Large charge expansion

Oleg Antipin

Institute Rudjer Boskovic, Croatia

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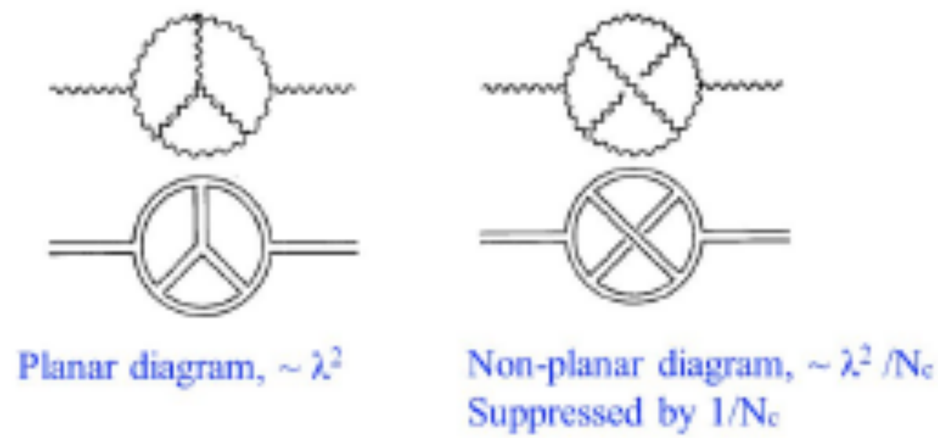
We want to solve QFT

Which tools do we have?

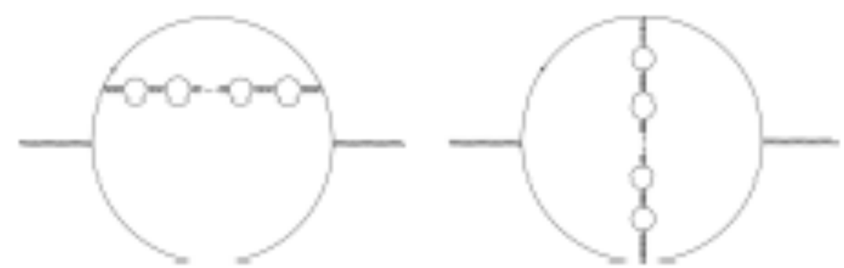
Perturbation theory…

Examples

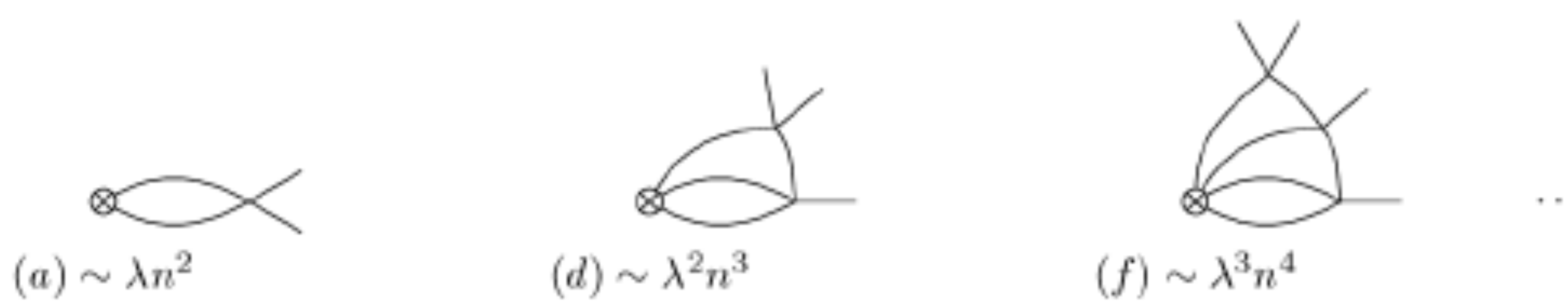
- Perturbative loop expansion in small coupling (Feynman diagrams)
- Large-N_c in $SU(N_c)$ gauge theories: Planar limit ($1/N_c$ expansion)



- Large-N_f: Bubble diagrams ($1/N_f$ expansion)



- Large-charge expansion (topic of this talk) ($1/Q$ expansion)



Reorganizing perturbative expansion

For a well-defined limit need to introduce 't Hooft coupling $\mathcal{A}$

- Large-N_c : Planar limit : $A_c \equiv g^2 N_c = fixed$
- Large-N_f : Bubble diagrams : $A_f \equiv g^2 N_f = fixed$
- Large-charge expansion : $A_Q \equiv g^2 Q = fixed$

Then we have

$$observable \sim \sum_{l=loops} g^l P_l(N) = \sum_k \frac{1}{N^k} F_k(\mathcal{A})$$

$$N = \{N_c, N_f, Q\}$$

Let us now see explicitly how this 't Hooft coupling emerges…

Perturbative loop expansion: semiclassical approach

Consider the two-point function in the $U(1)$ complex scalar model

$$\mathcal{S} = \int d^4x \left[\partial\bar{\phi}\partial\phi + \frac{\lambda_0}{4}\left(\bar{\phi}\phi\right)^2\right]$$

Rescale the field as $\phi \to \phi/\sqrt{\lambda_0}$:

$$\langle\bar{\phi}(x_f)\phi(x_i)\rangle \equiv \frac{\int D\phi D\bar{\phi}\,\bar{\phi}(x_f)\phi(x_i)e^{-S}}{\int D\phi D\bar{\phi}\,e^{-S}} = \frac{1}{\lambda_0}\frac{\int D\phi D\bar{\phi}\,\bar{\phi}(x_f)\phi(x_i)e^{-\frac{S}{\lambda_0}}}{\int D\phi D\bar{\phi}\,e^{-\frac{S}{\lambda_0}}}$$

Ordinary loop expansion with λ_0 the loop counting parameter. For $\lambda_0 \ll 1$ the path integral is dominated by the extrema of S.

Evaluate via a saddle point expansion by expanding the action around the stationary configuration $\phi_0 = 0$

$$S = S(\phi_0) + \frac{1}{2}(\phi - \phi_0)^2 S''(\phi_0) + \ldots$$

ϕ_0 is the solution of the classical EOM

Large charge expansion: Semiclassical approach

The operators ϕ^Q ($\bar{\phi}^Q$) carry $U(1)$ charge $+Q$ ($-Q$)

Consider the two-point function $\langle \bar{\phi}^Q \phi^Q \rangle$ and rescale the field as $\phi \to \phi\sqrt{Q}$

$$\langle \bar{\phi}^Q(x_f)\phi^Q(x_i)\rangle = Q^Q \frac{\int D\phi D\bar{\phi}\, \bar{\phi}^Q(x_f)\phi^Q(x_i) e^{-Q\mathcal{S}}}{\int D\phi D\bar{\phi}\, e^{-Q\mathcal{S}}}$$

ϕ^Q and $\bar{\phi}^Q$ can be brought up in the exponent, obtaining

$$\langle \bar{\phi}^Q(x_f)\phi^Q(x_i)\rangle = Q^Q \frac{\int D\phi D\bar{\phi}\; e^{-Q\left[\int \partial\bar{\phi}\partial\phi + \frac{Q\lambda_0}{4}(\bar{\phi}\phi)^2 - \ln\bar{\phi}(x_f) - \ln\phi(x_i)\right]}}{\int D\phi D\bar{\phi}\; e^{-Q\left[\int \partial\bar{\phi}\partial\phi + \frac{Q\lambda_0}{4}(\bar{\phi}\phi)^2\right]}} .$$

In a CFT

$$\langle \bar{\phi}^Q(x_f)\phi^Q(x_i)\rangle_{CFT} = \frac{1}{|x_f - x_i|^{2\Delta_{\phi^Q}}}$$

is physical (critical exponents)

$$\Delta_{\phi^Q} \equiv Q\left(\frac{d-2}{2}\right) + \gamma_{\phi^Q}$$

Large charge expansion: Semiclassical approach

$$\langle\bar{\phi}^Q(x_f)\phi^Q(x_i)\rangle = Q^Q \frac{\int D\phi D\bar{\phi}\; e^{-Q\left[\int \partial\bar{\phi}\partial\phi + \frac{Q\lambda_0}{4}(\bar{\phi}\phi)^2 - \ln\bar{\phi}(x_f) - \ln\phi(x_i)\right]}}{\int D\phi D\bar{\phi}\; e^{-Q\left[\int \partial\bar{\phi}\partial\phi + \frac{Q\lambda_0}{4}(\bar{\phi}\phi)^2\right]}}$$

The dependence on λ_0 and Q shows that **we can perform the path integral via a saddle point expansion** around the stationary points of

$$\mathcal{S}_{\text{eff}} \equiv \int d^dx \left[\int \partial\bar{\phi}\partial\phi + \frac{Q\lambda_0}{4}(\bar{\phi}\phi)^2 - \ln\bar{\phi}(x_f) - \ln\phi(x_i)\right]$$

in the limit of large Q, while keeping $\lambda_0 Q$ fixed. Q counts loops.
The result is a 't Hooft-like expansion in the coupling $\mathcal{A}_0 = \lambda_0 Q$.
The scaling dimension of ϕ^Q takes the **large charge expansion** form

$$\boxed{\Delta_{\phi^Q} = \sum_{k=-1} \frac{1}{Q^k}\Delta_k(\mathcal{A}_0)}$$

Δ_k is the $(k+1)$-loop correction in the saddle point expansion.

$$\langle\bar{\phi}^Q(x_f)\phi^Q(x_i)\rangle_{CFT} = \frac{1}{|x_f - x_i|^{2\Delta_{\phi^Q}}}$$

Small $\lambda_0 Q$: Reorganizing perturbative expansion

g can be quartic Yukawa gauge or coupling	**1-loop**	**2-loop**	**3-loop**	
Δ_{-1}	$Q^2\lambda_0$	$Q^3\lambda_0^2$	$Q^4\lambda_0^3$	$\dots$
Δ_0	$Q\lambda_0$	$Q^2\lambda_0^2$	$Q^3\lambda_0^3$	$\dots$
Δ_1		$Q\lambda_0^2$	$Q^2\lambda_0^3$	$\dots$
Δ_2			$Q\lambda_0^3$	$\dots$
$\vdots$				

Small λ₀Q: This work computation

g can be quartic Yukawa gauge or coupling	1-loop	2-loop	3-loop	
Δ_{-1}	$Q^2\lambda_0$	$Q^3\lambda_0^2$	$Q^4\lambda_0^3$	....
Δ_0	$Q\lambda_0$	$Q^2\lambda_0^2$	$Q^3\lambda_0^3$	....
Δ_1		$Q\lambda_0^2$	$Q^2\lambda_0^3$	....
Δ_2			$Q\lambda_0^3$	....
$\vdots$				

Large $\lambda_0 Q$: Large charge limit

Orlando et al 2015

$$\Delta_Q = \sum_{k=-1} \frac{\Delta_k(\lambda_0 Q)}{Q^k}$$

$$\Delta_Q = Q^{\frac{d}{d-1}} \left[\alpha_1 + \alpha_2 Q^{\frac{-2}{d-1}} + \alpha_3 Q^{\frac{-4}{d-1}} + \ldots\right] + Q^0 \left[\beta_0 + \beta_1 Q^{\frac{-2}{d-1}} + \ldots\right] + \mathcal{O}\left(Q^{-\frac{d}{d-1}}\right)$$

EFT for phonons (superfluid phase)

Semiclassical computation

$$S = S(\phi_0) + \frac{1}{2}(\phi - \phi_0)^2 S''(\phi_0) + \ldots$$

Δ_{-1} $\quad$ Δ_0

Method

Rattazzi et al 2019

In a CFT

$$\langle \bar{\phi}^Q(x_f)\phi^Q(x_i)\rangle_{CFT} = \frac{1}{|x_f - x_i|^{2\Delta_{\phi^Q}}}$$

- Tune QFT to the perturbative fixed point
- Map the theory to the cylinder $\mathbb{R}^d \to \mathbb{R} \times S^{d-1}$
- Exploit operator/state correspondence for the 2-point function to relate anomalous dimension to the energy $E=\Delta/R$
- To compute this energy evaluate expectation value of the evolution operator in an arbitrary state with fixed charge Q

• Tune QFT to the perturbative fixed point

1) In D=d-$\boldsymbol{\varepsilon}$ dimensions, formal Wilson-Fisher fixed point exists

$$\beta(g) = -\epsilon g + \beta_{d=4}(g) = 0 \quad \Rightarrow \quad g^* = f(\epsilon)$$

2) In D=d dimensions, fixed point might exists with small parameter $\boldsymbol{\varepsilon}$ build from parameters of the model (e.g. numbers of colors, flavors, fields components, etc)

Example: Banks-Zaks FP in d=4 multi-flavor QCD, $\boldsymbol{\varepsilon}$=Nf/Nc

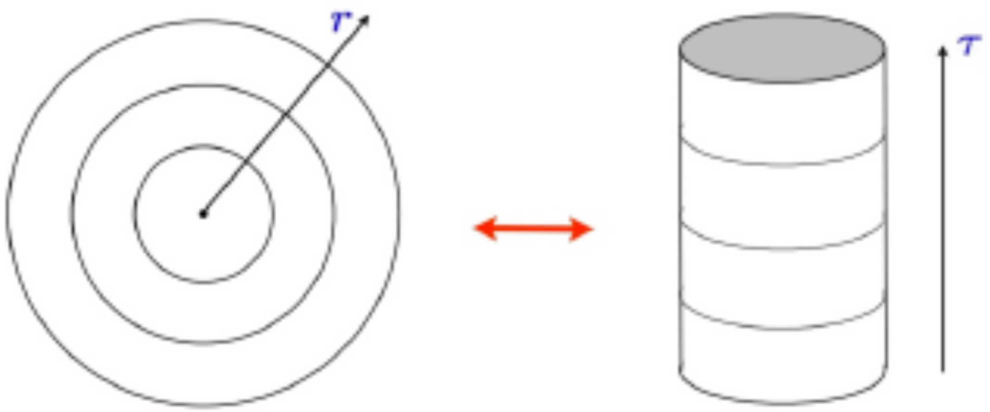
• Weyl map and operator/state correspondence

Working at the WF fixed point we can map the theory to the cylinder.

$$\mathbb{R}^d \to \mathbb{R} \times S^{d-1}, \qquad r = Re^{\tau/R}$$

In a CFT

$$\langle \bar{\phi}^Q(x_f)\phi^Q(x_i)\rangle_{CFT} = \frac{1}{|x_f - x_i|^{2\Delta_{\phi^Q}}}$$



The eigenvalues of the dilation charge, i.e. the scaling dimensions, become the energy spectrum on the cylinder.

$$\boxed{E_{\phi^Q} = \Delta_{\phi^Q}/R}$$

State-operator correspondence:
States and operators are in 1-to-1 correspondence.

Proof:

$$\langle \bar{\phi}^Q(x_f)\phi^Q(x_i)\rangle_{cyl} = |x_f|^{\Delta_{\phi^Q}}|x_i|^{\Delta_{\phi^Q}}\langle \bar{\phi}^Q(x_f)\phi^Q(x_i)\rangle_{flat} = \frac{|x_f|^{\Delta_{\phi^Q}}|x_i|^{\Delta_{\phi^Q}}}{|x_f - x_i|^{2\Delta_{\phi^Q}}} \overset{\tau_i \to -\infty}{=} e^{-E_{\Delta_{\phi^Q}}(\tau_f - \tau_i)}$$

• Computing the energy

$$\boxed{E_{\phi^Q} = \Delta_{\phi^Q}/R}$$

ϕ^Q is the charge-Q operator with the smallest scaling dimension $\Longrightarrow$

We have to compute the ground state energy.

Δ_{ϕ^Q} is given by the expectation value of the evolution operator e^{-HT} in an arbitrary state $|Q\rangle$ with fixed charge Q.

$$\langle Q| e^{-HT} |Q\rangle \underset{T\to\infty}{=} \mathcal{N}\, e^{-\frac{\Delta_{\phi^Q}}{R} T}$$

R is the radius of the sphere and H is the Hamiltonian.

To study system at fixed charge:

$$H \to H + \mu Q$$

μ is chemical potential

Example : O(N) model at WF fixed point

Lagrangian

$$\mathcal{S} = \int d^d x \left(\frac{(\partial \phi_i)^2}{2} + \frac{(4\pi)^2 g_0}{4!} (\phi_i \phi_i)^2 \right)$$

In $d = 4 - \epsilon$, this theory features an infrared Wilson Fisher fixed point.

$$g^*(\epsilon) = \frac{3\epsilon}{8+N} + \frac{9(3N+14)\epsilon^2}{(8+N)^3} + \mathcal{O}(\epsilon^3)$$

Weyl map the theory to the cylinder:

$$\mathcal{S}_{cyl} = \int d^d x \sqrt{g} \left(g_{\mu\nu} \partial^\mu \bar{\phi}_i \partial^\nu \phi_i + m^2 \bar{\phi}_i \phi_i + \frac{(4\pi)^2 g_0}{6} (\bar{\phi}_i \phi_i)^2 \right)$$

$$m^2 = \left(\frac{d-2}{2R}\right)^2$$

stemming from the coupling to Ricci scalar

$O(N)$ charges

In the $O(N)$ vector model with even N we can fix up to $\frac{N}{2}$ charges, which is the **rank** of the $O(N)$ group.
We introduce complex field variables

$$\begin{aligned}
\varphi_1 &= \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2) = \frac{1}{\sqrt{2}}\sigma_1\, e^{i\chi_1}\,, \\
\varphi_2 &= \frac{1}{\sqrt{2}}(\phi_3 + i\phi_4) = \frac{1}{\sqrt{2}}\sigma_2\, e^{i\chi_2}\,, \\
\varphi_3 &= \dots
\end{aligned}$$

We fix $N/2$ charges through $N/2$ constraints $Q_i = \bar{Q}_i$, where $\{\bar{Q}_i\}$ is a set of fixed constants. φ_i ($\bar{\varphi}_i$) has charge $\bar{Q}_i = 1$ (-1). Then we map the theory to the cylinder.

Classical solution

$$S = S(\phi_0) + \frac{1}{2}(\phi - \phi_0)^2 S''(\phi_0) + \dots$$

The solution of the EOM with minimal energy is spatially homogeneous

$$\sigma_i = A_i \qquad , \ \chi_i = -i\mu\tau \qquad\qquad i = 1, \dots, N/2$$

where

μ is chemical potential

$$\mu^2 - m^2 = \frac{(4\pi)^2}{6} g_0 v^2 \qquad \text{EOM}$$

$$\frac{\bar{Q}}{\text{vol.}} = \mu v^2 \qquad \text{Noether charge}$$

$$v^2 \equiv \sum_{i=1}^{k} A_i^2 \qquad \text{Sum of the VeVs}$$

$$\bar{Q} \equiv \sum_{i=1}^{k} \bar{Q}_i \qquad \text{Sum of the charges}$$

There is only a single chemical potential μ, even if the charges $\bar{Q}_i$ are all different.

Symmetry breaking pattern

We fix $N/2$ charges.

1. Since there is a single chemical potential the system preserves the $U(N/2)$ symmetry.
2. Then the vacuum of the theory spontaneously breaks $U(N/2)$ to $U(N/2-1)$. In fact it is possible to rotate the ground state as

$$\frac{1}{\sqrt{2}}(A_1, \ldots, A_{N/2}) \longrightarrow \big(\underbrace{0, \ldots, 0}_{N/2-1}, \tfrac{v}{\sqrt{2}}\big)$$

The symmetry breaking pattern is

$$\boxed{U(N/2) \rightarrow U(N/2-1)}$$

The sum of the charges acts as a single charge!

Effective action

$$\langle \bar{Q}| \, e^{-HT} \, |\bar{Q}\rangle = \frac{1}{\mathcal{Z}} \int_{\sigma_{N/2}=v}^{\sigma_{N/2}=v} D^n\sigma \; D^n\chi \; e^{-\mathcal{S}_{\text{eff}}}$$

$$\mathcal{S}_{\text{eff}} = \int_{-T/2}^{T/2} d\tau \int d\Omega_{d-1} \left(\frac{1}{2}\partial\sigma_i\partial\sigma_i + \frac{1}{2}\sigma_i^2(\partial\chi_i\partial\chi_i) + \frac{m^2}{2}\sigma_i^2 + \frac{(4\pi)^2}{24} g_0(\sigma_i\sigma_i)^2 + \frac{i}{\text{vol.}} \bar{Q} \, \dot{\chi}_{N/2} \right)$$

The red term fixes the charge of initial and final states to Q.

$$H \to H + \mu Q$$

$$S = S(\phi_0) + \frac{1}{2}(\phi - \phi_0)^2 S''(\phi_0) + \dots$$

$$\sigma_i = A_i \quad , \ \chi_i = -i\mu\tau$$

$$S_{eff} = \int_{-T/2}^{T/2} d\tau \int d\Omega_{d-1} \Big(\frac{1}{2}\partial\sigma_i\partial\sigma_i + \frac{1}{2}\sigma_i^2(\partial\chi_i\partial\chi_i) + \frac{m^2}{2}\sigma_i^2 + \frac{(4\pi)^2}{24}g_0(\sigma_i\sigma_i)^2 + \frac{i}{\text{vol.}}\bar{Q}\,\dot{\chi}_{N/2} \Big)$$

$$\frac{S_{eff}}{T} = \frac{\bar{Q}}{2}\left(\frac{3}{2}\mu + \frac{1}{2}\frac{m^2}{\mu}\right)$$

$$\mu^2 - m^2 = \frac{(4\pi)^2}{6}g_0 v^2$$

$$\frac{\bar{Q}}{\text{vol.}} = \mu v^2$$

$$\mu(\mu^2 - m^2) = \frac{g_0\bar{Q}}{4R^{D-1}\Omega_{D-1}}$$

$$m^2 = \left(\frac{d-2}{2R}\right)^2$$

Leading order: Δ_{-1}

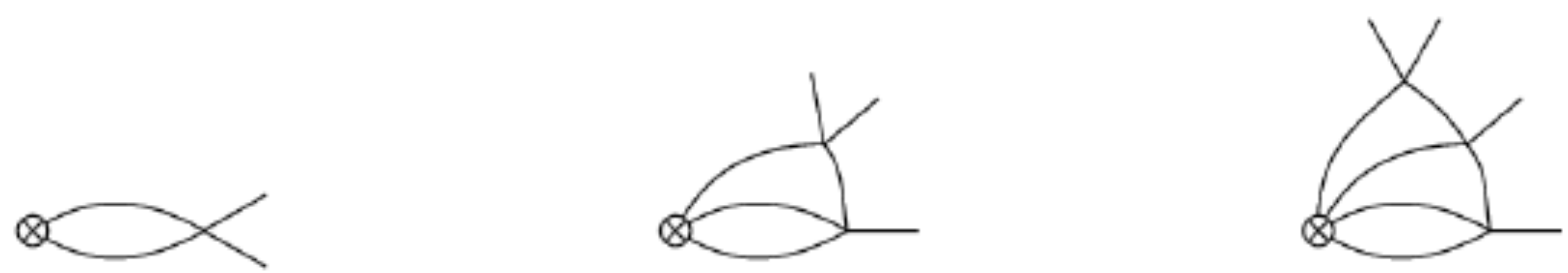
Δ_{-1} is given by the effective action evaluated on the classical trajectory at the fixed point

$$S_{eff}R = E_{-1}R = \Delta_{-1}$$

$$\frac{4\Delta_{-1}}{g^*\bar{Q}} = \frac{3^{\frac{2}{3}}\left(x+\sqrt{-3+x^2}\right)^{\frac{1}{3}}}{3^{\frac{1}{3}}+\left(x+\sqrt{-3+x^2}\right)^{\frac{2}{3}}} + \frac{3^{\frac{1}{3}}\left(3^{\frac{1}{3}}+\left(x+\sqrt{-3+x^2}\right)^{\frac{2}{3}}\right)}{\left(x+\sqrt{-3+x^2}\right)^{\frac{1}{3}}}$$

where $x \equiv 6g^*\bar{Q}$.

This classical result resums an infinite number of Feynman diagrams!



$(a) \sim \lambda n^2$ $(d) \sim \lambda^2 n^3$ $(f) \sim \lambda^3 n^4$

Small x:

$$\frac{\Delta_{-1}}{g^*} = \bar{Q}\left[1+\frac{1}{3}g^*\bar{Q}-\frac{2}{9}(g^*\bar{Q})^2+\frac{8}{27}(g^*\bar{Q})^3+\mathcal{O}\left((g^*\bar{Q})^4\right)\right]$$

separated by value of μ

Large x:

$$\frac{\Delta_{-1}}{g_*} = \frac{3}{4g_*}\left[\frac{3}{4}\left(\frac{4g_*\bar{Q}}{3}\right)^{\frac{4}{3}}+\frac{1}{2}\left(\frac{4g_*\bar{Q}}{3}\right)^{\frac{2}{3}}+\mathcal{O}(1)\right]$$

LO result

g is quartic coupling	**1-loop**	**2-loop**	**3-loop**	
Δ_{-1}	$Q^2 g$	$Q^3 g^2$	$Q^4 g^3$	$\ldots.$
Δ_0	Qg	$Q^2 g^2$	$Q^3 g^3$	$\ldots.$
Δ_1		Qg^2	$Q^2 g^3$	$\ldots.$
Δ_2			Qg^3	$\ldots.$
$\vdots$				

Leading quantum correction:

$$S = S(\phi_0) + \frac{1}{2}(\phi - \phi_0)^2 S''(\phi_0) + \dots$$

$$\begin{cases} \chi_i = -i\mu t \ + \frac{1}{v} p_i(x) \,, & i = 1, \dots, \frac{N}{2} - 1 \,, \\ \chi_{N/2} = -i\mu t \ + \frac{1}{v}\pi(x) \,, & \\ \sigma_i = s_i(x) \,, & i = 1, \dots, \frac{N}{2} - 1 \,, \\ \sigma_{N/2} = v + r(x) & \end{cases}$$

Expand to quadratic order in fluctuations:

$$\mathcal{L}_2 = \frac{1}{2}(\partial\pi)^2 + \frac{1}{2}(\partial r)^2 + (\mu^2 - m^2) r^2 - 2 \; i \; \mu \; r \; \dot{\pi} + \frac{1}{2}\partial s_i \partial s_i + \frac{1}{2}\partial p_i \partial p_i - 2 \; i \; \mu \; s_i \; \dot{p}_i$$

Gaussian integral of the action (B is a NxN matrix)

$$\int \mathcal{D}r \mathcal{D}\pi \mathcal{D}s_i \mathcal{D}p_i \, e^{-S^{(2)}} = \frac{1}{\det \ B}$$

Fluctuations spectrum

Phonon

- One relativistic (Type I) Goldstone boson (the conformal mode) and one massive state with mass $\sqrt{6\mu^2 - 2m^2}$.

$$\omega_\pm(l) = \sqrt{J_\ell^2 + 3\mu^2 - m^2 \pm \sqrt{4J_\ell^2\mu^2 + (3\mu^2 - m^2)^2}}$$

- $\frac{N}{2} - 1$ non-relativistic (Type II) Goldstone bosons and $\frac{N}{2} - 1$ massive states with mass 2μ

$$\omega_{\pm\pm}(l) = \sqrt{J_\ell^2 + \mu^2} \pm \mu$$

$J_\ell^2 = \ell(\ell + d - 2)/R^2$ is the eigenvalue of the Laplacian on the sphere.

One-loop correction: Δ_0 **(sum of zero point energies)**

The one-loop correction Δ_0 is determined by the fluctuation determinant around the classical trajectory. It reads

$$\Delta_0 = \frac{R}{2}\sum_{\ell=0}^{\infty} n_\ell \left[\omega_+(\ell) + \omega_-(\ell) + (\tfrac{N}{2} - 1)(\omega_{++}(\ell) + \omega_{--}(\ell))\right]$$

where n_ℓ is the multiplicity of the Laplacian on the $(d-1)$-dimensional sphere and the ω_i are the dispersion relations of the fluctuations counted with their multiplicity.

Small x:

$$\Delta_0(g^*\bar{Q}) = -\left(\frac{5}{3} + \frac{N}{6}\right) g^*\bar{Q} + \left(\frac{1}{3} - \frac{N}{18}\right) (g^*\bar{Q})^2 + \frac{1}{27}[N - 36 + 28\ \zeta(3) + 2N\ \zeta(3)]\ (g^*\bar{Q})^3 + \mathcal{O}\left((g^*\bar{Q})^4\right)$$

Large x:

$$\Delta_0 = \left[\alpha + \frac{N+8}{48}\ln\left(\frac{4g^*\bar{Q}}{3}\right)\right]\left(\frac{4g^*\bar{Q}}{3}\right)^{\frac{4}{3}} + \left[\beta - \frac{N+8}{72}\ln\left(\frac{4g^*\bar{Q}}{3}\right)\right]\left(\frac{4g^*\bar{Q}}{3}\right)^{\frac{2}{3}} + \mathcal{O}(1).$$

$$\alpha = -0.4046 - 0.0854N$$

$$\beta = -0.8218 - 0.0577N$$

EFT regimes

Solve:

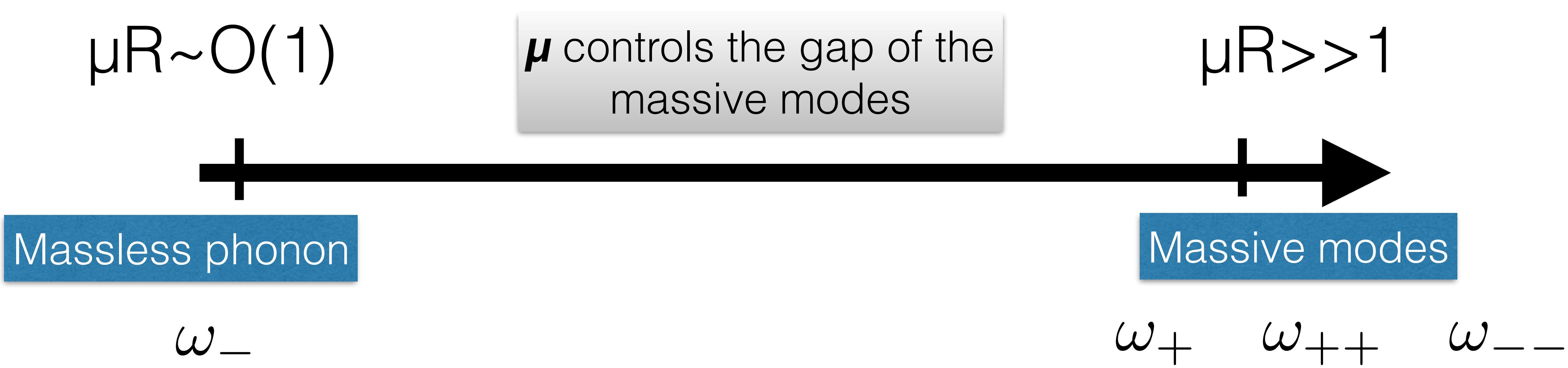
$$\mu(\mu^2 - m^2) = \frac{g_0 \bar{Q}}{4R^{D-1}\Omega_{D-1}}$$

Small g_0Q:

$$\mu R = 1 + \frac{g_0 \bar{Q}}{16\pi^2} + \dots$$

Large g_0Q:

$$\mu R = \frac{(g_0 \bar{Q})^{1/3}}{2\pi^{2/3}} + \dots$$



NLO result

g is quartic coupling	1-loop	2-loop	3-loop	
Δ_{-1}	$Q^2 g$	$Q^3 g^2$	$Q^4 g^3$	$\ldots\ldots$
Δ_0	Qg	$Q^2 g^2$	$Q^3 g^3$	$\ldots\ldots$
Δ_1		Qg^2	$Q^2 g^3$	$\ldots\ldots$
Δ_2			Qg^3	$\ldots\ldots$
$\vdots$				

Identify the operator

We want the smallest dimension operator carrying a total charge $\bar{Q}$

1. Derivatives increase the scaling dimension $\Longrightarrow$ we consider operator without derivatives.
2. The latter belong to the fully symmetric $O(N)$ space $\Longrightarrow$ m-index traceless symmetric tensors, $T^{(m)}_{(i_1\ldots i_m)}\phi^{2p}$. They have charge m and classical dimension $m+2p \Longrightarrow p=0$.
3. **Thus our operator is the $\bar{Q}$-index traceless symmetric tensor with classical dimension $\bar{Q}$.** It can be represented as a $\bar{Q}$-boxes Young tableau with one row.

$$\mathcal{O}_{\bar{Q}} = \underbrace{\Box\Box\Box\Box\ldots\Box}_{Q}$$

$\Delta_{\bar{Q}}$ define a set of **crossover (critical) exponent** which measures the stability of the system (e.g. critical magnets) against anisotropic perturbations (e.g. crystal structure).

Boosting perturbation theory to 4-loops

We can expand our result for small 't Hooft coupling $g\bar{Q}$ and obtain the conventional loop expansion

$$\begin{aligned}
\Delta_{\bar{Q}} &= \bar{Q} + \left(-\frac{\bar{Q}}{2} + \frac{\bar{Q}(\bar{Q}-1)}{8+N}\right)\epsilon - \left[\frac{2}{(8+N)^2}\bar{Q}^3 + \frac{(N-22)(N+6)}{2(8+N)^3}\bar{Q}^2 + \frac{184+N(14-3N)}{4(8+N)^3}\bar{Q}\right]\epsilon^2 \\
&+ \left[\frac{8}{(8+N)^3}\bar{Q}^4 + \frac{-456-64N+N^2+2(8+N)(14+N)\zeta(3)}{(8+N)^4}\bar{Q}^3\right. \\
&- \frac{-31136-8272N-276N^2+56N^3+N^4+24(N+6)(N+8)(N+26)\zeta(3)}{4(N+8)^5}\bar{Q}^2 \\
&\left.+ \frac{-65664-8064N+4912N^2+1116N^3+48N^4-N^5+64(N+8)(178+N(37+N))\zeta(3)}{16(N+8)^5}\bar{Q}\right]\epsilon^3 \\
&+ \left[c_5\bar{Q}^5 + c_4\bar{Q}^4 + c_3\bar{Q}^3 + c_2\bar{Q}^2 + c_1\bar{Q}\right]\epsilon^4 + \mathcal{O}\left(\epsilon^5\right)
\end{aligned}$$

Red terms: obtained via the semiclassical large charge expansion.
Black terms: obtained by combining the knowledge of the red ones with the known perturbative results for the $\bar{Q}=1$, $\bar{Q}=2$ and $\bar{Q}=4$ cases.

$\bar{Q}$=1 and N=4 is the anomalous dimension of the Higgs field

Boosting perturbation theory to all-loops

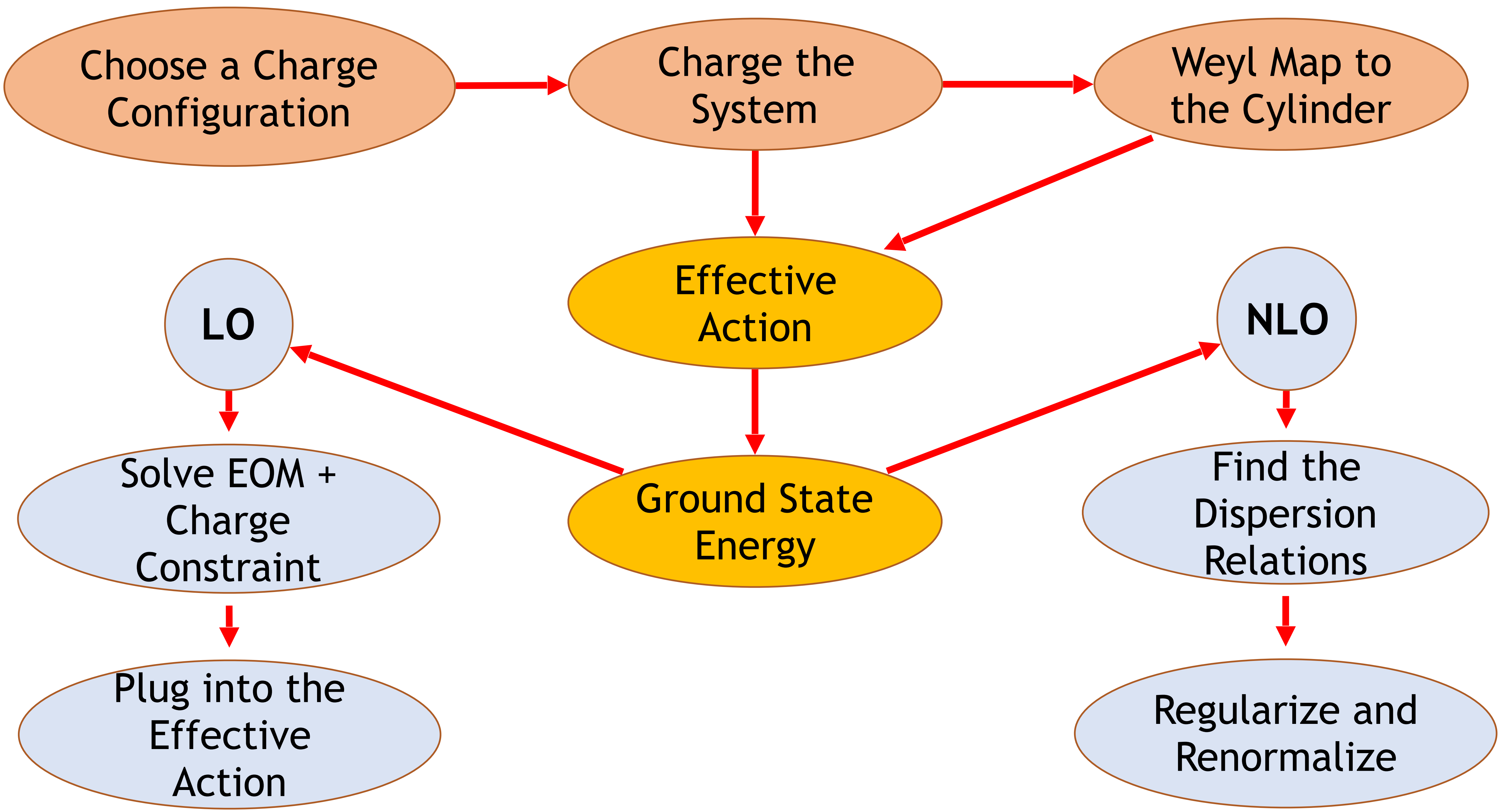
Our results resum the leading and next to leading order terms in the large charge expansion to all-orders in the coupling.

We can use them to predict terms at arbitrary high-loop orders in the standard diagrammatic approach.

6-loops: $\left(-\frac{572}{243}\bar{Q} + \frac{2}{279}[10191 - 64N - 2\zeta(3)(1327 + 160N)\right.$
$$\left. - 2\zeta(5)(1441 + 80N) - 70\zeta(7)(46 + N) - 21\zeta(9)(126 + N)](g^*\bar{Q})^6\right.$$

An independent diagrammatic check of our prediction (up to 6-loop) appeared in *I. Jack and D. R. T. Jones, arXiv: 2101.09820 [hep-th].*

Summary



Choose a Charge Configuration
Charge the System
Weyl Map to the Cylinder
Effective Action
LO
NLO
Solve EOM + Charge Constraint
Ground State Energy
Find the Dispersion Relations
Plug into the Effective Action
Regularize and Renormalize

Yukawa interactions: NJLY model

$$\mathcal{L}_{\mathrm{NJLY}} = \partial_\mu \bar{\phi} \partial^\mu \phi + \bar{\psi}_j \partial\!\!\!/ \psi^j + g \bar{\psi}_{Rj} \bar{\phi} \psi^j_L + g \bar{\psi}_{Lj} \phi \psi^j_R + \frac{\lambda}{24} \left(\bar{\phi}\phi\right)^2$$

$$\phi = f e^{i\chi}$$

$$\chi = -i\mu\tau$$

Remove phases from Yukawa term via:

$$\psi_L \to \psi_L\, e^{\mu\tau/2}\,, \qquad \psi_R \to \psi_R\, e^{-\mu\tau/2}$$

Classically: $\psi^{cl}_{L,R} = 0$ → Δ_{-1} is O(2) model result

Quadratic in fluctuations:

$$S^{(2)} = \int_{-T/2}^{T/2} d\tau \int d\Omega_{d-1} \Big[\frac{1}{2}(\partial r)^2 + \frac{1}{2}(\partial \pi)^2 - 2i\mu\, r \partial_\tau \pi + (\mu^2 - m^2) r^2$$

$$+ i\mu\, \bar{\psi}_j \gamma^0 \psi^j + \bar{\psi}^j \nabla\!\!\!\!/_{\mathcal{M}} \psi^j + g\, f \bar{\psi}_{Lj} \psi^j_R + g\, f \bar{\psi}_{Rj} \psi^j_L \Big]$$

Gaussian integral

$$\int \mathcal{D}r \mathcal{D}\pi \mathcal{D}\bar{\psi} \mathcal{D}\psi\, e^{-S^{(2)}} = \frac{\det\ F}{\det\ B}$$

Fermionic dispersions

$$\omega_{f\pm}(\ell) = \sqrt{\frac{3g^2\left(\mu^2 - m^2\right)}{8\pi^2\lambda} + \left(\frac{\mu}{2} + \lambda_{f\pm}\right)^2}$$

Leading quantum correction

$$\Delta_0 = \frac{1}{2}\sum_{\ell=0}^{\infty}\left[n_\ell(\omega_+(\ell) + \omega_-(\ell)) - N_f n_{f,\ell}(\omega_{f+}(\ell) + \omega_{f-}(\ell))\right]$$

$$\Delta_0^{(f)} = Q\left(\frac{g^2}{8\pi^2} - \frac{3g^4}{32\pi^4\lambda}\right) + Q^2\left(\frac{g^2\lambda}{12\pi^2} - \frac{g^4}{32\pi^4}\right) + Q^3\left(\frac{g^6\zeta(3)}{64\pi^6} - \frac{g^2\lambda^2}{18\pi^2} + g^4\lambda\frac{1-3\zeta(3)}{48\pi^4}\right)$$

$+ \ldots.$

Gauge interactions: scalar QED

$$S = \int d^4x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + (D_\mu \phi)^\dagger D_\mu \phi + \frac{\lambda_0}{24} (\bar{\phi}\phi)^2 \right)$$

Complex WF fixed point in 4-ε dimensions

$$\lambda^* = \frac{3}{20}\left(19\epsilon \pm i\sqrt{719}\epsilon\right) , \qquad e^{*2} = 24\pi^2\epsilon$$

Classically: $A_\mu = 0$ → Δ_{-1} is O(2) model result

Quadratic in fluctuations:

$$\mathcal{L}^{(2)} = -\frac{1}{2} A_\mu \left(-g^{\mu\nu}\nabla^2 + \mathcal{R}^{\mu\nu} + \left(1 - \frac{1}{\xi}\right) \nabla^\mu \nabla^\nu - (ef)^2 g^{\mu\nu} \right) A_\nu$$
$$+ \frac{1}{2}(\partial_\mu r)^2 - \frac{1}{2} 2(m^2 - \mu^2) r^2 + \frac{1}{2}(\partial_\mu \pi)^2 - \frac{\xi}{2}(ef)^2 \pi^2 - 2i\mu\, r\partial_\tau \pi - 2if\mu r A^0$$

Dispersions

	Field	d_ℓ	ε_ℓ	ℓ_0
Spatial	B_i	$n_A(\ell)$	$\sqrt{\lambda_A^2 + (d-2) + e^2v^2}$	1
	C_i	$n_B(\ell)$	$\sqrt{\lambda_B^2 + e^2v^2}$	1
Ghosts	$(c, \bar{c})$	$-2n_B(\ell)$	$\sqrt{\lambda_B^2 + e^2v^2}$	0
Temporal	A_0	$n_B(\ell)$	$\sqrt{\lambda_B^2 + e^2v^2}$	0
Complex Scalar	ϕ	$n_B(\ell)$	$\sqrt{\lambda_B^2 + 3\mu^2 - m^2 + \frac{1}{2}e^2v^2 \pm \sqrt{\left(3\mu^2 - m^2 - \frac{1}{2}e^2v^2\right)^2 + 4\lambda_B^2\mu^2}}$	0

Leading quantum correction

$$\Delta_0 = Q\left(-\frac{9e^4}{128\pi^4\lambda} + \frac{3e^2}{16\pi^2} - 2\lambda\right) + Q^2\left(\frac{e^4}{256\pi^4} - \frac{e^2\lambda}{12\pi^2} + \frac{2\lambda^2}{9}\right)$$
$$+ Q^3\left(\frac{e^6(9\zeta(3)-1)}{1024\pi^6} - \frac{e^4\lambda(3\zeta(3)+1)}{96\pi^4} + \frac{e^2\lambda^2(3-2\zeta(3))}{12\pi^2} + \frac{2}{27}\lambda^3(16\zeta(3)-17)\right)$$

Pheno application: Higgsplosion

Multi-boson production

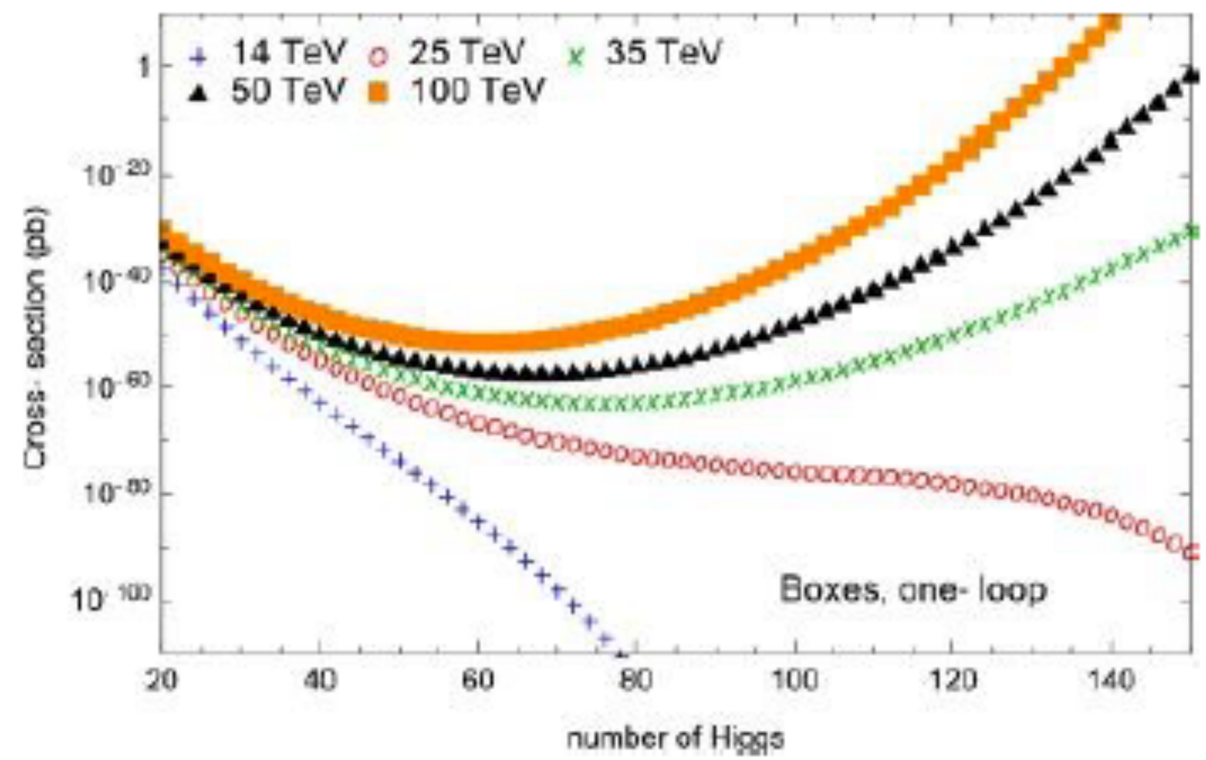
$$\lambda\phi^4$$

Consider the $1 \to n$ amplitude

$$A^{tree} = n!\lambda^{\frac{n-1}{2}} e^{-\frac{5}{6}En}$$

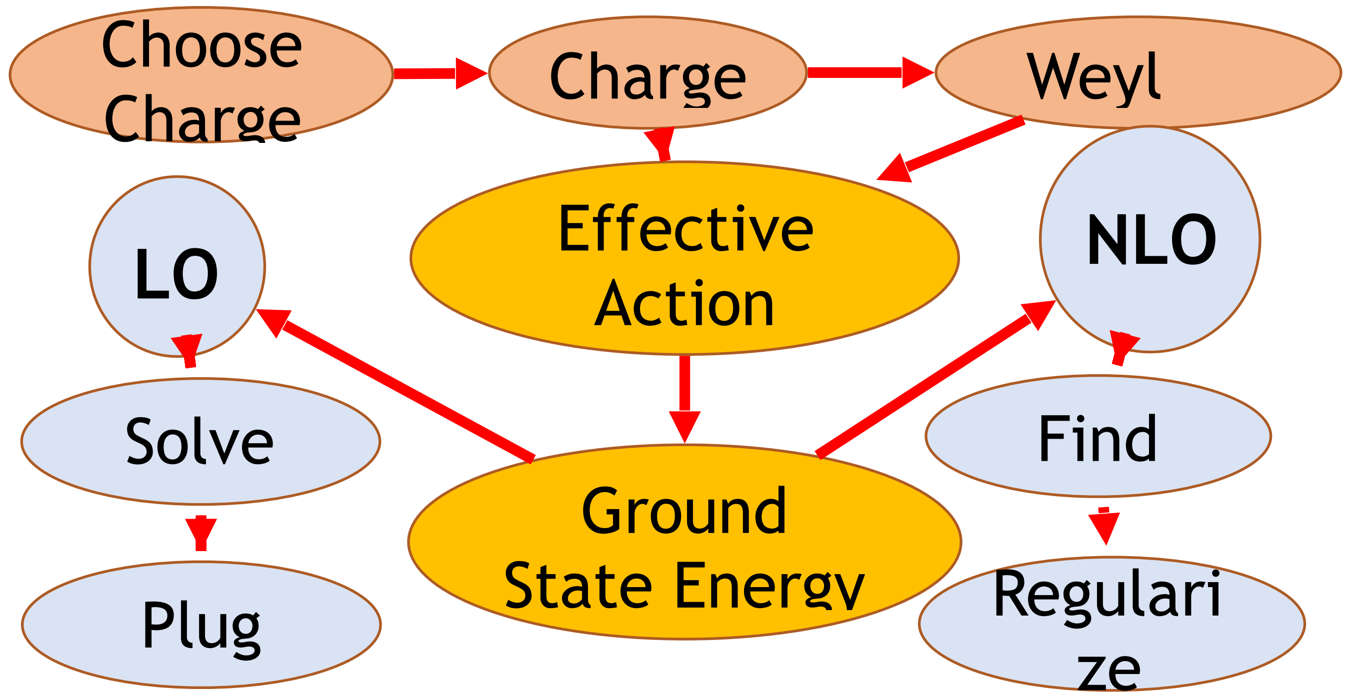
$$A = A^{tree} e^{B\lambda n}$$

$$\sigma(1 \to n) = e^{F(\lambda n, E)}$$



[Degrande, Khoze, Mattelaer, 2016]

$$n \approx \sqrt{s}/m$$



Ground State

Inhomogeneous
(operators with derivaties, spin)

$(H^\dagger D_\mu H)^2$ for W mass anomaly

Fermionic (superconductivity)

Baryons

$\mathbb{R}^d \to \mathbb{R} \times S^{d-1}$

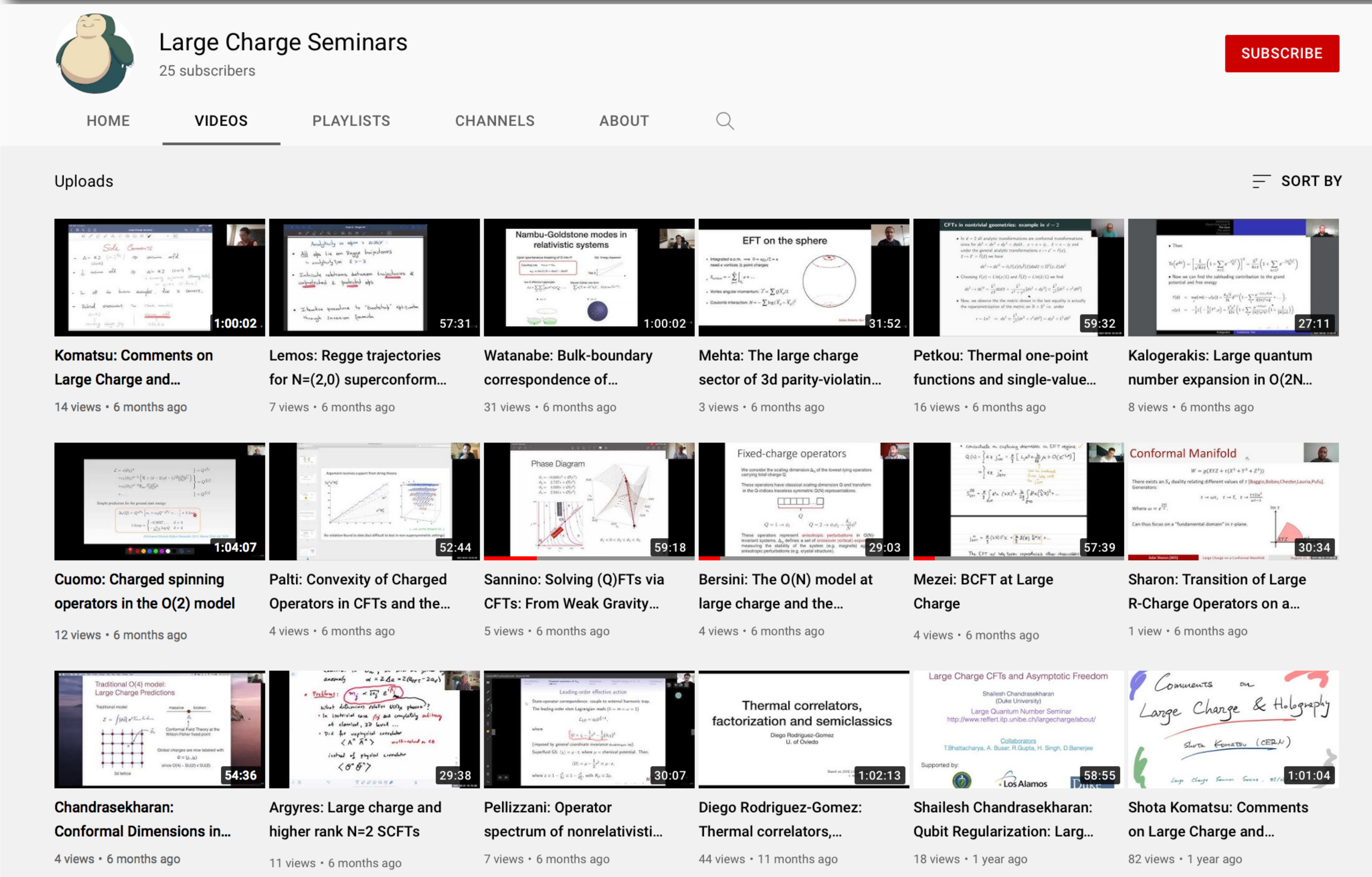
Weyl Map to the Cylinder

Other manifolds
(e.g.torus, finite temperature)

Other directions/aspects

- Large order behaviour of the series (resurgence)
- Dualities (particle-vortex, fermion-bosons, ….)
- Other (higher) correlations functions
- Holography (charged black holes)
- Non-relativistic CFTs
- …….

Youtube series



Thank you!

References

1) Hellerman et al	*JHEP 12 (2015) 071*	Original paper
2) Rattazzi et al	*JHEP 11 (2019) 110*	Semiclassical method
3) Orlando et al	Phys.Rept. 933 (2021)	Mini-review
4) Jack and Jones	*Phys.Rev.D* 103 (2021) 8, 085013	Higher loops checks
5) Antipin et al	*Phys.Rev.D* 102 (2020) 4, 045011	O(N) model

Counting of Goldstones

The symmetry breaking pattern is $U\left(\frac{N}{2}\right) \to U\left(\frac{N}{2}-1\right)$. Then the expected number of Goldstone bosons is

$$\dim\left(U\left(\frac{N}{2}\right)/U\left(\frac{N}{2}-1\right)\right) = N-1$$

We have only $N/2$ Goldstones!

Solution $\Longrightarrow$ fixing the charge **we broke Lorentz symmetry**. This modifies some of the Type I ($\equiv$ relativistic) Goldstone bosons into fewer Type II ($\equiv$ nonrelativistic) Goldstones which **count double**.

$$\text{Counting} \qquad 1+2\times\left(\frac{N}{2}-1\right) = N-1$$

Chada-Nielsen Theorem: H. B. Nielsen and S. Chadha, "On how to count Goldstone bosons", Nucl.Phys.B105 (1976).