \Box V^\nu - \partial^\nu(\partial_\mu V^\mu)$$

$$\therefore \begin{cases} (\Box + m^2)V^\mu = 0, \\ \partial_\mu V^\mu = 0 \end{cases}$$

no scalar mode @on-shell ($p^2=m^2$)

$$\epsilon_i^\mu \ (i = 1,2,3) \quad p_\mu \epsilon^\mu = 0 \qquad \sum_{i=1,2,3} \epsilon_i^\mu \epsilon_i^{*\nu} = -g^{\mu\nu} + \frac{p^\mu p^\nu}{m^2}$$

It is easy to do the polarization sum in the rest frame.

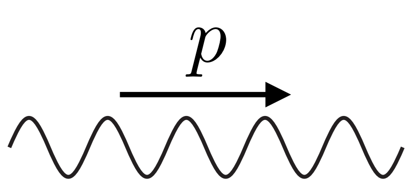
$$\sum_{i=1,2,3} \epsilon_i^\mu \epsilon_i^{\nu *} = \begin{pmatrix} 0 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} + \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix} = -g^{\mu\nu} + \frac{p^\mu p^\nu}{m^2}$$

The same result is obtained in the z-boosted frame and any frame.

Massive spin-1 vector = massless vector & massless scalar.

helicity (±1,0) helicity ±1 helicity 0

propagator



$$\frac{-i}{p^2 - m^2 + i\epsilon}\left[-g^{\mu\nu} + \frac{p^\mu p^\nu}{m^2}\right]$$

@off-shell ($p^2 \neq m^2$ $p^2>0$)

scalar mode can propagate

$$p^\mu = \begin{pmatrix} \sqrt{p^2} \\ 0 \\ 0 \\ 0 \end{pmatrix} \longrightarrow -g^{\mu\nu} + \frac{p^\mu p^\nu}{m^2} = \begin{pmatrix} \boxed{\frac{p^2-m^2}{m^2}} & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

"rest frame"

spin-1 field must be gauge field (see later).

Interactions

Possible interactions

Renormalizability is assumed. -> mass dim. <= 4 $D_\mu = \partial_\mu + igA_\mu$

	spin 0 ϕ	spin 1/2 ψ	spin 1 A_μ
spin 0 ϕ	$\phi^3,\ \phi^4,\ \lvert\phi\rvert^4$ scalar	$\bar{\psi}\psi\phi$ Yukawa	$\lvert D_\mu\phi\rvert^2$ gauge
spin 1/2 ψ	$\bar{\psi}\psi\phi$ Yukawa	N/A	$\bar{\psi}i\gamma^\mu D_\mu\psi$ gauge
spin 1 A_μ	$\lvert D_\mu\phi\rvert^2$ gauge	$\bar{\psi}i\gamma^\mu D_\mu\psi$ gauge	$F_{\mu\nu}F^{\mu\nu}$ gauge

Particle content and gauge symmetry are specified, all interactions are determined.

Lecture 2

U(1) gauge theory

Quantum ElectroDynamics (QED)

$$\mathcal{L}_{\text{QED}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\gamma^\mu D_\mu - m)\psi$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad D_\mu = \partial_\mu + ieQA_\mu.$$

e=|e|: positron charge

Q=-1 for electron

U(1) transformation

$$\psi(x) \to \psi'(x) = e^{iQ\theta(x)}\psi(x),$$
$$\bar{\psi}(x) \to \bar{\psi}'(x) = \bar{\psi}(x)e^{-iQ\theta(x)}.$$

Dψ is transformed as

$$\begin{aligned} D_\mu\psi \to D'_\mu\psi' &= (\partial_\mu + i|e|QA'_\mu)e^{iQ\theta(x)}\psi \\ &= e^{iQ\theta(x)}\left[\partial_\mu + i|e|Q\left(A'_\mu + \frac{1}{|e|}\partial_\mu\theta(x)\right)\right]\psi \\ &\equiv e^{iQ\theta(x)}(\partial_\mu + i|e|QA_\mu)\psi = e^{iQ\theta(x)}D_\mu\psi \end{aligned}$$

Therefore, A_μ must transform as $A'_\mu = A_\mu - \frac{1}{e}\partial\theta_\mu(x)$ [N.B.] $m^2 A_\mu A^\mu$ is forbidden.

Equation of Motion (EOM) $\partial_\mu\left(\frac{\partial\mathcal{L}_{\text{QED}}}{\partial(\partial_\mu A_\nu)}\right) - \frac{\partial\mathcal{L}_{\text{QED}}}{\partial A_\nu} = 0 \rightarrow \partial_\mu F^{\mu\nu} = eQ\bar{\psi}\gamma^\nu\psi = eQj^\nu.$

Note

Let us define the 4 vector potential and current as

$$A^\mu(x) = \big(\phi(x), \boldsymbol{A}(x)\big), \quad j^\mu(x) = \bar{\psi}(x)\gamma^\mu\psi(x) = \big(n(x), \boldsymbol{j}(x)\big).$$

electrostatic potential

electron number density

$$-\mathcal{L}_{\rm QED} \ni |e|Qj^\mu(x)A_\mu(x) \ni -|e|j^0(x)A_0(x) = -|e|n(x)\phi(x)$$

this sign must be minus.

Therefore, the notations of the covariant derivative are either

$$D_\mu = \partial_\mu - ieA_\mu \quad (e = |e|)$$ e is positron charge

or $$D_\mu = \partial_\mu + ieA_\mu \quad (e = -|e|)$$ e is electron charge

e.g. Peskin-Schroeder's book

SU(N) gauge theory

Lagrangian

$$\mathcal{L} = -\frac{1}{2}\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) + \bar{\psi}(i\gamma^\mu D_\mu - m)\psi,$$

$$D_\mu = \partial_\mu + igA_\mu, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu].$$

Note that $A_\mu = T^a A_\mu^a$ with $[T^a, T^b] = if^{abc}T^c$. $\quad A_\mu^\dagger = A_\mu,\ \mathrm{tr}A_\mu = 0$

ψ and A_μ transform as

$$\psi \to \psi' = U\psi = e^{i\theta^a T^a}\psi, \quad \bar{\psi} \to \bar{\psi}' = \bar{\psi}U^{-1},$$

$$A_\mu \to A'_\mu = UA_\mu U^{-1} + \frac{i}{g}(\partial_\mu U)U^{-1}.$$

$U^{-1} = U^\dagger$

Strong interaction: $SU(3)_C$, Weak interaction: $SU(2)_L$

Exercises

(1) Show that $A'^\dagger_\mu = A'_\mu, \quad \mathrm{tr}A'_\mu = 0.$

(2) Derive EOM for A_μ.

SU(3)c

Hadron particles that have strong interaction

$$p=(uud),\ n=(udd),\ \pi^+=(u\bar{d})\ ,\pi^-=(d\bar{u}),\ \text{etc.}$$

quark

$$q=\begin{pmatrix} q^1 \\ q^2 \\ q^3 \end{pmatrix} \begin{matrix} \text{R} \\ \text{G} \\ \text{B} \end{matrix}$$

$$q \to g' = Uq$$

$$U = e^{i\frac{\lambda^s}{2}\theta^s}$$

antiquark

$$\bar{q} = \begin{pmatrix} \bar{q}_1 & \bar{q}_2 & \bar{q}_3 \end{pmatrix}$$

$$\bar{q} \to \bar{q}' = \bar{q}U^\dagger$$

meson $\bar{q}q \to \bar{q}'q = \bar{q}U^\dagger Uq = \bar{q}q$

baryon $\epsilon_{ijk}q^iq^jq^k \to \epsilon_{ijk}q'^iq'^jq'^k = \epsilon_{ijk}U^i{}_\ell U^j{}_m U^k{}_n q^\ell q^m q^n$

$$= \det U \epsilon_{ijk}q^iq^jq^k = \epsilon_{ijk}q^iq^jq^k$$

Gauge fixing

Gauge fixing is necessary. Otherwise,

e.g., U(1) gauge theories.

$$\int [dA] e^{iS} = \int [dA] \exp\left[i \int d^4x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right) \right]$$
$$= \int [dA] \exp\left[i \int d^4x \left\{ \frac{1}{2} A_\mu(x) \left(\partial^2 g^{\mu\nu} - \partial^\mu \partial^\nu \right) A_\nu(x) \right\} \right]$$
$$= \int [dA] \exp\left[i \int \frac{d^4k}{(2\pi)^4} \left\{ \underline{\frac{1}{2} A_\mu(k) \left(-k^2 g^{\mu\nu} + k^\mu k^\nu \right) A_\nu(-k)} \right\} \right]$$

gauge inv. $A_\mu(k) \to A_\mu(k) + k_\mu \alpha(k)$

This functional integral is badly divergent for the $k_\mu \alpha(k)$ part.

Also, when you define propagator of A_μ,

$$(\partial^2 g_{\mu\nu} - \partial_\mu \partial_\nu) D^{\nu\rho}(x-y) = i \delta_\mu{}^\rho \delta^{(4)}(x-y)$$

$$\xrightarrow[\text{momentum space}]{} \quad (-k^2 g_{\mu\nu} + k_\mu k_\nu) D^{\nu\rho}(k) = i \delta_\mu{}^\rho$$

no solution since $\det(-k^2 g_{\mu\nu} + k_\mu k_\nu) = 0$

Those issues are due to gauge invariance.

Gauge fixing

We add a gauge-fixing term in the Lagrangian. E.g., $\mathcal{L} \ni -\frac{1}{2\xi}(\partial^\mu A_\mu)^2$

$$\left[-k^2 g_{\mu\nu} + \left(1 - \frac{1}{\xi}\right) k_\mu k_\nu\right] D^{\nu\rho}(k) = i\delta_\mu{}^\rho$$

Now we have a solution

$$D^{\mu\nu}(k) = \frac{-i}{k^2 + i\epsilon}\left(g^{\mu\nu} - (1-\xi)\frac{k^\mu k^\nu}{k^2}\right)$$

ξ =0 Landau gauge

ξ =1 Feynman gauge

Physical observables do not depend on ξ (see later).

In the SM, the gauge fixing terms are not simple. My favorite method is "gauge fixing based on BRS invariance."

Gauge fixing

T. Kugo and S. Uehara, NPB197 (1982) 378.

Gauge fixing based on BRS invariance (global symmetry).

$$\boldsymbol{\delta}_B \Phi(x) = igc(x)\Phi(x),$$

$$\boldsymbol{\delta}_B A_\mu(x) = \partial_\mu c(x) + ig\Big[c(x), A_\mu(x)\Big],$$

$$\boldsymbol{\delta}_B c^a(x) = \frac{ig}{2}\Big[c(x), c(x)\Big],$$

$$\boldsymbol{\delta}_B \bar{c}^a(x) = ib^a(x),$$

$$\boldsymbol{\delta}_B b^a(x) = 0.$$

$c(x) = c^a(x)T^a$: ghost field, $\bar{c}(x) = \bar{c}^a(x)T^a$: anti-ghost field,
$b(x) = b^a(x)T^a$: auxiliary field

$$\boxed{\mathcal{L}_{\mathrm{GF+FP}} = -i\boldsymbol{\delta}_B(\bar{c}^a F^a)}$$

F^a is a gauge-fixing function

When F^a does not contain ghost or anti-ghost fields

$$\boxed{\begin{aligned} \mathcal{L}_{\mathrm{GF}} &= b^a F^a, \\ \mathcal{L}_{\mathrm{FP}} &= i\bar{c}^a(x)(\boldsymbol{\delta}_B F^a). \end{aligned}}$$

Gauge fixing

E.g., SU(N) gauge theory

gauge-fixing function: $$\boxed{F^a = \partial^\mu A^a_\mu + \frac{\xi}{2} b^a}$$

$$\mathcal{L}_{\rm GF} = b^a F^a = b_a \partial^\mu A^a_\mu + \frac{\xi}{2}(b^a)^2 = \frac{\xi}{2}\left(b^a + \frac{1}{\xi}\partial^\mu A^a_\mu\right)^2 - \frac{1}{2\xi}(\partial^\mu A^a_\mu)^2$$

Eliminating the b-field by EOM, one has $\mathcal{L}_{\rm GF} = -\dfrac{1}{2\xi}(\partial^\mu A^a_\mu)^2.$

The FP ghost term is

$$\mathcal{L}_{\rm FP} = i\bar{c}^a \partial^\mu D_\mu c^a. \qquad c^{a\dagger} = c^a, \quad \bar{c}^{a\dagger} = \bar{c}^a$$

QED $$\mathcal{L}_{\rm GF} = -\frac{1}{2\xi}(\partial^\mu A_\mu)^2, \quad \mathcal{L}_{\rm FP} = i\bar{c}^a \Box c^a.$$

Gauge fixing

E.g., SM

gauge-fixing function:

$$\boxed{\begin{aligned} F_3^s(x) &= \partial^\mu G_\mu^s(x) + \frac{\xi_3}{2} b_3^s(x), \\ F_2^a(x) &= \partial^\mu A_\mu^a(x) - i\xi_2 g_2 \left(\phi^\dagger(x) \frac{\tau^a}{2} \varphi - \varphi^\dagger \frac{\tau^a}{2} \phi(x) \right) + \frac{\xi_2}{2} b_2^a(x), \\ F_1(x) &= \partial^\mu B_\mu(x) - i\xi_1 \frac{g_1}{2} \left(\phi^\dagger(x) \varphi - \varphi^\dagger \phi(x) \right) + \frac{\xi_1}{2} b_1(x). \end{aligned}}$$

where the Higgs doublet field is defined as $\Phi(x) = \varphi + \phi(x)$.

$$\begin{aligned} \mathcal{L}_{\rm GF} &= b_3^s(x) F_3^s(x) + b_2^a(x) F_2^a(x) + b_1(x) F_1(x) \\ &\to -\frac{1}{2\xi_3} \left(\partial^\mu G_\mu^s(x) \right)^2 - \frac{1}{2\xi_2} \left[\partial^\mu A_\mu^a(x) - i\xi_2 g_2 \left(\phi^\dagger(x) \frac{\tau^a}{2} \varphi - \varphi^\dagger \frac{\tau^a}{2} \phi(x) \right) \right]^2 \\ &\qquad - \frac{1}{2\xi_1} \left[\partial^\mu B_\mu(x) - i\xi_1 \frac{g_1}{2} \left(\phi^\dagger(x) \varphi - \varphi^\dagger \phi(x) \right) \right]^2 \end{aligned}$$

[N.B.] $\xi_2 = \xi_1$ simplifies calculations.

$$\mathcal{L}_{\rm FP} = i\bar{c}_3^s(x) \left[\boldsymbol{\delta}_B F_3^s(x) \right] + i\bar{c}_2^a(x) \left[\boldsymbol{\delta}_B F_2^a(x) \right] + i\bar{c}_1(x) \left[\boldsymbol{\delta}_B F_1(x) \right]$$

You may think

OK, the gauge fixing is necessary. Then, the gauge symmetry is broken at the Lagrangian level. So, is it possible to have SSB? 🤔

Elitzur's theorem

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Impossibility of spontaneously breaking local symmetries

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It is argued that a spontaneous breaking of local symmetry for a symmetrical gauge theory without gauge fixing is impossible. The argument is demonstrated in a simple system of Abelian gauge fields on a lattice.

- The local symmetry must be broken first explicitly by a gauge-fixing term leaving only global symmetry (BRS symmetry)

- This remaining global symmetry can be broken spontaneously.

Particle content of the SM

Quarks and Leptons	$SU(3)_C \times SU(2)_L \times U(1)_Y$	Q	B	L
$q_{iL} = \begin{pmatrix} u_{iL} \\ d_{iL} \end{pmatrix}$	$\left(\mathbf{3}, \mathbf{2}, \frac{1}{6}\right)$	$\begin{pmatrix} 2/3 \\ -1/3 \end{pmatrix}$	$1/3$	0
u_{iR}	$\left(\mathbf{3}, \mathbf{1}, \frac{2}{3}\right)$	$2/3$	$1/3$	0
d_{iR}	$\left(\mathbf{3}, \mathbf{1}, -\frac{1}{3}\right)$	$-1/3$	$1/3$	0
$\ell_{iL} = \begin{pmatrix} \nu_{iL} \\ e_{iL} \end{pmatrix}$	$\left(\mathbf{1}, \mathbf{2}, -\frac{1}{2}\right)$	$\begin{pmatrix} 0 \\ -1 \end{pmatrix}$	0	1
e_{iR}	$(\mathbf{1}, \mathbf{1}, -1)$	-1	0	1

i: generation indices, L: left-handed fermions, R: right-handed fermions $\quad Q = T^3 + Y$

B: baryon number, $\quad L$: lepton number global U(1) symmetries

gauge fields

SU(3)$_C$ $\quad G^s_\mu(x) \quad (\mathbf{8}, \mathbf{1}, 0) \quad s = 1, 2, \cdots, 8$ gluon

SU(2)$_L$ $\quad A^a_\mu(x) \quad (\mathbf{1}, \mathbf{3}, 0) \quad a = 1, 2, 3$

U(1)$_Y$ $\quad B_\mu(x) \quad (\mathbf{1}, \mathbf{1}, 0)$

Higgs field

$\Phi(x) \quad \left(\mathbf{1}, \mathbf{2}, \frac{1}{2}\right)$

Lagrangian of the SM

$$\mathcal{L}_{\rm SM} = \mathcal{L}_g + \mathcal{L}_f + \mathcal{L}_Y + \mathcal{L}_H,$$

$$\begin{aligned}
\mathcal{L}_g &= -\frac{1}{4}G^s_{\mu\nu}G^{s\mu\nu} - \frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu},\\
\mathcal{L}_f &= \bar{q}_{iL} i\gamma^\mu D_\mu q_{iL} + \bar{u}_{iR} i\gamma^\mu D_\mu u_{iR} + \bar{d}_{iR} i\gamma^\mu D_\mu d_{iR} + \bar{\ell}_{iL} i\gamma^\mu D_\mu \ell_{iL} + \bar{e}_{iR} i\gamma^\mu D_\mu e_{iR},\\
-\mathcal{L}_Y &= Y^u_{ij}\bar{q}_{iL}\tilde{\Phi}u_{jR} + Y^d_{ij}\bar{q}_{iL}\Phi d_{jR} + Y^e_{ij}\bar{\ell}_{iL}\Phi e_{jR} + \text{h.c.},\\
\mathcal{L}_H &= (D_\mu\Phi)^\dagger D^\mu\Phi - V(\Phi), \quad V(\Phi) = -\mu^2\Phi^\dagger\Phi + \lambda\big(\Phi^\dagger\Phi\big)^2.
\end{aligned}$$

Field strengths

$$\begin{aligned}
G^s_{\mu\nu} &= \partial_\mu G^s_\nu - \partial_\nu G^s_\mu + g_3 f^{stu} G^t_\mu G^u_\nu,\\
F^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g_2 \epsilon^{abc} A^b_\mu A^c_\nu,\\
B_{\mu\nu} &= \partial_\mu B_\nu - \partial_\nu B_\mu.
\end{aligned}$$

Covariant derivative

$$D_\mu = \partial_\mu + i g_3 \underbrace{\Lambda^s}_{\frac{\lambda^s}{2}} G^s_\mu(x) + i g_2 \underbrace{T^a}_{\frac{\tau^a}{2}} A^a_\mu(x) + i g_1 Y B_\mu(x)$$

Higgs field

$$\Phi(x) = \begin{pmatrix} G^+(x) \\ \frac{1}{\sqrt{2}}\big(v + h(x) + iG^0(x)\big) \end{pmatrix}, \quad \tilde{\Phi}(x) = i\tau^2\Phi^*(x) = \begin{pmatrix} \frac{1}{\sqrt{2}}\big(v + h(x) - iG^0(x)\big) \\ -G^-(x) \end{pmatrix} \quad G^-(x) = \big(G^+(x)\big)^*$$

Parameters of the SM

	notations	number of parameters
gauge couplings	$g_3,\ g_2,\ g_1$	3
quark masses	$m_u,\ m_c,\ m_t$ $m_d,\ m_s,\ m_b$	6
lepton masses	$m_e,\ m_\mu,\ m_\tau$	3
CKM matrix	$\theta_{12},\ \theta_{13},\ \theta_{23},\ \delta$	4
Higgs sector	$v,\ m_h$ $(\mu^2,\ \lambda)$	2

tot=18

All parameters are known with certain errors.

1 more parameter

The following operator is gauge invariant and renormalizable.

$$\mathcal{L}_g \ni \frac{g_3^2}{32\pi^2}\theta_{\rm QCD} G^s_{\mu\nu}\tilde{G}^{s\mu\nu}, \quad \tilde{G}^{s\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}G^s_{\rho\sigma} \quad \epsilon^{1234} = -\epsilon_{1234} = 1$$

- CP violating.

- Contributions to Electric Dipole Moment (EDM) of the neutron.

$$\bar{\theta} = \theta_{\rm QCD} - \arg\det(\underset{\substack{\uparrow \\ \text{Yukawa couplings of quarks}}}{Y^u Y^d})$$

Experimental upper bound $\bar{\theta}_{\rm EXP} < 10^{-10}$ extremely small! Why?

called strong CP problem

solutions: Peccei-Quinn mechanism, Nelson-Barr mechanism, etc.

Exercise

The following operator is also gauge invariant and renormalizable.

$$\mathcal{L}_g \ni \frac{g_2^2}{32\pi^2}\theta_{\rm EW} F^a_{\mu\nu}\tilde{F}^{a\mu\nu}, \quad \tilde{F}^{a\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F^a_{\rho\sigma}$$

However, $\theta_{\rm EW}$ is not physical in this case. Why??

Global symmetries

$$\mathcal{L}_f = \bar{q}_{iL} i\gamma^\mu D_\mu q_{iL} + \bar{u}_{iR} i\gamma^\mu D_\mu u_{iR} + \bar{d}_{iR} i\gamma^\mu D_\mu d_{iR} + \bar{\ell}_{iL} i\gamma^\mu D_\mu \ell_{iL} + \bar{e}_{iR} i\gamma^\mu D_\mu e_{iR}$$

invariant under 5 global U(3)

$$q_L \to U_{q_L} q_L,\ u_R \to U_{u_R} u_R,\ d_R \to U_{d_R} d_R,\ \ell_L \to U_{\ell_L} \ell_L,\ e_R \to U_{e_R} e_R$$

However, violated by the Yukawa interactions

$$-\mathcal{L}_Y = Y^u_{ij} \bar{q}_{iL} \tilde{\Phi} u_{jR} + Y^d_{ij} \bar{q}_{iL} \Phi d_{jR} + Y^e_{ij} \bar{\ell}_{iL} \Phi e_{jR} + \text{h.c.}$$

Only subgroups of $[U(3)]^5$ are invariant in the full Lagrangian.

$U(1)_B$ $\quad q_L \to e^{i\theta/3} q_L,\ u_R \to e^{i\theta/3} u_R,\ d_R \to e^{i\theta/3} d_R,$

$U(1)_L$ $\quad \ell_L \to e^{-i\theta} \ell_L,\ e_R \to e^{-i\theta} e_R$

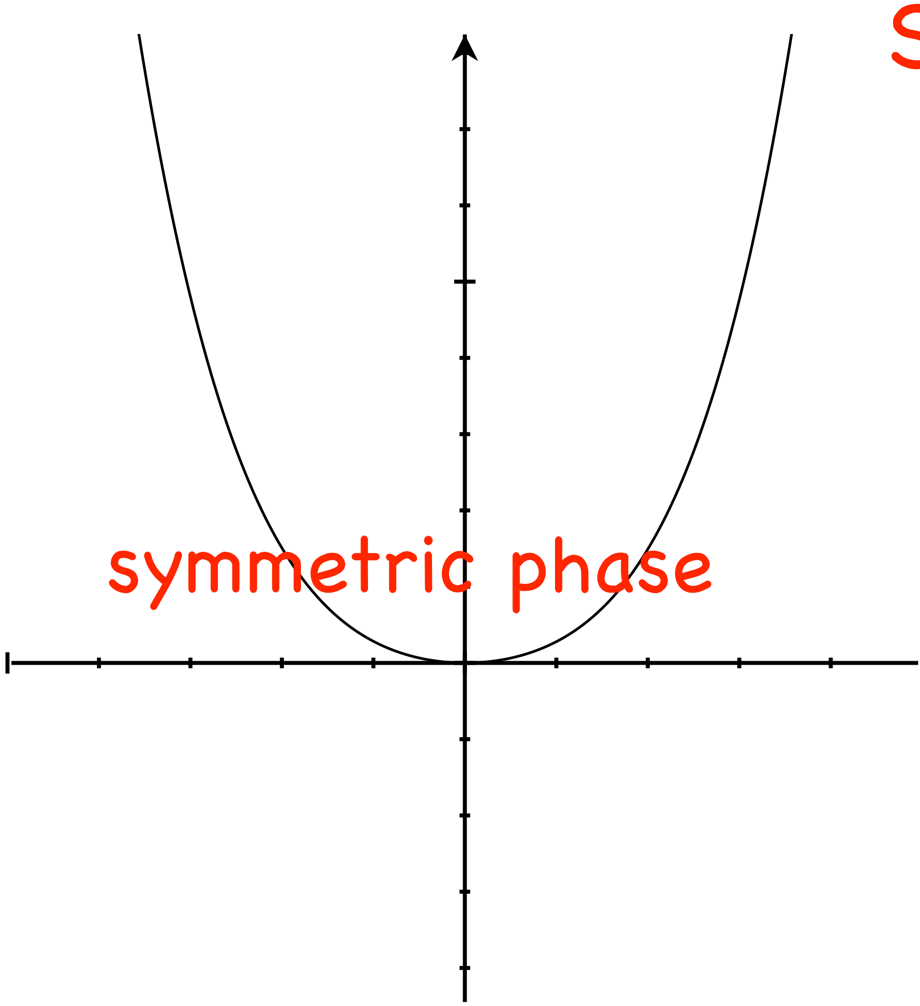
However, $U(1)_B$ and $U(1)_L$ are violated at quantum level (by chiral anomaly). Nevertheless, $U(1)_{B-L}$ is still unbroken.

$SU(3)_C \times SU(2)_L \times U(1)_Y$ -> $SU(3)_C \times U(1)_{em}$

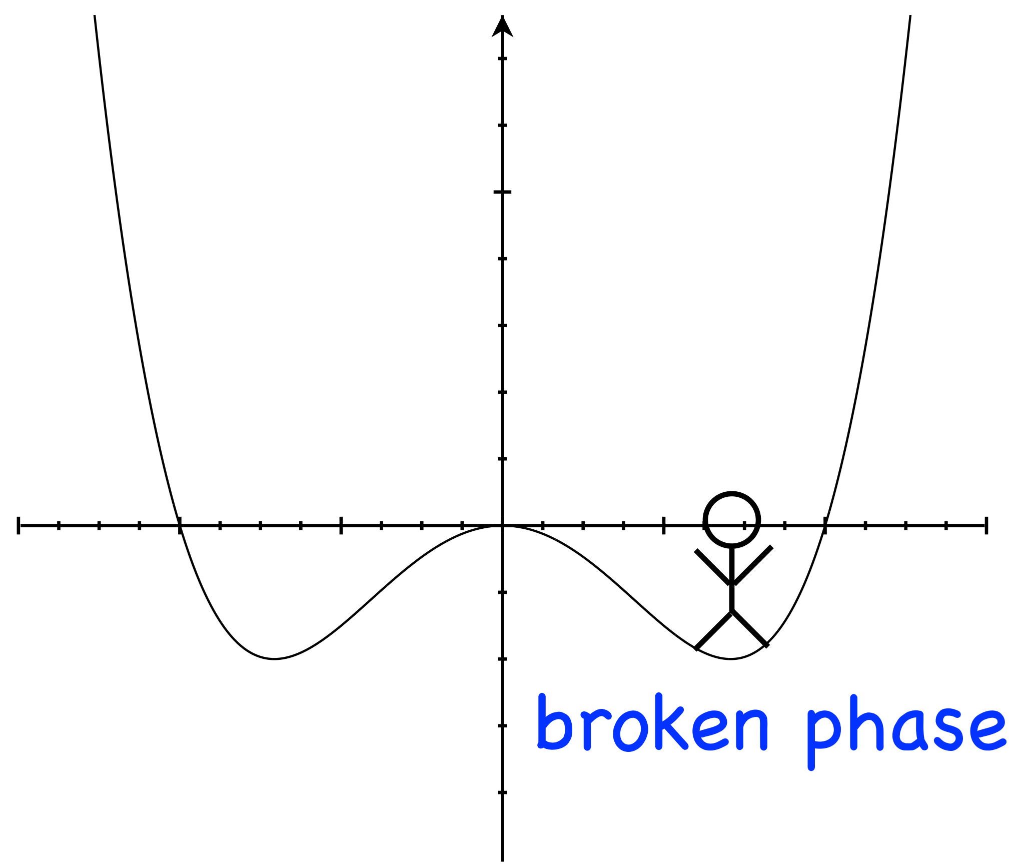
- EW symmetry is spontaneously broken by Higgs.

$$V(\Phi) = -\mu^2 \Phi^\dagger \Phi + \lambda (\Phi^\dagger \Phi)^2$$

SSB is induced by $-\mu^2$.



$$\langle \Phi \rangle = \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix} \longrightarrow$$



SM cannot answer why $-\mu^2$ rather than $+\mu^2$.

2 roles of Higgs

(1) symmetry breaker

$SU(3)_C \times SU(2)_L \times U(1)_Y \rightarrow SU(3)_C \times U(1)_{em}$

(2) Mass generator

Mass generation of all the elementary particles

Let us look into each of them.

Gauge boson mass generation

The gauge boson masses arise from $\mathcal{L} \ni (D_\mu \Phi)^\dagger (D^\mu \Phi)$

$$D_\mu \Phi = \left[\partial_\mu + i g_2 \frac{\tau^a}{2} A^a_\mu + i g_1 \frac{1}{2} B_\mu\right] \Phi$$

$$= \left[\partial_\mu + i\frac{g_2}{\sqrt{2}} \begin{pmatrix} 0 & W^+_\mu \\ W^-_\mu & 0 \end{pmatrix} + i\frac{g_2}{\cos\theta_W} \begin{pmatrix} (\frac{1}{2} - \sin^2\theta_W) Z_\mu & 0 \\ 0 & (-\frac{1}{2}) Z_\mu \end{pmatrix} + ie \begin{pmatrix} A_\mu & 0 \\ 0 & 0 \end{pmatrix}\right] \begin{pmatrix} G^+ \\ \frac{1}{\sqrt{2}}(v + h + iG^0) \end{pmatrix}$$

$$W^\pm_\mu = \frac{1}{\sqrt{2}}(A^1_\mu \mp i A^2_\mu), \quad \begin{pmatrix} Z_\mu \\ A_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta_W & -\sin\theta_W \\ \sin\theta_W & \cos\theta_W \end{pmatrix} \begin{pmatrix} A^3_\mu \\ B_\mu \end{pmatrix}$$

weak mixing (Winberg) angle

$$\cos\theta_W = \frac{g_2}{\sqrt{g_2^2 + g_1^2}}, \quad \sin\theta_W = \frac{g_1}{\sqrt{g_2^2 + g_1^2}}, \quad e = \frac{g_1 g_2}{\sqrt{g_2^2 + g_1^2}} = \sqrt{g_2^2 + g_1^2} \sin\theta_W \cos\theta_W = g_2 \sin\theta_W = g_1 \cos\theta_W$$

$$e = |e|$$

Gauge boson masses

$$(D_\mu \Phi)^\dagger (D^\mu \Phi) \ni m_W^2 W^+_\mu W^{-\mu} + \frac{m_Z^2}{2} Z_\mu Z^\mu, \quad m_W^2 = \frac{g_2^2}{4} v^2, \quad m_Z^2 = \frac{g_2^2 + g_1^2}{4} v^2$$

Photon remains massless owing to unbroken $U(1)_{em}$.

↓ provided by $G^{0,\pm}$

Massive vector field = 2 dof (transverse) + 1 dof (longitudinal)

Fermion mass generation

The fermion masses come from the Yukawa interactions.

$$-\mathcal{L}_Y = Y^u_{ij}\bar{q}_{iL}\tilde{\Phi}u_{jR} + Y^d_{ij}\bar{q}_{iL}\Phi d_{jR} + Y^e_{ij}\bar{\ell}_{iL}\Phi e_{jR} + \text{h.c.}$$

$$\Phi = \begin{pmatrix}\phi^+ \\ \phi^0\end{pmatrix} \underset{\text{unitary gauge}}{\to} \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}}(v+h(x))\end{pmatrix}, \quad \tilde{\Phi} = i\tau^2\Phi^* = \begin{pmatrix}\frac{1}{\sqrt{2}}(v+h(x)) \\ 0\end{pmatrix}$$

Rotate the fields as

$$u_L = V^u_L u'_L, \quad u_R = V^u_R u'_R; \quad d_L = V^d_L d'_L, \quad d_R = V^d_R d'_R; \quad e_L = V^e_L e'_L, \quad e_R = V^e_R e'_R$$

Fermion masses

$(Y^{u,d,e})_{ij}$ can be diagonalized by bi-unitary transformations.

$$V_L^{u,d,e\dagger}(Y^{u,d,e})V_R^{u,d,e} = \begin{pmatrix} y_1^{u,d,e} & & \\ & y_2^{u,d,e} & \\ & & y_3^{u,d,e}\end{pmatrix} \qquad -\mathcal{L}_Y \ni m_f\bar{f}f, \quad m_f = \frac{y_f}{\sqrt{2}}v$$

The fermion mass spectra are reflections of Yukawa structures.

Exercises

(1) Show that an arbitrary complex matrix can be diagonalized by bi-unitary transformation:

$$S^\dagger M U = \mathrm{diag}(m_1, m_2, \cdots, m_n)$$

$$U^\dagger U = U U^\dagger = 1, \; S^\dagger S = S S^\dagger = 1$$

(2) 2-by-2 matrix case

$$M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}, \quad S^\dagger M U = \begin{pmatrix} m_1 & \\ & m_2 \end{pmatrix}$$

Express $m_{1,2}$, S and U in terms of the matrix elements m_{ij}.

Higgs mass generation

The Higgs mass come from the Higgs potential.

$$V(\Phi) = -\mu^2 \Phi^\dagger \Phi + \lambda (\Phi^\dagger \Phi)^2$$

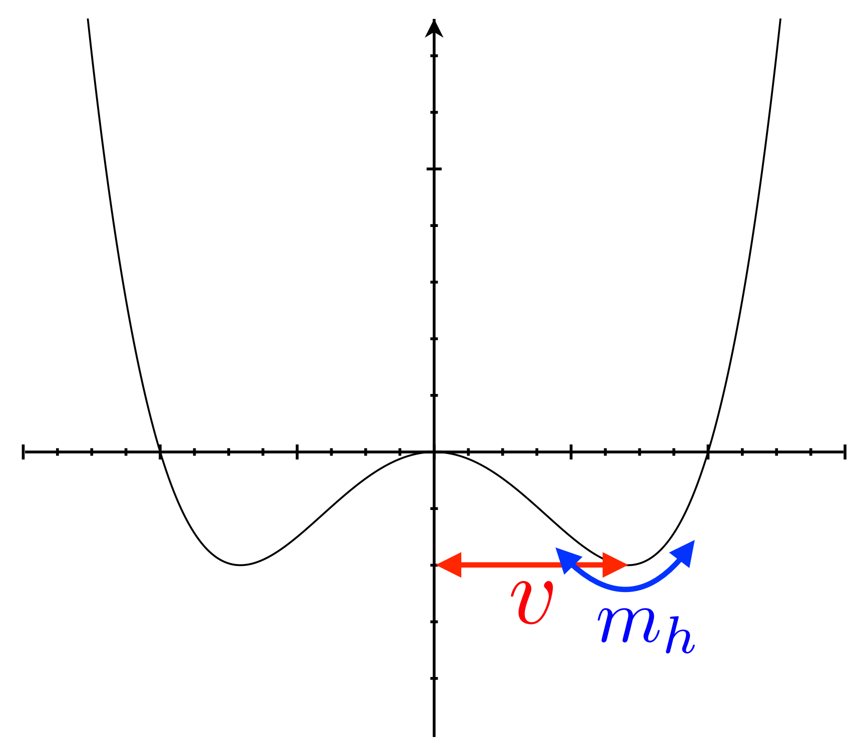
$$V(\Phi) \ni -\frac{\mu^2}{2} v^2 + \frac{\lambda}{4} v^4 - v(\mu^2 - \lambda v^2) h - \frac{1}{2}(\mu^2 - 3\lambda v^2) h^2 + \lambda v h^3 + \frac{\lambda}{4} h^4$$

vacuum condition

$$\left. \frac{\partial V}{\partial h} \right|_{h=0} = 0 \longrightarrow \mu^2 = \lambda v^2$$

Higgs mass

$$m_h^2 = \left. \frac{\partial^2 V}{\partial h^2} \right|_{h=0} = -\mu^2 + 3\lambda v^2 \underset{\mu^2 = \lambda v^2}{=} 2\lambda v^2$$



Experimental values:

$$v = 246 \text{ GeV}, \; m_h = 125 \text{ GeV} \longrightarrow \lambda = 0.13, \; \mu^2 = (88.4 \text{ GeV})^2$$

Exercises

(1) Derive m_{G^0} and $m_{G^\pm}$, where

$$V(\phi) \ni \frac{1}{2} m_{G^0}^2 (G^0)^2 + m_{G^\pm}^2 G^+ G^-$$

(2) Derive the couplings $\lambda_{hG^+G^-}$ and $\lambda_{hG^0G^0}$, where

$$V(\phi) \ni \frac{1}{2} \lambda_{hG^0G^0} h (G^0)^2 + \lambda_{hG^+G^-} h G^+ G^-$$

(3) Define $\Sigma = (G_1, G_2, G_3)^T$, where $G^\pm = (G_1 \pm i G_2)/\sqrt{2}$, $G_3 = G^0$. Show that V_0 is invariant under the global $SO(3)$ transformation, $\Sigma \to \Sigma' = O_3 \Sigma$.

Backup

References

incomplete list

There are many good text books and review articles.

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