

YANGIANS AND INTEGRABILITY IN ADS/CFT

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PLAN OF THE TALK

- S-matrix and Hopf algebra
- Non-abelian symmetries and the Yangian
- Quest for universal R-matrix and the quantum Double
- Classical r-matrix
- A secret symmetry
- Yangian representations and the Bound State S-matrix
- Applications and Open problems

*... a Journey through Symmetries*¹...

¹Sincere apologies to many beautiful papers which have not been included

ADS/CFT SPECTRAL PROBLEM

4D $\mathcal{N} = 4$ SYM $\leftrightarrow$ IIB Superstrings in $AdS_5 \times S^5$

Superconformal algebra $\mathfrak{psu}(2, 2|4)$

Single-trace composite operators $\mathcal{O}_{\{\alpha\}}(x) = \text{tr}[\phi_{i_1}(x) \dots \phi_{i_L}(x)]$

$\longrightarrow$ closed string states

Diag. of mixing matrix $\langle \mathcal{O}_{\{\alpha\}}(x), \mathcal{O}_{\{\beta\}}(y) \rangle = \frac{\delta_{\{\alpha\}, \{\beta\}}}{|x-y|^{2\Delta_{\{\alpha\}}}}$

• Anomalous dimension $\leftrightarrow$ Energy of string state

Planar lim.: Equivalent to diag. the Hamiltonian of an *integrable* spin-chain ($\exists$ extra local integrals of motion)

[Minahan-Zarembo '02]

Example: $\mathfrak{su}(2)$ sector

$$Z = \phi_1 + i\phi_2 \quad W = \phi_3 + i\phi_4$$

$$\mathcal{O}_{\{\alpha\}}(x) = \text{tr}[Z \dots Z W \dots W Z \dots W] \longrightarrow \downarrow \dots \downarrow \uparrow \dots \uparrow \downarrow \dots \uparrow$$

AdS/CFT Case

Choose a vacuum $\downarrow = Z$

$$\dots \downarrow \dots \downarrow \dots \downarrow \uparrow \downarrow \dots \downarrow \dots \downarrow \dots \downarrow \dots \downarrow$$

$$\mathfrak{psu}(2, 2|4) \longrightarrow \mathfrak{psu}(2|2) \quad (\text{excitations on the vacuum})$$

Two bosonic and two fermionic spin-polarizations;
even part of $\mathfrak{psu}(2|2)$ is $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$

[Frappat-Sciarrino-Sorba '96; Götz-Quella-Schomerus '05]

$\mathfrak{psu}(2|2)$ is the only simple classic Lie superalgebra
admitting up to 3 central extensions

[Iohara-Koga '01]

Its representation theory with triple c.e. was not much
studied, little was known about quantum deformations

[Beisert '06; Kac priv comm]

Next thing you knew... physics picks that $\longrightarrow$

S-MATRIX and HOPF ALGEBRA

Elementary excitations (magnons) scatter through S-matrix

[Staudacher '04; Beisert '05]

$$R : V_1 \otimes V_2 \longrightarrow V_1 \otimes V_2$$

$$S = PR \quad P = (\textit{graded}) \textit{perm}$$

V_i carries a representation of (two-copies of) centrally-extended $\mathfrak{psu}(2|2) := A$

S-matrix encodes info on dynamics, therefore its symmetries are important

Action of symmetry generators on 2-particle states ('in') given by 'coproduct'

$$\Delta : A \longrightarrow A \otimes A$$

such that $[\Delta(a), \Delta(b)] = \Delta([a, b])$ (homomorphism) and

$$(P\Delta)R = R\Delta$$

$P\Delta$ is called the 'opposite' coproduct Δ^{op} ('out')

To begin with: LIE SUPERALGEBRA SYMMETRY

$$\begin{aligned}
[\mathbb{L}_a^b, \mathbb{J}_c] &= \delta_c^b \mathbb{J}_a - \frac{1}{2} \delta_a^b \mathbb{J}_c \\
[\mathbb{L}_a^b, \mathbb{J}^c] &= -\delta_a^c \mathbb{J}^b + \frac{1}{2} \delta_a^b \mathbb{J}^c \\
\{\mathbb{Q}_\alpha^a, \mathbb{Q}_\beta^b\} &= \epsilon_{\alpha\beta} \epsilon^{ab} \mathbb{C} \\
\{\mathbb{Q}_\alpha^a, \mathbb{G}_b^\beta\} &= \delta_b^a \mathbb{R}_\alpha^\beta + \delta_\alpha^\beta \mathbb{L}_b^a + \frac{1}{2} \delta_b^a \delta_\alpha^\beta \mathbb{H}
\end{aligned}
\qquad
\begin{aligned}
[\mathbb{R}_\alpha^\beta, \mathbb{J}_\gamma] &= \delta_\gamma^\beta \mathbb{J}_\alpha - \frac{1}{2} \delta_\alpha^\beta \mathbb{J}_\gamma \\
[\mathbb{R}_\alpha^\beta, \mathbb{J}^\gamma] &= -\delta_\alpha^\gamma \mathbb{J}^\beta + \frac{1}{2} \delta_\alpha^\beta \mathbb{J}^\gamma \\
\{\mathbb{G}_a^\alpha, \mathbb{G}_b^\beta\} &= \epsilon^{\alpha\beta} \epsilon_{ab} \mathbb{C}^\dagger
\end{aligned}$$

Dynamical Spin-Chain Picture

$$\mathbb{H} |p\rangle = \epsilon(p) |p\rangle$$

$$\mathbb{C} |p\rangle = c(p) |p Z^-\rangle \qquad \mathbb{C}^\dagger |p\rangle = \bar{c}(p) |p Z^+\rangle$$

$Z^{+(-)}$: one site of the chain is added (removed)

On 2-particle states the action is non-local:

$$\mathbb{C} \otimes 1 |p_1\rangle \otimes |p_2\rangle =$$

$$\mathbb{C} \otimes 1 \sum_{n_1 < n_2} e^{i p_1 n_1 + i p_2 n_2} | \dots Z Z \phi_1 \underbrace{Z \dots Z}_{n_2 - n_1 - 1} \phi_2 Z \dots \rangle$$

$$(\text{rescale } n_2) = c(p_1) e^{i p_2} |p_1\rangle \otimes |p_2\rangle$$

$$S \Delta(\mathbb{C}) = S [\mathbb{C} \otimes 1 + 1 \otimes \mathbb{C}] = S [e^{i p_2} \mathbb{C}_{local} \otimes 1 + 1 \otimes \mathbb{C}_{local}]$$

$$\Delta(\mathbb{C}_{local}) = \mathbb{C}_{local} \otimes e^{i p} + 1 \otimes \mathbb{C}_{local}$$

[Gomez-Hernandez '06; Plefka-Spill-AT '06]

Similar coproduct arises for the other (super)charges, controlled by a quantum number $[[Q]]$ s.t.

$$\Delta(Q) = Q \otimes e^{i[[Q]]p} + 1 \otimes Q \quad \neq \quad \underline{\Delta^{op}(Q)}!!!$$

$$[[\mathbb{R}, \mathbb{L}, \mathbb{H}]] = 0, [[Q]] = \frac{1}{2}, [[Q^\dagger]] = -\frac{1}{2}, [[\mathbb{C}]] = 1, [[\mathbb{C}^\dagger]] = -1$$

Consistency requirement from centrality of $\Delta(C)$:

$$\Delta^{op}(C) R = R \Delta(C) = \Delta(C) R$$

therefore (R invertible)

$$\Delta^{op}(C) = \Delta(C)$$

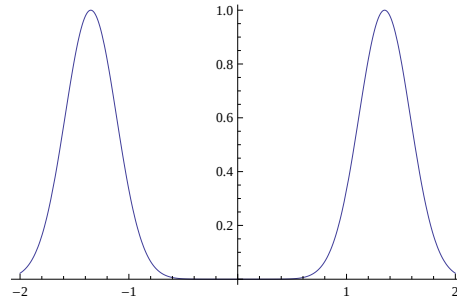
(coproduct is said to be *co-commutative*)

In our case, this is guaranteed by physical request

$$e^{ip} = \kappa \mathbb{C} + 1$$

for a constant $\kappa = ig$ [$\longrightarrow \Delta(\mathbb{C}) = \kappa \mathbb{C} \otimes \mathbb{C} + \mathbb{C} \otimes 1 + 1 \otimes \mathbb{C}$]

- One can check all the axioms of Hopf algebras are satisfied



STRING WORLDSHEET PICTURE

Coproduct reproduced from Bernard-LeClair procedure

[Klose-McLoughlin-Roiban-Zarembo '06]

Alternative classical argument. Light-cone worldsheet supercharges have non-locality

$$Q = \int_{-\infty}^{\infty} d\sigma J(\sigma) e^{i \int_{-\infty}^{\sigma} d\sigma' \partial x^-(\sigma')}$$

[Arutyunov-Frolov-Plefka-Zamaklar '06]

Imagine two well-separated soliton excitations (“scattering state”). Define semiclassical action of charge

$$\begin{aligned} Q_{|profile} &= \int_{-\infty}^{\infty} d\sigma J(\sigma)_{|profile} e^{i \int_{-\infty}^{\sigma} d\sigma' \partial x^-(\sigma')_{|profile}} \\ &= \int_{-\infty}^0 d\sigma J(\sigma) e^{i \int_{-\infty}^{\sigma} d\sigma' \partial x^-(\sigma')} \\ &\quad + \int_0^{\infty} d\sigma J(\sigma) e^{i \int_{-\infty}^0 d\sigma' \partial x^-(\sigma')} e^{i \int_0^{\sigma} d\sigma' \partial x^-(\sigma')} \\ &\sim Q_1 + e^{ip_1} Q_2 \longrightarrow \Delta(Q) = Q \otimes 1 + e^{ip} \otimes Q \end{aligned}$$

CROSSING SYMMETRY

Hopf-algebra antipode $\Sigma : A \longrightarrow A$ is defined as

$$m(\Sigma \otimes 1)\Delta(Q) = \text{“0”}$$

where

$$m(a \otimes b) = ab$$

Derive from it antiparticle representation $\tilde{Q}$:

$$\Sigma(Q) = C^{-1} \tilde{Q}^{st} C$$

with C charge-conjugation matrix.

Possible to write down crossing symmetry of S-matrix

[Janik '06]

$$(\Sigma \otimes 1) R = (1 \otimes \Sigma^{-1}) R = R^{-1}$$

directly from the dynamical generators: $\Sigma(Q) = -e^{-ip}Q$

- Reformulation as a Faddeev-Zamolodchikov algebra

[Arutyunov-Frolov-Zamaklar '06]

$$A_1 A_2 = S A_2 A_1$$

YANGIANS

$\exists$ Lie superalgebra Q^A . Suppose $\exists$ additional charges $\hat{Q}^A$

$$[Q^A, Q^B] = if_C^{AB} Q^C \qquad [Q^A, \hat{Q}^B] = if_C^{AB} \hat{Q}^C$$

(plus Serre) with coproducts

$$\begin{aligned} \Delta(Q^A) &= Q^A \otimes 1 + 1 \otimes Q^A \\ \Delta(\hat{Q}^A) &= \hat{Q}^A \otimes 1 + 1 \otimes \hat{Q}^A + \frac{i}{2} f_{BC}^A Q^B \otimes Q^C \end{aligned}$$

[Drinfeld '86]

(Infinite) Spin-Chain [Dolan-Nappi-Witten '03, Agarwal-Rajeev '04,
Zwiebel '06, Beisert-Zwiebel '07]

Classical String [Mandal-Suryanarayana-Wadia '02, Bena-Polchinski-
-Roiban '03, Hatsuda-Yoshida '04, Das-Maharana-Melikyan-Sato '04]

S-matrix Yangian

[Beisert '07]

We know modification

$$\Delta(Q^A) = Q^A \otimes 1 + e^{i[[A]]p} \otimes Q^A$$

Additionally, $\exists$ centrally-extended $\mathfrak{psu}(2|2)$ Yangian

$$\Delta(\hat{Q}^A) = \hat{Q}^A \otimes 1 + e^{i[[A]]p} \otimes \hat{Q}^A + \frac{i}{2} f_{BC}^A Q^B e^{i[[C]]p} \otimes Q^C$$

REMARKS

- Evaluation representation:

$$\hat{Q}^A = u Q^A = ig \left(x^+ + \frac{1}{x^+} - \frac{i}{2g} \right) Q^A$$

(satisfies comm. rel.s)

- f_{BC}^A needs f_C^{AB} and inverse Killing form G_{AB}^{-1}

For $\mathfrak{psu}(2|2)$, this does not exist, yet table of coproducts can be fully determined (cf. extension by automorph.s)

[Spill 'dipl.thesis, Beisert '06])

- Yangian coproduct is non-local and... quadratic (!)
- Traditionally, Yangian symmetry in evaluation representation implies difference form

$$S = S(u_1 - u_2)$$

S-matrix is known *NOT* to possess this symmetry (u depends on $x^\pm$), but let us keep it in mind...

- For higher bound-states, either YBE or Yangian symmetry have to be used to completely fix S-matrix

[Arutyunov-Frolov '08, de Leeuw '08]

DO WE HAVE A CONTINUUM PICTURE?

Take a 2D classical field theory, with local currents

$$J_\mu = J_\mu^A T_A \quad \partial^\mu J_\mu^A = 0 \quad Q^A = \int_{-\infty}^{\infty} dx J_0^A$$

satisfying flatness (Lax pair)

$$\partial_0 J_1 - \partial_1 J_0 + [J_0, J_1] = 0$$

The following non-local current is conserved

$$\hat{J}_\mu^A = \epsilon_{\mu\nu} J^{\nu,A} + \frac{i}{2} f_{BC}^A J_\mu^B \int_{-\infty}^x dx' J_0^C(x')$$

$$\frac{d}{dt} \hat{Q}^A = \frac{d}{dt} \int_{-\infty}^{\infty} dx \hat{J}_0^A(x) = 0$$

Prototype: **Principal Chiral Model**

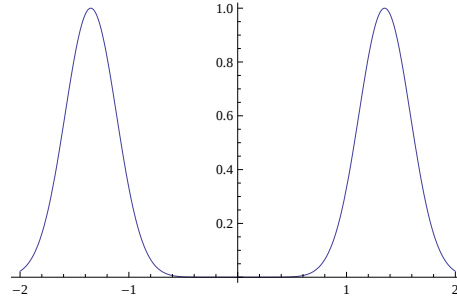
$$L = \text{Tr}[\partial_\mu g^{-1} \partial^\mu g] \quad g \in \text{Lie}$$

(left,right) global symmetry $g \longrightarrow e^{i\lambda} g, g e^{i\lambda}$

Noether current is flat

$$J^{L,R} = (\partial_\mu g) g^{-1}, g^{-1} (\partial_\mu g) \in \text{lie}$$

Classical integrability $\rightarrow$ *something analog happens for string*



Semiclassical argument [*Lüscher-Pohlmeyer '78; MacKay '92*]:

$$\begin{aligned} \text{Local charges:} \quad \int_{-\infty}^{\infty} dx J_0^A|_{profile}(x) &= \int_{-\infty}^0 J_0^A + \int_0^{\infty} J_0^A \\ &\longrightarrow \Delta(Q^A) = Q^A \otimes 1 + 1 \otimes Q^A \end{aligned}$$

Non-local charges:

$$\hat{Q}^A = \int_{-\infty}^{\infty} dx J_1^A(x) + \frac{i}{2} f_{BC}^A \int_{-\infty}^{\infty} dx J_0^B(x) \int_{-\infty}^x dx' J_0^C(x')$$

Evaluating on profile

$$\begin{aligned} \hat{Q}_{|profile}^A &= \int_{-\infty}^0 J_1^A + \frac{i}{2} f_{BC}^A \int_{-\infty}^0 J_0^B \int_{-\infty}^x J_0^C \\ &\quad + \int_0^{\infty} J_1^A + \frac{i}{2} f_{BC}^A \int_0^{\infty} J_0^B \int_0^x J_0^C \\ &\quad + \frac{i}{2} f_{BC}^A \int_0^{\infty} J_0^B \int_{-\infty}^0 J_0^C \\ &\longrightarrow \Delta(\hat{Q}^A) = \hat{Q}^A \otimes 1 + 1 \otimes \hat{Q}^A + \frac{i}{2} f_{BC}^A Q^B \otimes Q^C \end{aligned}$$

Quantization of this action in absence of anomalies [*Lüscher '78*]

$\longrightarrow$ **Hopf algebra rep on Hilbert space** [*Bernard-LeClair '92*]

$$[Q^A, Q^B] = i f_C^{AB} Q^C \qquad [Q^A, \hat{Q}^B] = i f_C^{AB} \hat{Q}^C$$

SOME EXPECTED CONSEQUENCES

Yangian is an infinite-dimensional non-abelian symmetry algebra

- (Semiclassical and quantum) S-matrix gets very constrained
- Spectrum degeneracies are organized in Yangian irreps:
spectrum generating algebra [cf. angular momentum]

$$H \hat{Q} |\psi\rangle = \hat{Q} H |\psi\rangle = \epsilon \hat{Q} |\psi\rangle$$

- Transfer matrix may enjoy Kirillov-Reshetikhin benefits

————→

- Whole mathematics of Yangian doubles and rational R-matrices enters the game (quantum groups, in general)

[Chari-Pressley '94, Etingof-Schiffman '98, MacKay '04, Molev '07,

Annecy ∞]

————→

YANGIANS AND BETHE ANSATZ

[Kirillov-Reshetikhin '86, '87]

Given a rational solution of YBE $R_{12}(u)$, Yangian Y can be generated by T_{ij}^k , $k \geq 1$ and $i, j = 1, \dots, N$, s.t.

$$R_{12}(u) T_1(u+v) T_2(v) = T_2(v) T_1(u+v) R_{12}(u)$$

$$T(u) = 1 + \sum_{n \geq 1} u^{-n} T_{ij}^k E^{ij}$$

with $(E^{ij})_{kl} = \delta_k^i \delta_l^j$ and Hopf algebra coproduct

$$\Delta(T_{ij}(u)) = T_{ik}(u+v) \otimes T_{kj}(u)$$

$\exists$ abelian subalgebra $\mathcal{T}$ generated by $t_k(u)$, obtained antisymmetrizing k -th tensor product of $T_{ij}(u)$

‘Quantum det’ $t_N(u)$ generates center of the Yangian

- **Example:** $gl(N)$ $T_{ij}(u) = \delta_{ij} + \frac{E_{ij}}{u-v}$

Common eigenvectors of $\mathcal{T}$ are in one-to-one with solutions of Bethe equations

$\mathcal{T}$ commutes with $gl(N) \subset Y$, therefore it classifies multiplicities of irreps in tensor products of $gl(N)$

UNIVERSAL R-MATRIX

Given H non *co*-commutative Hopf algebra ($P\Delta \neq \Delta$),
suppose $\exists$ abstract solution $R \in H \otimes H$ of

$$(P\Delta) R = R \Delta$$

Universal means independent of representations in each
factors of $\otimes$

Stand. Yangian is one such H : *“There is so much symmetry,
that S -matrix can be written purely in terms of generators of
symmetry algebra!”*

Theorem (Drinfeld): if R satisfies Quasi-Triangularity
(rep-independent version of bootstrap principle)

$$(\Delta \otimes 1)R = R_{13} R_{23}$$

$$(1 \otimes \Delta)R = R_{13} R_{12}$$

then it also satisfies YBE and Crossing

•

Direct proof of properties of S -matrix

Complete solution to scattering problem reduces to:

*find the abstract tensor R given H , and then project it into your
favorite (bound-state) rep*

More than that: LeClair-Smirnov

Solve/Construct the model from universal R-matrix, its intertwining properties, and the representation theory of the associated quantum group

[LeClair-Smirnov '92]

Can we answer Staudacher's Strings '08 question:

“What is ultimately the model we are diagonalizing?”

using the representation theory of the Yangian?

Hic sunt leones. **What would Drinfeld do?**

Study perturbation of YBE around identity, and classify

[Belavin-Drinfeld '82]

Suppose

$$R \sim 1 \otimes 1 + \hbar r + \mathcal{O}(\hbar^2)$$

$r \in \mathfrak{lie} \otimes \mathfrak{lie}$ (cf. exponential map) is the *classical r-matrix* and satisfies simpler equation to study: **ClassicalYBE**

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0$$

CLASSICAL r-MATRIX

Classical limit encodes info on quantum

Theorem (Belavin-Drinfeld): if $r(u_1 - u_2) \in \text{lie} \otimes \text{lie}$ solves CYBE and [...] has a simple pole in $u_1 - u_2 = 0$ with residue $C_2 = \text{quadratic Casimir of } \text{lie} \otimes \text{lie}$, then

→ then r is unitary, i.e. $r_{12}(u_1 - u_2) = -r_{21}(u_2 - u_1)$, meromorphic in plane $u = u_1 - u_2$, all poles are simple and form a lattice Γ

- $\dim \Gamma = 2$ → elliptic r-matrix/elliptic q.group
- $\dim \Gamma = 1$ → trigonom. r-matrix/(affine) q.group
- $\dim \Gamma = 0$ → rational r-matrix/Yangian

How do we see this? Factorization:

Yang's example ('67)

$r = \frac{C_2}{u_2 - u_1}$ solves CYBE (by def $[C_2, Q^A \otimes 1 + 1 \otimes Q^A] = 0$)

$$r = \frac{C_2}{u_2 - u_1} = \frac{Q^A \otimes Q_A}{u_2 - u_1} = \sum_{n \geq 0} Q^A u_1^n \otimes Q_A u_2^{-n-1} = \sum_{n \geq 0} Q_n^A \otimes Q_{A, -n-1}$$

$[Q_n^A, Q_m^B] = i f_C^{AB} Q_{n+m}^C$: loop algebra ... = classical Yangian

Theorem II:

[Belavin-Drinfeld '83]

if r not of difference form, but dual Coxeter number of lie nonzero, $\exists$ change of variables to difference form

REMARK

- r -matrix controls Poisson brackets of classical L -oper and contains seed of quantization

$$(R \sim 1 \otimes 1 + \hbar r + \text{reconstructible})$$

WHAT ABOUT AdS/CFT S-MATRIX?

$$R \sim 1 \otimes 1 + \hbar r \quad \text{in near BMN limit}$$

- $\hbar$ is $1/g$ (with g related to 't Hooft coupling)
- r is tree-level string r -matrix

[Klose-McLoughlin-Roiban-Zarembo '06]

- Classical representation variable

$$x^\pm(x) = x \left(\sqrt{1 - \frac{1}{g^2(x - \frac{1}{x})^2}} \pm \frac{i}{g(x - \frac{1}{x})} \right) \rightarrow x$$

[Arutyunov-Frolov '06]

LET US TRY

[AT '07]

- Algebra tends to a limiting centrally-extended $\mathfrak{psu}(2|2)$
- Classical r-matrix is not diff-form (Belavin-Drinfeld ² not applicable), *nevertheless* has a simple pole at origin $x_1 - x_2 = 0$ ($\dim \Gamma = 0$, suggests Yangians...)
- Th. [easy]: To satisfy CYBE with such pole, residue *must be* Casimir of $\mathfrak{lie} \otimes \mathfrak{lie}$

For centr-ext $\mathfrak{psu}(2|2)$ we don't have it...

How to get away?

Residue at origin is quadr. Casimir of $\mathfrak{gl}(2|2)!$

*Just on pole, borrow an extra generator $B = \text{diag}(1, 1, -1, -1)$ from nearest non-degenerate superalgebra, to respect Th. [easy]
Away from residue B is broken*

- Only on pole, $\exists$ change of variables to diff-form (consistent with Bel.-Drinf. Th. II)

Such *borrowing* reminds of math prescription for universal R-matrices based on degenerate Cartan matrices

[Khoroshkin-Tolstoy '91]

²Super-version $\longrightarrow$ consult [Leites-Serganova '84; Zhang-Gould-Bracken '91; Karaali '04, '06]

- Khoroshkin-Tolstoy prescription: universal R-matrix goes

$$R = \prod_{\text{roots}} e^{E^+ \otimes E^-} e^{a_{ij}^{-1} H^i \otimes H^j} \prod_{\text{roots}} e^{E^- \otimes E^+}$$

$E^\pm$ are roots of $\mathfrak{lie}$, H_i Cartan generators and a_{ij} Cartan matrix

“If degenerate, add H_k ’s until you can invert it”

- Expected we had to call in an $H_4 = B$ soon or later

We may have a chance of factorizing $\longrightarrow$

- Remark: $\mathfrak{gl}(2|2)$ Casimir is well-known from opposite regime $g = 0$

$$R_{1loop} \sim 1 \otimes 1 + \frac{C_2^{\mathfrak{gl}(2|2)}}{u_1 - u_2}$$

(up to twists). Yang’s *quantum* R-matrix, prototype for QYBE

YANGIAN DOUBLES

- Remember Yang

$$r = \frac{C_2}{u_2 - u_1} = G_{AB}^{-1} \frac{Q^A \otimes Q^B}{u_2 - u_1} = \sum_{n \geq 0} G_{AB}^{-1} Q_n^A \otimes Q_{-n-1}^B$$

- $\exists$ way (surpassed by history)

[Moriyama-AT '07]

$$\begin{aligned} r = & \sum_{n \geq 0} \mathbb{G}_{a,n}^\alpha \otimes \hat{\mathbb{Q}}_{\alpha,-n-1}^a - \mathbb{Q}_{\alpha,n}^a \otimes \hat{\mathbb{G}}_{a,-n-1}^\alpha + \mathbb{H}_n \otimes \hat{\mathbb{B}}_{-n-1} + \mathbb{B}_n \otimes \hat{\mathbb{H}}_{-n-1} \\ & + (\mathbb{L}_{b,n}^a \otimes \hat{\mathbb{L}}_{a,-n-1}^b - \mathbb{L}_{b,-n-1}^a \otimes \hat{\mathbb{L}}_{a,n}^b) - (\mathbb{R}_{\beta,n}^\alpha \otimes \hat{\mathbb{R}}_{\alpha,-n-1}^\beta - \mathbb{R}_{\beta,-n-1}^\alpha \otimes \hat{\mathbb{R}}_{\alpha,n}^\beta) \end{aligned}$$

for an enlarged Cartan matrix $a_{ij}^{-1} H^i H^j = 4\mathbb{H}\mathbb{B} + \mathbb{L}^2 - \mathbb{R}^2$

Bonus: $B_n = \frac{1}{2}(x^n - x^{-n})diag(1, 1, -1, -1)$ vanishes at $n = 0$

(indeed, $\exists$ no such Lie algebra symmetry)

- For higher n (higher Yangian generators), suggests $\exists$ of additional (non-local) symmetry of B -type

Is this additional symmetry confirmed at quantum level?

YES

Matsumoto-Moriyama-AT '07, Beisert-Spill '07

SECRET SYMMETRY

$\exists$ (first level) Yangian symmetry of quantum S-matrix

$$\Delta(\hat{B}) = \hat{B} \otimes 1 + 1 \otimes \hat{B} + \frac{i}{2g} (\mathbb{G}_a^\alpha \otimes \mathbb{Q}_\alpha^a + \mathbb{Q}_\alpha^a \otimes \mathbb{G}_a^\alpha)$$

$$\Sigma(\hat{B}) = -\hat{B} + \frac{2i}{g} \mathbb{H}$$

$$\hat{B} = \frac{1}{4} (x^+ + x^- - 1/x^+ - 1/x^-) \text{diag}(1, 1, -1, -1)$$

[Do not be misled by appearances: formula is exact $\forall g$]

Generates through comm new type of Yangian susys.

Consistent with classical limits, both obsolete ($\leftarrow$)
and new

New [Beisert-Spill '07]

$$r = \frac{\mathcal{T} - \tilde{B} \otimes \mathbb{H} - \mathbb{H} \otimes \tilde{B}}{i(u_1 - u_2)} - \frac{\Sigma \otimes \mathbb{H}}{iu_2} + \frac{\mathbb{H} \otimes \Sigma}{iu_1} - \frac{\mathbb{H} \otimes \mathbb{H}}{\frac{2iu_1u_2}{u_1 - u_2}}$$

$$\mathcal{T} = 2 (\mathbb{R}_\beta^\alpha \otimes \mathbb{R}_\alpha^\beta - \mathbb{L}_b^a \otimes \mathbb{L}_a^b + \mathbb{G}_a^\alpha \otimes \mathbb{Q}_\alpha^a - \mathbb{Q}_\alpha^a \otimes \mathbb{G}_a^\alpha)$$

$$\tilde{B} = \frac{1}{2ad + bc} \text{diag}(1, 1, -1, -1)$$

Confirmed for bound states

[de Leeuw '08, Arutyunov-de Leeuw-AT '09]

Quantization is open problem.

Connection wt [Dorey-Vicedo '06, Mikhailov - Schaefer-Nameki '08, Magro '08]?

A STUBBORN BOY: THE DIFFERENCE FORM

- $\hat{B} = \frac{u}{\epsilon(p)} \text{diag}(1, 1, -1, -1)$ also goes $B_n \sim u^n B$

- r has pretty $\frac{1}{u_1 - u_2}$'s in nice places

- $\exists$ Drinfeld's second realization of the Yangian

Evaluation representation is still of type $Q_n \sim (u + y)^n Q$

[Spill-AT '08]

- Connection with exceptional Lie algebra $D(2, 1; \alpha)$ should help localizing $u_1 - u_2$ dependence

[Beisert '05, Matsumoto-Moriyama '08, '09]

- Looks like difference form $u_1 - u_2$ is almost there, because Yangian calls for it (Th. Also Easy)

Upon rep, it is then masked by additional dependence on u of representation labels $a(u), b(u), c(u), d(u)$ entering the S-matrix

see also [Bazhanov, talk at IGST '09]

- *Universal R-matrix should disentangle it!*

Provocative rewriting of fundamental R-matrix [AT '08]
(like Khoroshkin-Tolstoy's? What has future told?)

THE BOUND STATE S-MATRIX

[Arutyunov-de Leeuw-AT '09]

- On one hand, desire of complete set of finite dimensional rep S-matrices, to figure out universal R-matrix (dreaming of a group-theoretic solution with nice ensuing math)
- On the other hand, we want finite-size. Integrability dictates: finite-size is obtained once we know the *entire* asymptotic data, including *all* scattering matrices
- So far, powerful conjectures for transfer matrix eigenvalues based on standard treatments (cf. Bazhanov-Reshetikhin)

[Beisert '07, Gromov-Kazakov-Vieira '09]

and superbe success of Luescher's corrections

[Fiamberti-Santambrogio-Sieg-Zanon '07, Bajnok-Janik '08, Bajnok-Janik-Lukowski '08]

have nurtured fascinating constructions

[Gromov-Kazakov-Vieira '09, Arutyunov-Frolov '09, Bombardelli-Fioravanti-Tateo '09, Frolov-Suzuki '09]

But centrally-extended $\mathfrak{psu}(2|2)$ is very special, and a complete mathematical proof is still missing... Can we prove these conjectures through alternative path?

- Let us follow a direct S-matrix approach $\longrightarrow$

EXPLICIT CONSTRUCTION

We use superspace formalism (atypical totally symm rep)

[Arutyunov-Frolov '08]

$$\begin{aligned}\Phi(w, \theta) &= \sum_{\ell=0}^{\infty} \Phi_{\ell}(w, \theta) \\ \Phi_{\ell} &= \phi^{a_1 \dots a_{\ell}} w_{a_1} \dots w_{a_{\ell}} + \phi^{a_1 \dots a_{\ell-1} \alpha} w_{a_1} \dots w_{a_{\ell-1}} \theta_{\alpha} + \\ &\quad \phi^{a_1 \dots a_{\ell-2} \alpha \beta} w_{a_1} \dots w_{a_{\ell-2}} \theta_{\alpha} \theta_{\beta}\end{aligned}$$

$$\begin{aligned}\mathbb{L}_a^b &= w_a \frac{\partial}{\partial w_b} - \frac{1}{2} \delta_a^b w_c \frac{\partial}{\partial w_c} & \mathbb{R}_{\alpha}^{\beta} &= \theta_{\alpha} \frac{\partial}{\partial \theta_{\beta}} - \frac{1}{2} \delta_{\alpha}^{\beta} \theta_{\gamma} \frac{\partial}{\partial \theta_{\gamma}} \\ \mathbb{Q}_{\alpha}^a &= a \theta_{\alpha} \frac{\partial}{\partial w_a} + b \epsilon^{ab} \epsilon_{\alpha \beta} w_b \frac{\partial}{\partial \theta_{\beta}} & \mathbb{G}_a^{\alpha} &= d w_a \frac{\partial}{\partial \theta_{\alpha}} + c \epsilon_{ab} \epsilon^{\alpha \beta} \theta_{\beta} \frac{\partial}{\partial w_b}\end{aligned}$$

$$\begin{aligned}\mathbb{C} &= ab \left(w_a \frac{\partial}{\partial w_a} + \theta_{\alpha} \frac{\partial}{\partial \theta_{\alpha}} \right) & \mathbb{C}^{\dagger} &= cd \left(w_a \frac{\partial}{\partial w_a} + \theta_{\alpha} \frac{\partial}{\partial \theta_{\alpha}} \right), \\ \mathbb{H} &= (ad + bc) \left(w_a \frac{\partial}{\partial w_a} + \theta_{\alpha} \frac{\partial}{\partial \theta_{\alpha}} \right)\end{aligned}$$

$$\begin{aligned}a &= \sqrt{\frac{g}{2\ell}} \eta & b &= \sqrt{\frac{g}{2\ell}} \frac{i\zeta}{\eta} \left(\frac{x^+}{x^-} - 1 \right) \\ c &= -\sqrt{\frac{g}{2\ell}} \frac{\eta}{\zeta x^+} & d &= \sqrt{\frac{g}{2\ell}} \frac{x^+}{i\eta} \left(1 - \frac{x^-}{x^+} \right)\end{aligned}$$

$$\eta = e^{i\frac{p}{4}} \sqrt{ix^- - ix^+}$$

$$x^+ + \frac{1}{x^+} - x^- - \frac{1}{x^-} = \frac{2i\ell}{g}$$

The integer ℓ is the number of bound-state constituents

[Dorey '06, Chen-Dorey-Okamura '06, Roiban '06]

INVARIANT SUBSPACES

Because of $\mathfrak{su}(2) \times \mathfrak{su}(2)$ invariance, S -matrix is block-diagonal

Case I a, b : $2 \times \ell_1 \ell_2$ vectors $\in V^I$ (a, b for $\alpha = 3, 4$ resp.)

$$|k, l\rangle^I \equiv \underbrace{\theta_\alpha w_1^{\ell_1-k-1} w_2^k}_{\text{Space1}} \underbrace{\vartheta_\alpha v_1^{\ell_2-l-1} v_2^l}_{\text{Space2}}$$

Case II a, b : $2 \times 4\ell_1 \ell_2$ vectors $\in V^{II}$ (a, b for $\alpha = 3, 4$ resp.)

$$\begin{aligned} |k, l\rangle_1^{II} &\equiv \underbrace{\theta_\alpha w_1^{\ell_1-k-1} w_2^k}_{\text{Space1}} \underbrace{v_1^{\ell_2-l} v_2^l}_{\text{Space2}} \\ |k, l\rangle_2^{II} &\equiv \underbrace{w_1^{\ell_1-k} w_2^k}_{\text{Space1}} \underbrace{\vartheta_\alpha v_1^{\ell_2-l-1} v_2^l}_{\text{Space2}} \\ |k, l\rangle_3^{II} &\equiv \underbrace{\theta_\alpha w_1^{\ell_1-k-1} w_2^k}_{\text{Space1}} \underbrace{\vartheta_3 \vartheta_4 v_1^{\ell_2-l-1} v_2^{l-1}}_{\text{Space2}} \\ |k, l\rangle_4^{II} &\equiv \underbrace{\theta_3 \theta_4 w_1^{\ell_1-k-1} w_2^{k-1}}_{\text{Space1}} \underbrace{\vartheta_\alpha v_1^{\ell_2-l-1} v_2^l}_{\text{Space2}} \end{aligned}$$

Case III: $6\ell_1 \ell_2$ vectors $\in V^{III}$

$$\begin{aligned} |k, l\rangle_1^{III} &\equiv \underbrace{w_1^{\ell_1-k} w_2^k}_{\text{Space1}} \underbrace{v_1^{\ell_2-l} v_2^l}_{\text{Space2}} \\ |k, l\rangle_2^{III} &\equiv \underbrace{w_1^{\ell_1-k} w_2^k}_{\text{Space1}} \underbrace{\vartheta_3 \vartheta_4 v_1^{\ell_2-l-1} v_2^{l-1}}_{\text{Space2}} \\ |k, l\rangle_3^{III} &\equiv \underbrace{\theta_3 \theta_4 w_1^{\ell_1-k-1} w_2^{k-1}}_{\text{Space1}} \underbrace{v_1^{\ell_2-l} v_2^l}_{\text{Space2}} \\ |k, l\rangle_4^{III} &\equiv \underbrace{\theta_3 \theta_4 w_1^{\ell_1-k-1} w_2^{k-1}}_{\text{Space1}} \underbrace{\vartheta_3 \vartheta_4 v_1^{\ell_2-l-1} v_2^{l-1}}_{\text{Space2}} \\ |k, l\rangle_5^{III} &\equiv \underbrace{\theta_3 w_1^{\ell_1-k-1} w_2^k}_{\text{Space1}} \underbrace{\vartheta_4 v_1^{\ell_2-l} v_2^{l-1}}_{\text{Space2}} \\ |k, l\rangle_6^{III} &\equiv \underbrace{\theta_4 w_1^{\ell_1-k} w_2^{k-1}}_{\text{Space1}} \underbrace{\vartheta_3 v_1^{\ell_2-l-1} v_2^l}_{\text{Space2}} \end{aligned}$$

S-matrix is of block-diagonal form

$$R = \begin{pmatrix} \boxed{\mathcal{X}} & & & & \\ & \boxed{\mathcal{Y}} & & & \\ & & \boxed{\mathcal{Z}} & & \\ & 0 & & \boxed{\mathcal{Y}} & \\ & & & & \boxed{\mathcal{X}} \end{pmatrix}$$

$$\begin{aligned} \mathcal{X} : V^{\text{I}} &\longrightarrow V^{\text{I}} \\ |k, l\rangle^{\text{I}} &\mapsto \sum_{m=0}^{k+l} \mathcal{X}_m^{k,l} |m, k+l-m\rangle^{\text{I}} \end{aligned}$$

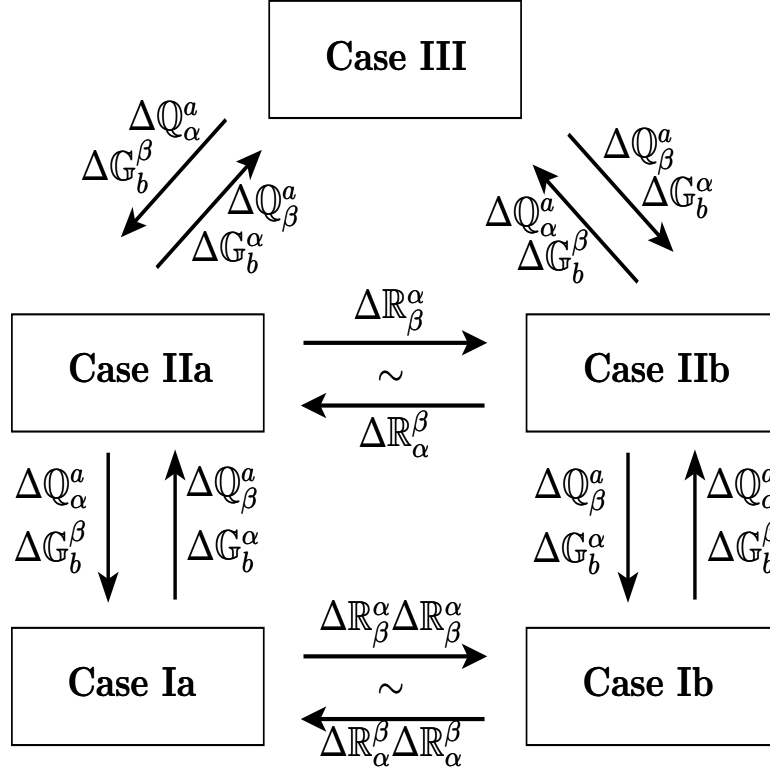
$$\begin{aligned} \mathcal{Y} : V^{\text{II}} &\longrightarrow V^{\text{II}} \\ |k, l\rangle_j^{\text{II}} &\mapsto \sum_{m=0}^{k+l} \sum_{j=1}^4 \mathcal{Y}_{m;i}^{k,l;j} |m, k+l-m\rangle_j^{\text{II}} \end{aligned}$$

$$\begin{aligned} \mathcal{Z} : V^{\text{III}} &\longrightarrow V^{\text{III}} \\ |k, l\rangle_j^{\text{III}} &\mapsto \sum_{m=0}^{k+l} \sum_{j=1}^6 \mathcal{Z}_{m;i}^{k,l;j} |m, k+l-m\rangle_j^{\text{III}} \end{aligned}$$

Full S-matrix is *two such copies times square of*

$$\begin{aligned} S_0(p_1, p_2) &= \left(\frac{x_1^-}{x_1^+} \right)^{\frac{\ell_2}{2}} \left(\frac{x_2^+}{x_2^-} \right)^{\frac{\ell_1}{2}} \sigma(x_1, x_2) \times \\ &\quad \times \sqrt{G(\ell_2 - \ell_1) G(\ell_2 + \ell_1)} \prod_{q=1}^{\ell_1-1} G(\ell_2 - \ell_1 + 2q) \end{aligned}$$

$$G(\ell) = \frac{u_1 - u_2 + \frac{\ell}{2}}{u_1 - u_2 - \frac{\ell}{2}} \quad u = \frac{g}{4i} \left(x^+ + \frac{1}{x^+} + x^- + \frac{1}{x^-} \right)$$



THE DERIVATION

Case I looks easiest, we start there $\longrightarrow$

We need exact solution for Case I. First, define a ‘vacuum’

$$|0\rangle \equiv w_1^{\ell_1} v_1^{\ell_2} \in V^{\text{III}}$$

such that $R|0\rangle = |0\rangle$, and then use $\Delta^{op} R = R \Delta$:

$$\begin{aligned}
 R|0, 0\rangle^{\text{I}} &= R \frac{\Delta(Q_3^1) \Delta(G_2^4) |0\rangle}{(a_2 c_1 - a_1 c_2) \ell_1 \ell_2} = \frac{\Delta^{op}(Q_3^1) \Delta^{op}(G_2^4) R|0\rangle}{(a_2 c_1 - a_1 c_2) \ell_1 \ell_2} \\
 &= \frac{x_1^- - x_2^+}{x_1^+ - x_2^-} \frac{e^{i \frac{p_1}{2}}}{e^{i \frac{p_2}{2}}} |0, 0\rangle^{\text{I}} \equiv \mathcal{D} |0, 0\rangle^{\text{I}}
 \end{aligned}$$

**Bound states are evaluation representations of Yangian,
with corresp. bound state parameter u**

[de Leeuw '08]

One generates entire Case I from $|0, 0\rangle^I$, using Yangian:

$$|k, l\rangle^I = \frac{\prod_{i=1}^k \left[\Delta(\hat{\mathbb{L}}_2^1) + \frac{\ell_1 - 2u_2 - 2i + 1}{2} \Delta(\mathbb{L}_2^1) \right] \prod_{j=1}^l \left[-\Delta(\hat{\mathbb{L}}_2^1) - \frac{1 + 2j - 2u_1 - \ell_2}{2} \Delta(\mathbb{L}_2^1) \right]}{\prod_{r=1}^k (\ell_1 - r) \prod_{p=1}^l (\ell_2 - p) \prod_{q=1}^{k+l} (\delta u + \frac{\ell_1 + \ell_2}{2} - q)} |0, 0\rangle^I$$

from which $\Delta^{op} R = R \Delta$ gives (for $\delta u = u_1 - u_2$)

$$R|k, l\rangle^I = \mathcal{D} \times \frac{\prod_{i=1}^k \left[\Delta^{op}(\hat{\mathbb{L}}_2^1) + \frac{\ell_1 - 2u_2 - 2i + 1}{2} \Delta^{op}(\mathbb{L}_2^1) \right] \prod_{j=1}^l \left[-\Delta^{op}(\hat{\mathbb{L}}_2^1) - \frac{1 + 2j - 2u_1 - \ell_2}{2} \Delta^{op}(\mathbb{L}_2^1) \right]}{\prod_{r=1}^k (\ell_1 - r) \prod_{p=1}^l (\ell_2 - p) \prod_{q=1}^{k+l} (\delta u + \frac{\ell_1 + \ell_2}{2} - q)} |0, 0\rangle^I$$

Explicit computation produces

$$R|k, l\rangle^I = \sum_{n=0}^{k+l} \mathcal{X}_n^{k,l} |n, k+l-n\rangle^I$$

$$\mathcal{X}_n^{k,l} = \mathcal{D} \frac{\prod_{i=1}^n (\ell_1 - i) \prod_{i=1}^{k+l-n} (\ell_2 - i)}{\prod_{r=1}^k (\ell_1 - r) \prod_{p=1}^l (\ell_2 - p) \prod_{q=1}^{k+l} (\delta u + \frac{\ell_1 + \ell_2}{2} - q)} \times \\ \times \sum_{m=0}^k \left\{ \binom{k}{k-m} \binom{l}{n-m} \prod_{p=1}^m \mathfrak{c}_p^+ \prod_{p=1-m}^{l-n} \mathfrak{c}_p^- \prod_{p=1}^{k-m} \mathfrak{d}_{\frac{k-p+2}{2}} \prod_{p=1}^{n-m} \tilde{\mathfrak{d}}_{\frac{k+l-m-p+2}{2}} \right\}$$

$$\begin{aligned} \mathfrak{c}_m^\pm &= \delta u \pm \frac{\ell_1 - \ell_2}{2} - m + 1 & \tilde{\mathfrak{c}}_m^\pm &= \delta u \pm \frac{\ell_1 + \ell_2}{2} - m + 1 \\ \mathfrak{d}_i &= \ell_1 + 1 - 2i & \tilde{\mathfrak{d}}_i &= \ell_2 + 1 - 2i \end{aligned}$$

Amplitude is restriction of Hypergeometric:

$$\begin{aligned} \mathcal{X}_n^{k,l} = & (-1)^{k+n} \pi D \frac{\sin[(k - \ell_1)\pi] \Gamma(l+1)}{\sin[\ell_1\pi] \sin[(k+l-\ell_2-n)\pi] \Gamma(l-\ell_2+1) \Gamma(n+1)} \\ & \times \frac{\Gamma(n+1-\ell_1) \Gamma(l + \frac{\ell_1-\ell_2}{2} - n - \delta u) \Gamma(1 - \frac{\ell_1+\ell_2}{2} - \delta u)}{\Gamma(k+l - \frac{\ell_1+\ell_2}{2} - \delta u + 1) \Gamma(\frac{\ell_1-\ell_2}{2} - \delta u)} \times \\ & {}_4\tilde{F}_3\left(-k, -n, \delta u + 1 - \frac{\ell_1 - \ell_2}{2}, \frac{\ell_2 - \ell_1}{2} - \delta u; 1 - \ell_1, \ell_2 - k - l, l - n + 1; 1\right) \end{aligned}$$

where ${}_4\tilde{F}_3(x, y, z, t; r, v, w; \tau) = {}_4F_3(x, y, z, t; r, v, w; \tau) / [\Gamma(r)\Gamma(v)\Gamma(w)]$

LUCKY SITUATION:

Our ${}_4F_3(a_i; b_j; 1)$ is ‘balanced’: $\sum a_i - \sum b_j = -1$

$\longrightarrow$ **6j-symbol**

$$\begin{aligned} {}_4F_3(a_1, a_2, a_3, a_4; b_1, b_2, b_3; 1) = & \frac{(-1)^{b_1+1} \Gamma(b_2) \Gamma(b_3) \sqrt{\Gamma(1-a_1) \Gamma(1-a_2) \Gamma(1-a_3)}}{\Gamma(1-b_1) \sqrt{\Gamma(b_2-a_1) \Gamma(b_2-a_2)}} \times \\ & \frac{\sqrt{\Gamma(1-a_4) \Gamma(a_1-b_1+1) \Gamma(a_2-b_1+1) \Gamma(a_3-b_1+1) \Gamma(a_4-b_1+1)}}{\sqrt{\Gamma(b_2-a_3) \Gamma(b_2-a_4) \Gamma(b_3-a_1) \Gamma(b_3-a_2) \Gamma(b_3-a_3) \Gamma(b_3-a_4)}} \times \\ & \left\{ \begin{array}{ccc} \frac{1}{2}(-a_1-a_4+b_3-1) & \frac{1}{2}(-a_1-a_3+b_2-1) & \frac{1}{2}(a_1+a_2-b_1-1) \\ \frac{1}{2}(-a_2-a_3+b_3-1) & \frac{1}{2}(-a_2-a_4+b_2-1) & \frac{1}{2}(a_3+a_4-b_1-1) \end{array} \right\} \end{aligned}$$

The relevant 6j-symbol $\left\{ \begin{array}{ccc} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{array} \right\}$ has coefficients

$$\begin{aligned} j_1 &= \frac{1}{2} \left(k + l - n + \frac{\ell_1 - \ell_2}{2} + \delta u \right) & j_2 &= \frac{1}{2} \left(\frac{\ell_1 + \ell_2}{2} - 2 - l - \delta u \right) \\ j_3 &= \frac{1}{2} (\ell_1 - 2 - k - n) & j_4 &= \frac{1}{2} \left(\frac{\ell_1 - \ell_2}{2} - 1 + l - \delta u \right) \\ j_5 &= \frac{1}{2} \left(\frac{\ell_1 + \ell_2}{2} - 1 - k - l + n + \delta u \right) & j_6 &= \frac{1}{2} (\ell_2 - 1) \end{aligned}$$

REMARKS

- Case I amplitude shows correct poles
- Case I states are highest weight w.r.t ‘fermionic’ $\mathfrak{su}(2)$, and carry a rep of the ‘bosonic’ $\mathfrak{su}(2)$ Yangian
- Case I S-matrix has difference-form (apart overall factor), and is a $6j$ -symbol with (half-)integer coefficients on physical poles (*namely, this block exhibits standard fusion*)

... Wait...

is it the representation of the universal R-matrix of the Yangian of ‘bosonic’ $\mathfrak{su}(2)$ in *arbitrary* evaluation representations (times an overall factor)?

YES

[Arutyunov-de Leeuw-AT '09]

- Since we are going to generate all other states and S-matrix blocks from case I, it looks like *one factor* of the full universal R-matrix is going to be:

Khoroshkin-Tolstoy's $\mathfrak{su}(2)$ -Yangian universal R-matrix, for the $\mathbb{L}$ -generators

OTHER CASES

How do we generate the other cases' S-matrix?

General strategy schematically as follows:

- **one one hand**

$$\begin{aligned} R \Delta(\mathbb{Q}) |Case II\rangle_i &= R Q_i |Case I\rangle &= Q_i R |Case I\rangle \\ &= Q_i \mathcal{X} |Case I\rangle \end{aligned}$$

- **on the other hand**

$$\begin{aligned} R \Delta(\mathbb{Q}) |Case II\rangle_i &= \Delta^{op}(\mathbb{Q}) R |Case II\rangle_i \\ &= R_i^j \Delta^{op}(\mathbb{Q}) |Case II\rangle_j = R_i^j Q_j^{op} |Case I\rangle \end{aligned}$$

From which

$$R_i^j = Q_i \mathcal{X} ([Q^{op}]^{-1})^j$$

Before being more specific, notice:

This construction automatically provides a ‘factorizing twist’ for the concrete S-matrix

$$R = F_{21} F^{-1} \quad \text{[Drinfeld '90]}$$

FULL THERAPY - CASE II

Define

$$\mathbb{S}|k, l\rangle_i^{\Pi} = \sum_{j=1}^4 \sum_{m=0}^{k+l} \mathcal{Y}_{m;i}^{k,l;j} |m, k+l-m\rangle_j^{\Pi}$$

and notice that

$$\Delta \mathbb{Q}_3^1 |k, l\rangle_j^{\Pi} = Q_j(k, l) |k, l\rangle^{\text{I}}$$

$$\begin{aligned} Q_1(k, l) &= a_2(l - \ell_2), & Q_2(k, l) &= a_1(\ell_1 - k) \\ Q_3(k, l) &= b_2, & Q_4(k, l) &= -b_1 \end{aligned}$$

Apply general strategy:

$$\begin{aligned} &^{\text{I}}\langle n, N-n | \Delta^{op} \mathbb{Q}_3^1 R |k, l\rangle_i^{\Pi} = \sum_{j=1}^4 \sum_{m=0}^{k+l} \mathcal{Y}_{m;i}^{k,l;j} ^{\text{I}}\langle n, N-n | \Delta^{op} \mathbb{Q}_3^1 |m, N-m\rangle_j^{\Pi} \\ &= \sum_{j=1}^4 \sum_{m=0}^{k+l} \mathcal{Y}_{m;i}^{k,l;j} Q_j^{op}(m, N-m) ^{\text{I}}\langle n, N-n | m, N-m \rangle^{\text{I}} \\ &= \sum_{j=1}^4 \mathcal{Y}_{n;i}^{k,l;j} Q_j^{op}(n, N-n) \\ &^{\text{I}}\langle n, N-n | \Delta^{op} \mathbb{Q}_3^1 R |k, l\rangle_i^{\Pi} = ^{\text{I}}\langle n, N-n | R \Delta \mathbb{Q}_3^1 |k, l\rangle_i^{\Pi} \\ &= Q_i(k, l) ^{\text{I}}\langle n, N-n | R |k, l\rangle^{\Pi} = Q_i(k, l) \sum_{m=0}^N \mathcal{X}_m^{k,l} ^{\text{I}}\langle n, N-n | m, N-m \rangle^{\text{I}} \\ &= Q_i(k, l) \mathcal{X}_n^{k,l} \end{aligned}$$

This gives four linear equations. Similarly, using $\Delta^{op}\mathbb{G}_2^4$ gives other four. Not enough, need Yangian

$$\begin{aligned}\Lambda_1 &= \Delta(\hat{\mathbb{Q}}_3^1) + \frac{2\Delta\hat{\mathbb{L}}_2^1\Delta(\mathbb{Q}_3^2)}{\ell_1 + \ell_2 - 2(N+1+\delta u)} - \frac{\ell_1 - \ell_2 + 2(N-2n+u_1+u_2)}{2(\ell_1 + \ell_2) - 4(N+1+\delta u)}\Delta\mathbb{L}_2^1\Delta(\mathbb{Q}_3^2) \\ \Lambda_2 &= \Delta(\hat{\mathbb{G}}_2^4) + \frac{2\Delta\hat{\mathbb{L}}_2^1\Delta(\mathbb{G}_1^4)}{\ell_1 + \ell_2 - 2(N+1+\delta u)} + \frac{\ell_1 - \ell_2 + 2(N-2n+u_1+u_2)}{2(\ell_1 + \ell_2) - 4(N+1+\delta u)}\Delta\mathbb{L}_2^1\Delta(\mathbb{G}_1^4)\end{aligned}$$

where $N = k + l$. These operators satisfy

$${}^I\langle n, N-n | \Lambda_a^{op} R | k, l \rangle_i^\Pi = \sum_{j=1}^4 \mathcal{Y}_{n;i}^{k,l;j} Q_{a,j}^{op}(n, N-n)$$

$$\begin{aligned}{}^I\langle n, N-n | \Lambda_a R | k, l \rangle_i^\Pi &= \\ \sum_{j=1}^4 \mathcal{Y}_{n;i}^{k,l;j} Q_{a,j}(n, N-n) &+ \mathcal{Y}_{n+1;i}^{k,l;j} Q_{a,j}^+(n, N-n) + \mathcal{Y}_{n-1;i}^{k,l;j} Q_{a,j}^-(n, N-n)\end{aligned}$$

Yangian makes the matrix equation invertible:

$$\mathcal{Y}_n^{k,l} \equiv \begin{pmatrix} \mathcal{Y}_{n;1}^{k,l;1} & \mathcal{Y}_{n;2}^{k,l;1} & \mathcal{Y}_{n;3}^{k,l;1} & \mathcal{Y}_{n;4}^{k,l;1} \\ \mathcal{Y}_{n;1}^{k,l;2} & \mathcal{Y}_{n;2}^{k,l;2} & \mathcal{Y}_{n;3}^{k,l;2} & \mathcal{Y}_{n;4}^{k,l;2} \\ \mathcal{Y}_{n;1}^{k,l;3} & \mathcal{Y}_{n;2}^{k,l;3} & \mathcal{Y}_{n;3}^{k,l;3} & \mathcal{Y}_{n;4}^{k,l;3} \\ \mathcal{Y}_{n;1}^{k,l;4} & \mathcal{Y}_{n;2}^{k,l;4} & \mathcal{Y}_{n;3}^{k,l;4} & \mathcal{Y}_{n;4}^{k,l;4} \end{pmatrix}$$

$$\begin{pmatrix} a_4 & a_3 & 0 & 0 \\ c_4 & c_3 & 0 & 0 \\ 0 & 0 & a_4 & a_3 \\ 0 & 0 & c_4 & c_3 \end{pmatrix} A \mathcal{Y}_n^{k,l} = \begin{pmatrix} a_2 & a_1 & 0 & 0 \\ c_2 & c_1 & 0 & 0 \\ 0 & 0 & a_2 & a_1 \\ 0 & 0 & c_2 & c_1 \end{pmatrix} \left\{ B^+ \mathcal{X}_n^{k+1,l-1} + B^- \mathcal{X}_n^{k-1,l+1} + B \mathcal{X}_n^{k,l} \right\}$$

$$A = \begin{pmatrix} N-n-\ell_2 & 0 & \frac{\mathcal{J}_{34}}{\mathcal{Q}_{34}} & \frac{1}{\mathcal{Q}_{43}} \\ 0 & \ell_1-n & \frac{1}{\mathcal{Q}_{43}} & \frac{\mathcal{J}_{43}}{\mathcal{Q}_{34}} \\ (N-n-\ell_2)(M-\delta u) & (n-\ell_1)\ell_2\mathcal{J}_{34} & \frac{(\delta u-M+\ell_2)\mathcal{J}_{34}}{\mathcal{Q}_{43}} & \frac{\delta u+M+\ell_1-\ell_2}{\mathcal{Q}_{43}}\mathcal{Q}_{34}\overline{\mathcal{Q}}_{34} \\ (N-n-\ell_2)(\ell_1\mathcal{J}_{43}) & (\ell_1-n)(\delta u+M) & \frac{M-\delta u-\ell_2+\ell_1}{\mathcal{Q}_{43}}\mathcal{Q}_{34}\overline{\mathcal{Q}}_{34} & \frac{(\delta u+M+\ell_1)\mathcal{J}_{43}}{\mathcal{Q}_{34}} \end{pmatrix}$$

$$\begin{aligned}
B^+ &= \frac{2(\ell_1 - k - 1)\mathbf{c}_{l-n}^-}{\tilde{\mathbf{c}}_{-N}^-} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ l & 0 & \frac{\mathcal{I}_{12}}{\mathcal{Q}_{12}} & 0 \\ 0 & 0 & \frac{1}{\mathcal{Q}_{21}} & 0 \end{pmatrix} & B^- &= \frac{2(\ell_2 - l - 1)\mathbf{c}_{n-l}^+}{\tilde{\mathbf{c}}_{-N}^-} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\mathcal{Q}_{12}} \\ 0 & k & 0 & \frac{\mathcal{I}_{21}}{\mathcal{Q}_{21}} \end{pmatrix} \\
B &= \begin{pmatrix} l-\ell_2 & 0 & \frac{\mathcal{I}_{12}}{\mathcal{Q}_{12}} & \frac{1}{\mathcal{Q}_{21}} \\ 0 & \ell_1-k & \frac{1}{\mathcal{Q}_{21}} & \frac{\mathcal{I}_{21}}{\mathcal{Q}_{12}} \\ (l-\ell_2)(N-\delta u) & (\ell_1-k)\ell_2\mathcal{I}_{12} & \frac{(N-\delta u-\ell_2)\mathcal{I}_{12}}{\mathcal{Q}_{12}} & \frac{N-\delta u-\ell_1-\ell_2}{\mathcal{Q}_{12}}\mathcal{Q}_{12}\overline{\mathcal{Q}}_{12} \\ (\ell_2-l)(\ell_1\mathcal{I}_{21}) & (\ell_1-k)(\delta u-N) & \frac{\delta u-N+\ell_1}{\mathcal{Q}_{12}}\mathcal{Q}_{12}\overline{\mathcal{Q}}_{12}+\ell_2 & \frac{(\delta u-N+\ell_1)\mathcal{I}_{21}}{\mathcal{Q}_{12}} \end{pmatrix} \\
&-2 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{l(1+n+k-\ell_1)(l-\ell_2)}{\tilde{\mathbf{c}}_{-N}^-} & 0 & \frac{(l-\ell_2)(1+n+k-\ell_1)\mathcal{I}_{12}}{\tilde{\mathbf{c}}_{-N}^-\mathcal{Q}_{12}} & \frac{(\ell_1-k)(1+N-n+l-\ell_2)}{\tilde{\mathbf{c}}_{-N}^-\mathcal{Q}_{21}} \\ 0 & \frac{k(1+N-n+l-\ell_2)(k-\ell_1)}{\tilde{\mathbf{c}}_{-N}^-} & \frac{(l-\ell_2)(1+n+k-\ell_1)}{\tilde{\mathbf{c}}_{-N}^-\mathcal{Q}_{21}} & \frac{(\ell_1-k)(1+N-n+l-\ell_2)\mathcal{I}_{21}}{\tilde{\mathbf{c}}_{-N}^-\mathcal{Q}_{12}} \end{pmatrix}
\end{aligned}$$

where we introduced

$$M = k + l - 2n$$

$$\mathcal{Q}_{ij} = a_i c_j - a_j c_i$$

$$\overline{\mathcal{Q}}_{ij} = b_i d_j - d_j b_i$$

$$\mathcal{I}_{ij} = a_i d_j - b_j c_i$$

Define

$$\begin{aligned}
\mathfrak{d}\mathfrak{a} \equiv \det A \frac{\mathcal{Q}_{34}^2}{(n-\ell_1)(N-n-\ell_2)} &= -4\delta u^2 + (\ell_1 - \ell_2)^2 + 4\ell_1\ell_2\mathcal{I}_{34}\mathcal{I}_{43} \\
&= -4\mathbf{c}_1^+\mathbf{c}_1^- + 4\ell_1\ell_2\mathcal{I}_{34}\mathcal{I}_{43}
\end{aligned}$$

then

$$\begin{aligned}
A^{-1} &= \frac{2}{\mathfrak{d}\mathfrak{a}} \begin{pmatrix} \frac{1}{N-n-\ell_2} & 0 & 0 & 0 \\ 0 & \frac{1}{n-\ell_1} & 0 & 0 \\ 0 & 0 & \mathcal{Q}_{43} & 0 \\ 0 & 0 & 0 & \mathcal{Q}_{43} \end{pmatrix} \times \\
&\times \begin{pmatrix} \frac{\mathfrak{d}\mathfrak{a}}{4} - [M + \frac{\ell_1-\ell_2}{2}][\mathbf{c}_1^- + \ell_1\mathcal{I}_{34}\mathcal{I}_{43}] & \mathcal{I}_{34}(\frac{\mathfrak{d}\mathfrak{a}}{4} - [M + \frac{\ell_1-\ell_2}{2}]\tilde{\mathbf{c}}_1^+) & \mathbf{c}_1^- + \ell_1\mathcal{I}_{34}\mathcal{I}_{43} & \tilde{\mathbf{c}}_1^+\mathcal{I}_{34} \\ -\mathcal{I}_{43}(\frac{\mathfrak{d}\mathfrak{a}}{4} - [M + \frac{\ell_1-\ell_2}{2}]\tilde{\mathbf{c}}_1^+) & -[M + \frac{\ell_1-\ell_2}{2}][\mathbf{c}_1^- + \ell_2\mathcal{I}_{34}\mathcal{I}_{43}] - \frac{\mathfrak{d}\mathfrak{a}}{4} & \tilde{\mathbf{c}}_1^+\mathcal{I}_{43} & \mathbf{c}_1^+ + \ell_2\mathcal{I}_{34}\mathcal{I}_{43} \\ -\ell_1[M + \frac{\ell_1-\ell_2}{2}]\mathcal{I}_{43} & \frac{\mathfrak{d}\mathfrak{a}}{4} - \mathbf{c}_1^+[M + \frac{\ell_1-\ell_2}{2}] & \ell_1\mathcal{I}_{43} & \mathbf{c}_1^+ \\ \frac{\mathfrak{d}\mathfrak{a}}{4} + \mathbf{c}_1^-[M + \frac{\ell_1-\ell_2}{2}] & \ell_2[M + \frac{\ell_1-\ell_2}{2}]\mathcal{I}_{34} & -\mathbf{c}_1^- & -\ell_2\mathcal{I}_{34} \end{pmatrix}
\end{aligned}$$

Therefore, final result

$$\mathcal{Y}_n^{k,l} = A^{-1} \begin{pmatrix} \frac{\mathcal{Q}_{32}}{\mathcal{Q}_{34}} & \frac{\mathcal{Q}_{31}}{\mathcal{Q}_{34}} & 0 & 0 \\ \frac{\mathcal{Q}_{42}}{\mathcal{Q}_{43}} & \frac{\mathcal{Q}_{41}}{\mathcal{Q}_{43}} & 0 & 0 \\ 0 & 0 & \frac{\mathcal{Q}_{32}}{\mathcal{Q}_{34}} & \frac{\mathcal{Q}_{31}}{\mathcal{Q}_{34}} \\ 0 & 0 & \frac{\mathcal{Q}_{42}}{\mathcal{Q}_{43}} & \frac{\mathcal{Q}_{41}}{\mathcal{Q}_{43}} \end{pmatrix} \{ \mathcal{X}_n^{k+1,l-1} B^+ + \mathcal{X}_n^{k-1,l+1} B^- + \mathcal{X}_n^{k,l} B \}$$

REMARKS

- (Apart perhaps from overall factor) final result purely depends only on δu , $\mathcal{Q}_{ij}$, $\overline{\mathcal{Q}}_{ij}$, $\mathcal{H}_{ij}$ and combinatorial factors involving integer bound-state components

[Still, putting this in a universal formula remains hard]

[but you never know]

- Case III is similarly generated from Case II.

S-matrix is *uniquely* determined

- We reproduce known S-matrices in the limit of small bound state numbers [Beisert '05, Arutyunov-Frolov '08, Bajnok-Janik '08]

- One can compute transfer matrix eigenvalues in arbitrary bound state representations *via* Algebraic Bethe Ansatz $\longrightarrow$ by restriction, conjectures of [Beisert '07] on quantum characteristic function are nicely confirmed!

[Arutyunov-de Leeuw-Suzuki-AT '09]

CONCLUSIONS

- A deep mathematical structure is there, in some aspects almost reducible to standard, in some others seemingly so much harder
- Nevertheless, blooming of developments allowed to unveil some of the most useful bits of it
- More progress expected as one digs deeper and deeper. Role of secret symmetry, derivation of quantum double, maybe one day universal R-matrix?
 - **Long reps** [Arutyunov-de Leeuw-AT arXiv:0912.0209]
- Beautiful relationship with (q -def) 1D Hubbard model
 - [Beisert-Koroteev '08; Beisert '10; Annecy '07 → *present*]
- Fascinating connections with Yangian and dual superconformal symmetries of scattering amplitudes await to be fully investigated
 - [Henn-Drummond-Plefka '09]

...Thank You