TOOLKIT FOR SCALAR FIELDS IN UNIVERSES WITH FINITE-DIMENSIONAL HILBERT SPACE

arXiv:2201.08405

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A Quantum Fueled Universe

1. Toolkit for scalar fields in universes with finite-dimensional Hilbert spaces
   …with Olivier Friedrich (Cambridge/LMU) and Olivier Doré (NASA JPL),
   arXiv:2201.08405

2. How low can vacuum energy go when your fields are finite-dimensional?
   …with Aidan Chatwin Davies (UBC) and Charles Cao (Maryland)
   Int. J. Mod. Phys. D Vol. 28, No. 14, 1944006 (2019), arXiv:2005.12938 [hep-th]

3. Modeling position and momentum in finite-dimensional Hilbert spaces with Generalized Pauli Operators
   … with Sean Carroll (Caltech), arXiv:1806.10134 [quant-ph]

A Quantum Fueled Universe

Quantum Mechanics ⟷ Cosmological Evolution of the Universe

- basic ideas in QM motivated by gravity, applied to cosmology
- creatively applying and critically thinking of first-principle QM
- "back of the envelope" style, yet realistic

- Local finite-dimensionality
- Scalar Field in an Expanding Universe
- Generalized Pauli Operators (GPOs)
- Parameterizations and Modeling Choices
- Implications for Cosmological Physics

Toolkit paper, introducing a paradigm!

Let’s start with our usual standard construction,

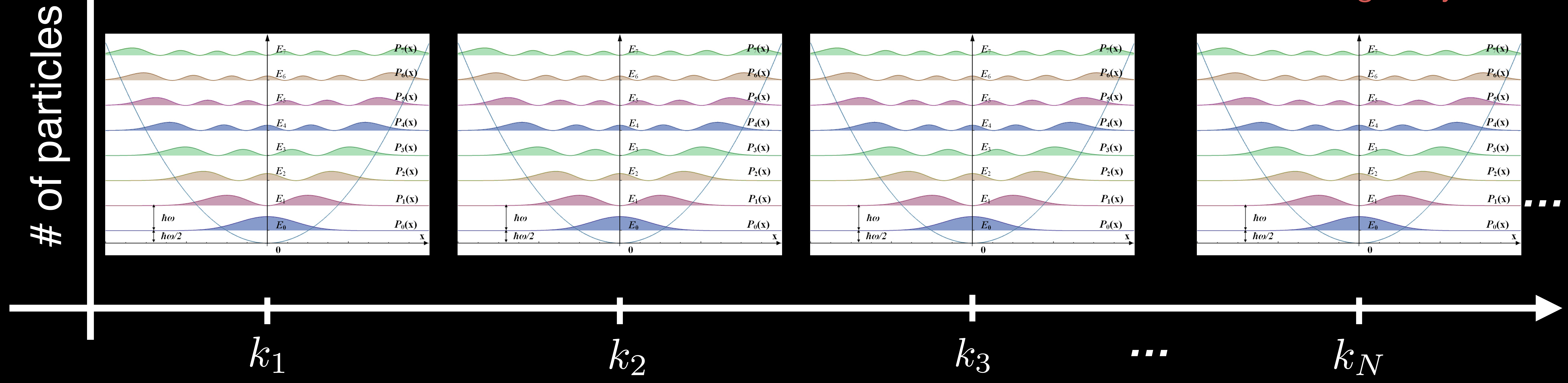
Quantum Field Theory

A Quantum Field Theoretic Construction

“collection of harmonic oscillators”

our current best understanding of physics **without* gravity

QM + Special Relativity = QFT



$$(\vec{k}_1, N_1)\ ,\ (\vec{k}_2, N_2)\ ,\ (\vec{k}_3, N_3)\ ,\ \ldots\ (\vec{k}_j, N_j)\ ,\ \ldots$$

Specify occupation of each wavenumber “mode”

Each wavenumber mode is a quantum harmonic oscillator

$$\left[\hat{\phi}_{\vec{k}}, \hat{\pi}_{\vec{k}}\right] = i\hbar$$

The Hilbert Space of QFT

Bao, Carroll & Singh 2017, arXiv:1704.00066

The Hilbert Space of QFT is *infinite*-dimensional.

- Arbitrary Long Wavelength Modes → IR Cutoff (~Hubble Size)

Lattice Regularization

- Arbitrary Short Wavelength Modes → UV Cutoff (~Planck Length)

No Bound on Bosonic Energy Excitations

Local Finite-Dimensionality

Bao, Carroll & Singh 2017, arXiv:1704.00066

The Hilbert Space of QFT is *infinite*-dimensional.

- Arbitrary Long Wavelength Modes → IR Cutoff (~Hubble Size)
- Arbitrary Short Wavelength Modes → UV Cutoff (~Planck Length)

(Lattice Regularization)

No Bound on Bosonic Energy Excitations

Include Gravity: Excite these degrees of freedom, many collapse into BHs

- Bekenstein Bound
- Holographic Principle

$$S_{BH} = \frac{A}{4G}$$

- Finite entropy
- Similar application to our dS horizon

Finite Entropy (hence, dofs) in a Finite Region of Space, suggests:

The Hilbert Space of Quantum Gravity is *locally finite-dimensional*

Our Working Hypothesis

The Hilbert Space of Quantum Gravity is *locally finite-dimensional*

Bao, Carroll & Singh 2017, arXiv:1704.00066

Local Finite-Dimensionality

Singh & Carroll 2018, arXiv:1806.10134

It Matters!

The Stone - von Neumann Theorem

Continuum Quantum Field Theory	Quantum Gravity
Infinite-Dimensional Hilbert Space for uncountably infinite dofs	Finite-Dimensional Hilbert Space for local regions of space
Non-Separable Hilbert spaces	Separable Hilbert spaces
Inequivalent representations of the Canonical Commutation Relations (CCRs)	Unique representation (upto unitary equivalence) of the CCRs
Additional choice/preferred algebraic state	Algebra is just “all hermitian operators”

Scalar Field in an Expanding Universe

$$\mathrm{d}s^2 = a^2\left(\mathrm{d}\eta^2 - \mathrm{d}\boldsymbol{x}^2\right) \ , \ [\cdot]' \equiv \frac{\partial}{\partial\eta}[\cdot]$$

Scale Factor

FRW Cosmology

“homogenous, isotropic expansion”

Action

$$S = \frac{1}{2}\int \mathrm{d}\eta \mathrm{d}^3x \ a^2 \ \left[(\phi')^2 - (\boldsymbol{\nabla}\phi)^2 - m^2a^2\phi^2\right]$$

$$= \frac{1}{2}\int \frac{\mathrm{d}\eta \mathrm{d}^3k}{(2\pi)^3} \ a^2 \ \left[|\phi'_{\boldsymbol{k}}|^2 - (|\boldsymbol{k}|^2 + m^2a^2)|\phi_{\boldsymbol{k}}|^2\right]$$

“Comoving Coordinates”

$$k = a\, k_{\mathrm{ph}}$$

mode. Expressing the Fourier transform of the field in terms of real and imaginary parts, $\phi_{\boldsymbol{k}} = A_{\boldsymbol{k}} + iB_{\boldsymbol{k}}$, we must have $A_{\boldsymbol{k}} = A_{-\boldsymbol{k}}$ and $B_{\boldsymbol{k}} = -B_{-\boldsymbol{k}}$ because ϕ is real. This allows us to define a new field

$$q_{\boldsymbol{k}} = \sqrt{2}\begin{cases} A_{\boldsymbol{k}} \text{ for } k_1 \leq 0 \\ B_{\boldsymbol{k}} \text{ for } k_1 > 0 \end{cases} \tag{2.4}$$

Scalar Field in an Expanding Universe

$$S = \int \frac{\mathrm{d}\eta \mathrm{d}^3 k}{(2\pi)^3} \left[\frac{a^2}{2} (q'_{\boldsymbol{k}})^2 - \frac{a^2(|\boldsymbol{k}|^2 + m^2 a^2)}{2} q_{\boldsymbol{k}}^2 \right] .$$

Put the universe in a box

$$L_c$$

Standard quantization

$$[\hat{q}_{\boldsymbol{k}}, \hat{p}_{\boldsymbol{k}'}] = i\delta_{\boldsymbol{k},\boldsymbol{k}'}$$

$$\hat{H}(t) = \sum_{\boldsymbol{k}} \left[\frac{L_c^3}{2a^3} \hat{p}_{\boldsymbol{k}}^2 + \frac{a^3}{L_c^3} \frac{(|\boldsymbol{k}|^2/a^2 + m^2)}{2} \hat{q}_{\boldsymbol{k}}^2 \right] .$$

Collection of time-dependent harmonic oscillators, one for each mode

Cosmological Connection

Governing Dynamics are Einstein's
Equations of General Relativity:

Curvature of Spacetime ⟷ Energy and Matter

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho(t)$$

"Friedmann Equation"

Governs Evolution of Universe based on Einstein's Theory of Gravity

Expansion history of the universe

Energy Density sourced from the quantum Hamiltonian

Sources of Infinite-Dimensionality

$$\hat{H}(t) = \sum_{\boldsymbol{k}} \left[\frac{L_c^3}{2a^3} \hat{p}_{\boldsymbol{k}}^2 + \frac{a^3}{L_c^3} \frac{(|\boldsymbol{k}|^2/a^2 + m^2)}{2} \hat{q}_{\boldsymbol{k}}^2 \right] .$$

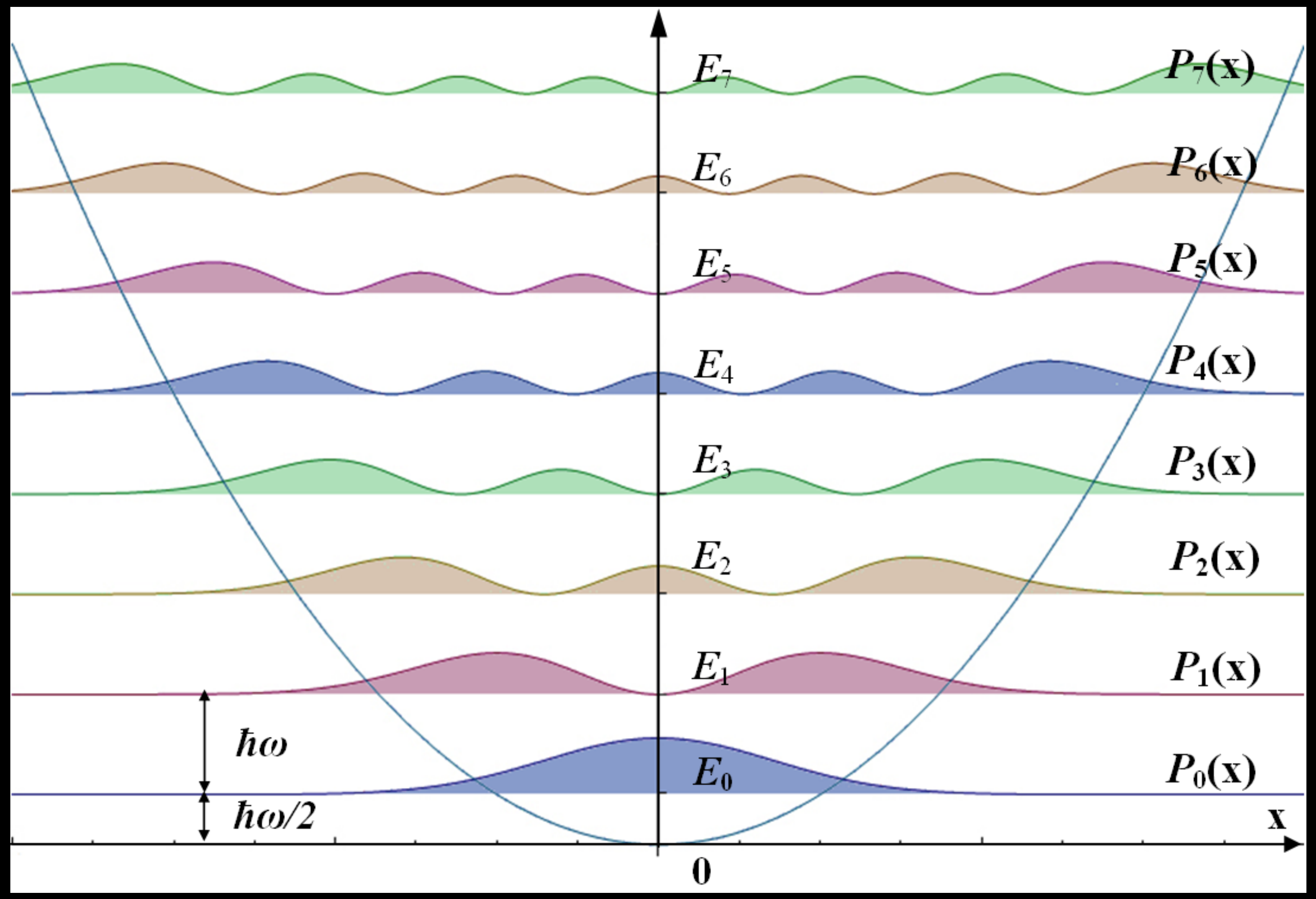
Collection of time-dependent harmonic oscillators

$$[\hat{q}_{\boldsymbol{k}}, \hat{p}_{\boldsymbol{k}'}] = i\delta_{\boldsymbol{k},\boldsymbol{k}'}$$

IR and UV scales

Latticize the theory in a box! L_c

$$k_{\mathrm{ph,max}} = \Lambda_{\mathrm{UV}}$$



$$E_0 = \frac{1}{2}\hbar\omega$$

"Zero Point Energy/ Vacuum Energy"

And still,

Energy Spectrum: $E_n = \left(n + \frac{1}{2}\right)\hbar\omega$

# of states: $\dim\mathcal{H} = \infty$

Finite-Dimensional Conjugate Variables

On a finite-dimensional Hilbert space,

$$\dim\mathcal{H} = d < \infty$$

Operators = Matrices!

Can we find matrices of size “d x d” which obey Heisenberg’s Commutation Relation? $[\hat{x}, \hat{p}] = i\hbar$

No!

It’s the Stone - von Neumann Theorem

$$[\hat{x}, \hat{p}] \neq i\hbar$$

Finite-Dimensional Conjugate Variables
Generalized Pauli Operators (GPOs)

Singh & Carroll 2018, arXiv:1806.10134

- Ubiquity of Conjugate variables in both Classical and Quantum Mechanics.

No finite-dimensional representations of Heisenberg CCR: Stone-von Neumann Theorem.

Rescaling to make our variables dimensionless:

$$Q_{\boldsymbol{k}} \equiv q_{\boldsymbol{k}}/L_c^2 \;,\; P_{\boldsymbol{k}} \equiv p_{\boldsymbol{k}} L_c^2$$

Finite-Dimensional Conjugate Variables
Generalized Pauli Operators (GPOs)

Singh & Carroll 2018, arXiv:1806.10134

Finite-dimensional Hilbert space

$$\dim \mathcal{H}_k = d_{\boldsymbol{k}} < \infty$$

$$Q_{\boldsymbol{k}} \equiv q_{\boldsymbol{k}}/L_c^2 \ , \ P_{\boldsymbol{k}} \equiv p_{\boldsymbol{k}} L_c^2$$

$$\hat{A}_{\boldsymbol{k}} = \exp\left(-i\alpha_{\boldsymbol{k}}\hat{P}_{\boldsymbol{k}}\right) \ , \ \hat{B}_{\boldsymbol{k}} = \exp\left(i\beta_{\boldsymbol{k}}\hat{Q}_{\boldsymbol{k}}\right) \ .$$

$$\hat{A}_{\boldsymbol{k}}\hat{B}_{\boldsymbol{k}} = \exp\left(\frac{-2\pi i}{d_{\boldsymbol{k}}}\right)\hat{B}_{\boldsymbol{k}}\hat{A}_{\boldsymbol{k}}$$

Weyl Commutation Relation

An equivalent commutation, albeit exponentiated

$$\alpha_{\boldsymbol{k}}, \beta_{\boldsymbol{k}} \in \mathbb{R}\backslash\{0\}$$

Hermann Weyl (1934)

$$\alpha_{\boldsymbol{k}}\beta_{\boldsymbol{k}} = \frac{2\pi}{d_{\boldsymbol{k}}}$$

to recover Heisenberg's commutation in the infinite limit

Finite-Dimensional Conjugate Variables
Generalized Pauli Operators (GPOs)

Singh & Carroll 2018, arXiv:1806.10134

Generalized Pauli Operators offer a way to realize Weyl's form of the CCR in a finite-dimensional Hilbert space, conjugate variables in finite dimensions

$$\hat{A}_{\boldsymbol{k}} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}_{d_{\boldsymbol{k}} \times d_{\boldsymbol{k}}}$$

$$d_{\boldsymbol{k}} = 2l_{\boldsymbol{k}} + 1$$

(can work for even dimensions too)

(translation operators)

$$\hat{B}_{\boldsymbol{k}} = \begin{pmatrix} \exp(\frac{2\pi i}{d_{\boldsymbol{k}}} l_{\boldsymbol{k}}) & 0 & \cdots & 0 \\ 0 & \exp(\frac{2\pi i}{d_{\boldsymbol{k}}} (l_{\boldsymbol{k}} - 1)) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \exp(\frac{-2\pi i}{d_{\boldsymbol{k}}} l_{\boldsymbol{k}}) \end{pmatrix}_{d_{\boldsymbol{k}} \times d_{\boldsymbol{k}}}$$

Finite-Dimensional Conjugate Variables

"Just what you would expect"

Singh & Carroll 2018, arXiv:1806.10134

$$\mathrm{Spec}(\hat{Q}_{\boldsymbol{k}}) = \{-\ell_{\boldsymbol{k}}\alpha_{\boldsymbol{k}} \ , \ \ldots \ , \ \ell_{\boldsymbol{k}}\alpha_{\boldsymbol{k}}\}$$

$$\mathrm{Spec}(\hat{P}_{\boldsymbol{k}}) = \{-\ell_{\boldsymbol{k}}\beta_{\boldsymbol{k}} \ , \ \ldots \ , \ \ell_{\boldsymbol{k}}\beta_{\boldsymbol{k}}\}$$

Lattice-like spectrum for the conjugate variables

$$e^{-i\alpha\hat{P}}\,|Q_j\rangle = |Q_{j+1}\rangle$$

$$e^{i\beta\hat{Q}}\,|P_j\rangle = |P_{j+1}\rangle$$

(cyclic shifts)

Recover Heisenberg in the limit:

$$d_{\boldsymbol{k}} \to \infty$$

$$\alpha_{\boldsymbol{k}}, \beta_{\boldsymbol{k}} \to 0$$

$$\alpha_{\boldsymbol{k}}\beta_{\boldsymbol{k}} = \frac{2\pi}{d_{\boldsymbol{k}}}$$

Free Parameters: $d_{\boldsymbol{k}}$ and $\alpha_{\boldsymbol{k}}$

Have I Seen This Before?

For d = 2

Yes!

$$\hat{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Pauli Matrices

$$\hat{B} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

“Clock and Shift Matrices”

“Generalized Clifford Algebra”

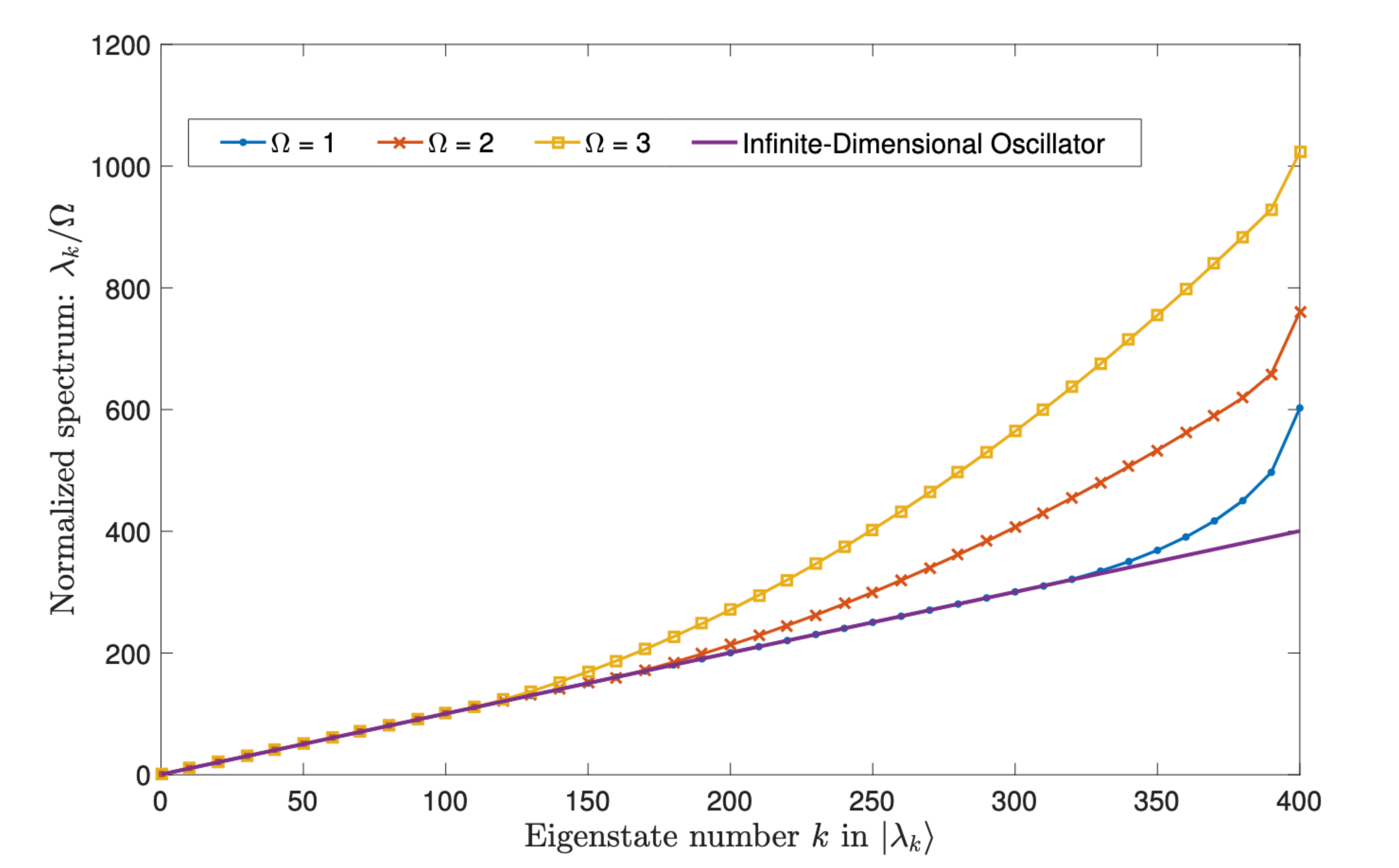
- Modification to Canonical Commutation, uncertainty principle.

- Modification of eigenspectra of operators, eg. the quantum harmonic oscillator

Modifications to Energy Levels

Singh & Carroll 2018, arXiv:1806.10134

$$\hat{H}_{\boldsymbol{k}} = \left[\frac{\hat{P}_{\boldsymbol{k}}^2}{2M} + \frac{M\Omega_{\boldsymbol{k}}^2}{2}\hat{Q}_{\boldsymbol{k}}^2\right]$$



Implications of Finite-Dimensionality

Suppresion of Zero Point Energy

Singh & Carroll 2018, arXiv:1806.10134

Zero-point energy of a single oscillator

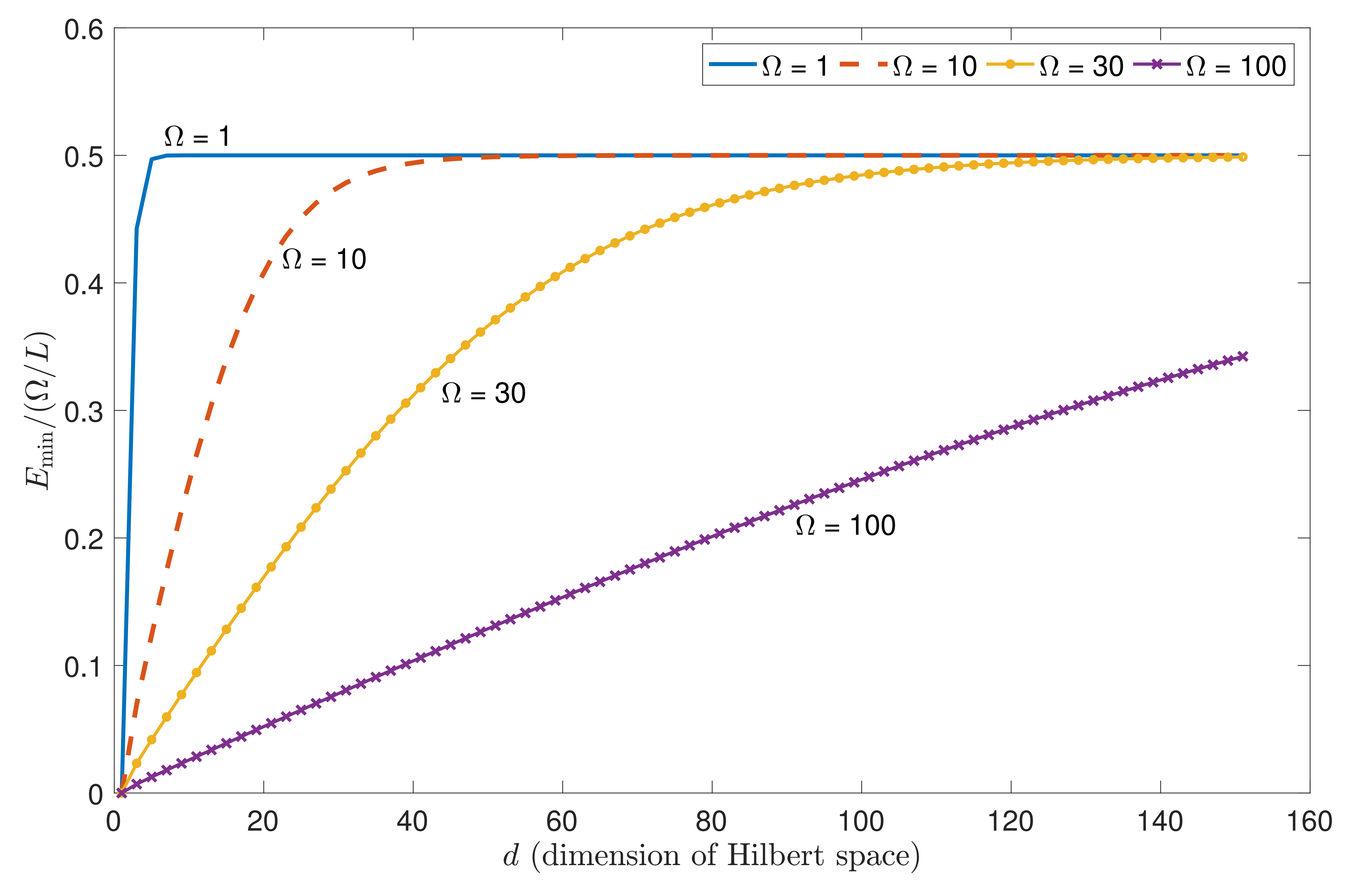
- Depends on both frequency and dimension

- Saturation to infinite-dim result

- High frequency and Low Dimension show suppression

Full expressions in paper

Accurate Approximations to Exact Matrix Calculations



Finite-Dimensional Hamiltonian for our Expanding Universe

$$\hat{H} = \sum_{|\boldsymbol{k}|<a\Lambda_{\rm UV}} \left[\frac{\hat{P}_{\boldsymbol{k}}^2}{2a^3 L_c} + \frac{a^3 L_c \left(|\boldsymbol{k}|^2/a^2 + m^2\right)}{2} \hat{Q}_{\boldsymbol{k}}^2 \right] \equiv \sum_{|\boldsymbol{k}|<a\Lambda_{\rm UV}} \left[\frac{\hat{P}_{\boldsymbol{k}}^2}{2M} + \frac{M\Omega_{\boldsymbol{k}}^2}{2} \hat{Q}_{\boldsymbol{k}}^2 \right]$$

GPO finite-dim conjugate variables

Collection of time-dependent harmonic oscillators, one for each mode

$$M = a^3 L_c$$

$$\Omega_{\boldsymbol{k}} = \sqrt{|\boldsymbol{k}|^2/a^2 + m^2}$$

Physical UV Cutoff

$$k_{\rm ph} = \Lambda_{\rm UV} \implies k_{\rm max} = a\Lambda_{\rm UV}$$

Free Parameters: $d_{\boldsymbol{k}}$ and $\alpha_{\boldsymbol{k}}$

Parametric Profile for Hilbert Space Dimension

"a simple power law ansatz"

$$d_{\boldsymbol{k}} = D\ (k/\Lambda_{\mathrm{UV}})^{n_D} + d_{\min}$$

$\{n_D, D\}$

Parameter Space

$d_{\min} = 2$

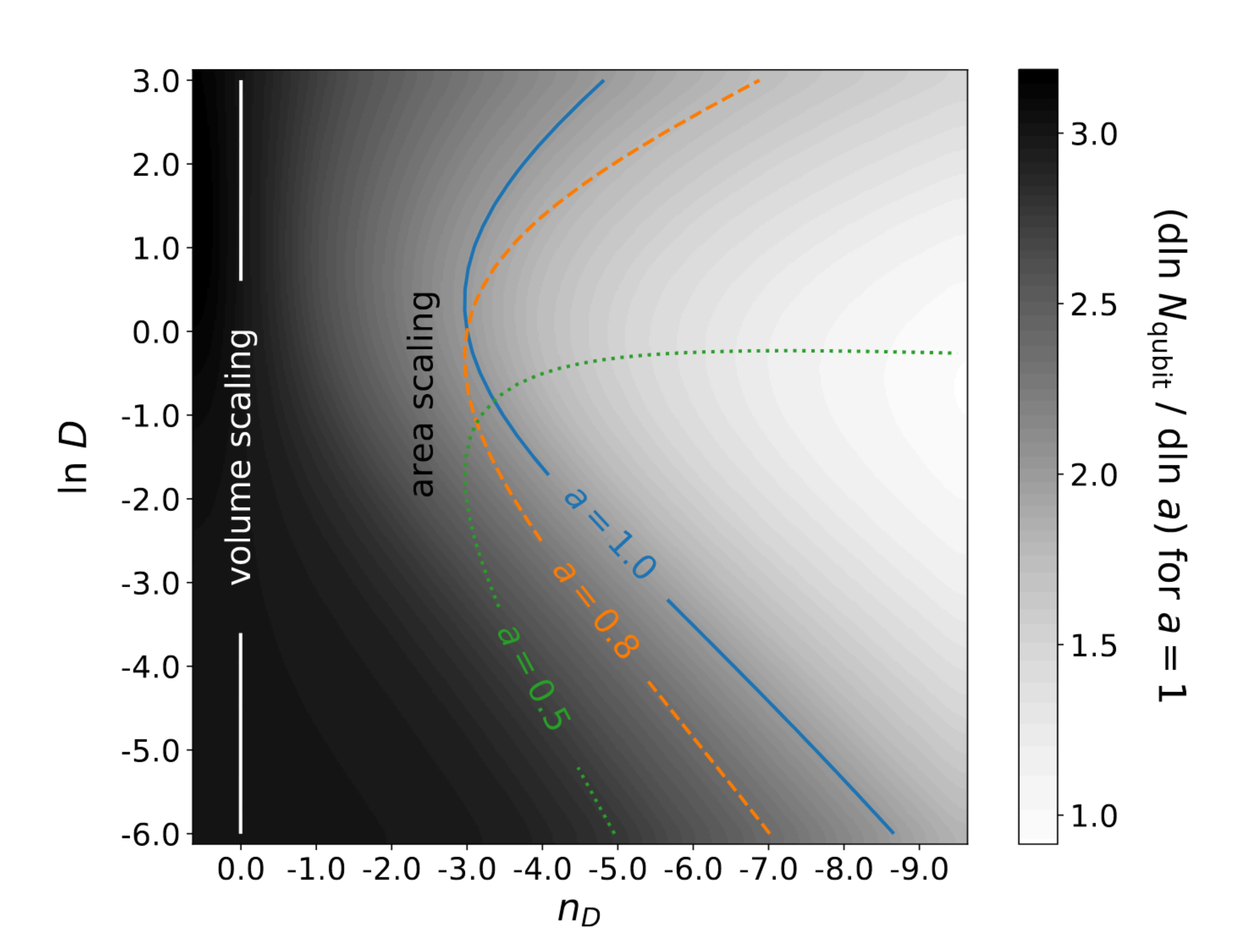
Each mode has the Hilbert space of at least a qubit

Scaling of Degrees of Freedom

Counting the number of degrees of freedom

$$N_{\text{qubit}} = \sum_{|\boldsymbol{k}|<a(t)\Lambda_{\text{UV}}} \log_2(d_{\boldsymbol{k}}) \approx \frac{L_c^3}{(2\pi)^3} \int_{|\boldsymbol{k}|<a(t)\Lambda_{\text{UV}}} d^3k \; \log_2(d_{\boldsymbol{k}})$$

Scaling of Degrees of Freedom



$n_D = 0$ Volume Scaling

$n_D < 0$ Sub-volume scaling

Implications for Holographic Principle

Choice of Eigenvalue Spacing

"Fiducial Choice"

At $t_{\boldsymbol{k}}$ when a mode enters the sum in the Hamiltonian $|\boldsymbol{k}| = a(t_{\boldsymbol{k}})\Lambda_{\mathrm{UV}}$

"mode initialization"

we demand,
maximization of ground
state energy

$$\frac{d}{d\alpha_{\boldsymbol{k}}} E_{\min,\boldsymbol{k}}(t_{\boldsymbol{k}}) = 0$$

We show in the paper:

Finite-dimensional effects
can only decrease the
ground state energy

Minimizes deviations from
infinite-dimensional results
in the low energy spectrum

$$\alpha_{\boldsymbol{k}} = \sqrt{\frac{2\pi}{d_{\boldsymbol{k}} M(t_{\boldsymbol{k}})\Omega_{\boldsymbol{k}}(t_{\boldsymbol{k}})}} \,, \quad \beta_{\boldsymbol{k}} = \sqrt{\frac{2\pi M(t_{\boldsymbol{k}})\Omega_{\boldsymbol{k}}(t_{\boldsymbol{k}})}{d_{\boldsymbol{k}}}}$$

(conservative effects of finite-dimensionality)

Choice of Eigenvalue Spacing

"Fiducial Choice"

$$\frac{d}{d\alpha_{\boldsymbol{k}}} E_{\min,\boldsymbol{k}}(t_{\boldsymbol{k}}) = 0$$

$$\alpha_{\boldsymbol{k}} = \sqrt{\frac{2\pi}{d_{\boldsymbol{k}} M(t_{\boldsymbol{k}})\Omega_{\boldsymbol{k}}(t_{\boldsymbol{k}})}} \, , \quad \beta_{\boldsymbol{k}} = \sqrt{\frac{2\pi M(t_{\boldsymbol{k}})\Omega_{\boldsymbol{k}}(t_{\boldsymbol{k}})}{d_{\boldsymbol{k}}}}$$

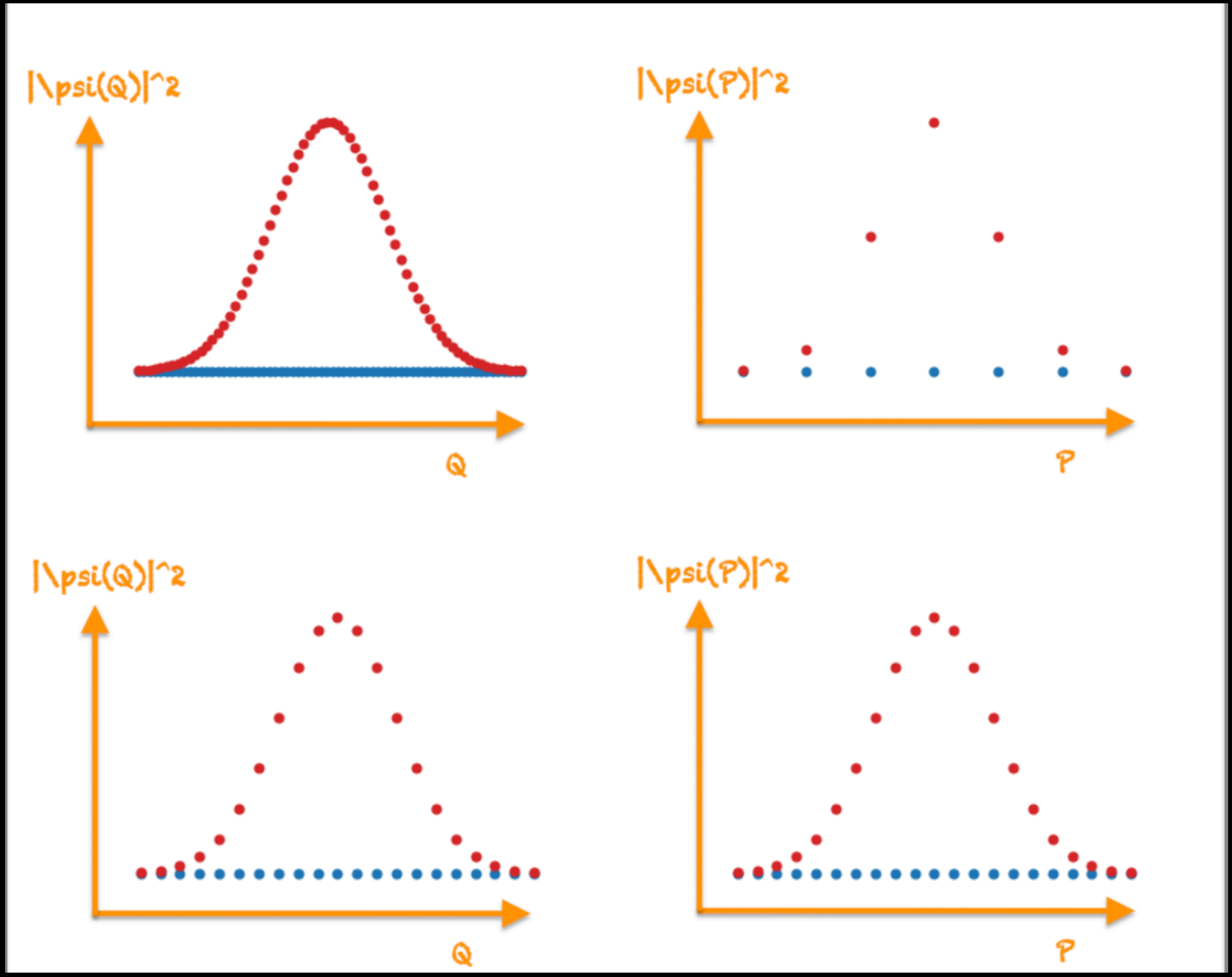
Expectations values are the same as the infinite-dimensional results

$$\langle 0(\boldsymbol{k}, t_{\boldsymbol{k}})| \hat{Q}^2_{\boldsymbol{k}} |0(\boldsymbol{k}, t_{\boldsymbol{k}})\rangle = \frac{1}{2M(t_{\boldsymbol{k}})\Omega_{\boldsymbol{k}}(t_{\boldsymbol{k}})}$$

$$\langle 0(\boldsymbol{k}, t_{\boldsymbol{k}})| \hat{P}^2_{\boldsymbol{k}} |0(\boldsymbol{k}, t_{\boldsymbol{k}})\rangle = \frac{M(t_{\boldsymbol{k}})\Omega_{\boldsymbol{k}}(t_{\boldsymbol{k}})}{2}$$

Implies an equal resolution in both variables

$$\frac{\langle 0(\boldsymbol{k}, t_{\boldsymbol{k}})| \hat{Q}^2_{\boldsymbol{k}} |0(\boldsymbol{k}, t_{\boldsymbol{k}})\rangle}{\alpha^2_{\boldsymbol{k}}} = \frac{\langle 0(\boldsymbol{k}, t_{\boldsymbol{k}})| \hat{P}^2_{\boldsymbol{k}} |0(\boldsymbol{k}, t_{\boldsymbol{k}})\rangle}{\beta^2_{\boldsymbol{k}}}$$



Choice of Eigenvalue Spacing

"An Alternate Choice"

$$\alpha_{\boldsymbol{k}} = \sqrt{\frac{2\pi}{d_{\boldsymbol{k}}}} = \beta_{\boldsymbol{k}}$$

More Algebraically Symmetric

(more extreme effects of finite-dimensionality)

Implications for Cosmological Physics

$$L_c \sim H_0^{-1}$$

constant comoving volume, coinciding with the Hubble horizon our universe is saturating to in the future

$$L_{\rm ph}(t) = a(t)\, L_c$$

(one possible choice of IR scale, we outline more possibilities in the paper)

Implications for Cosmological Physics

$$\epsilon_{\rm vac}(t) \qquad\qquad w_{\rm vac} = -\left(1 + \frac{1}{3}\frac{\mathrm{d}\ln\epsilon_{\rm vac}}{\mathrm{d}\ln a}\right)$$

Focus on vacuum energy, and its equation of state

Implications for expansion history of the universe

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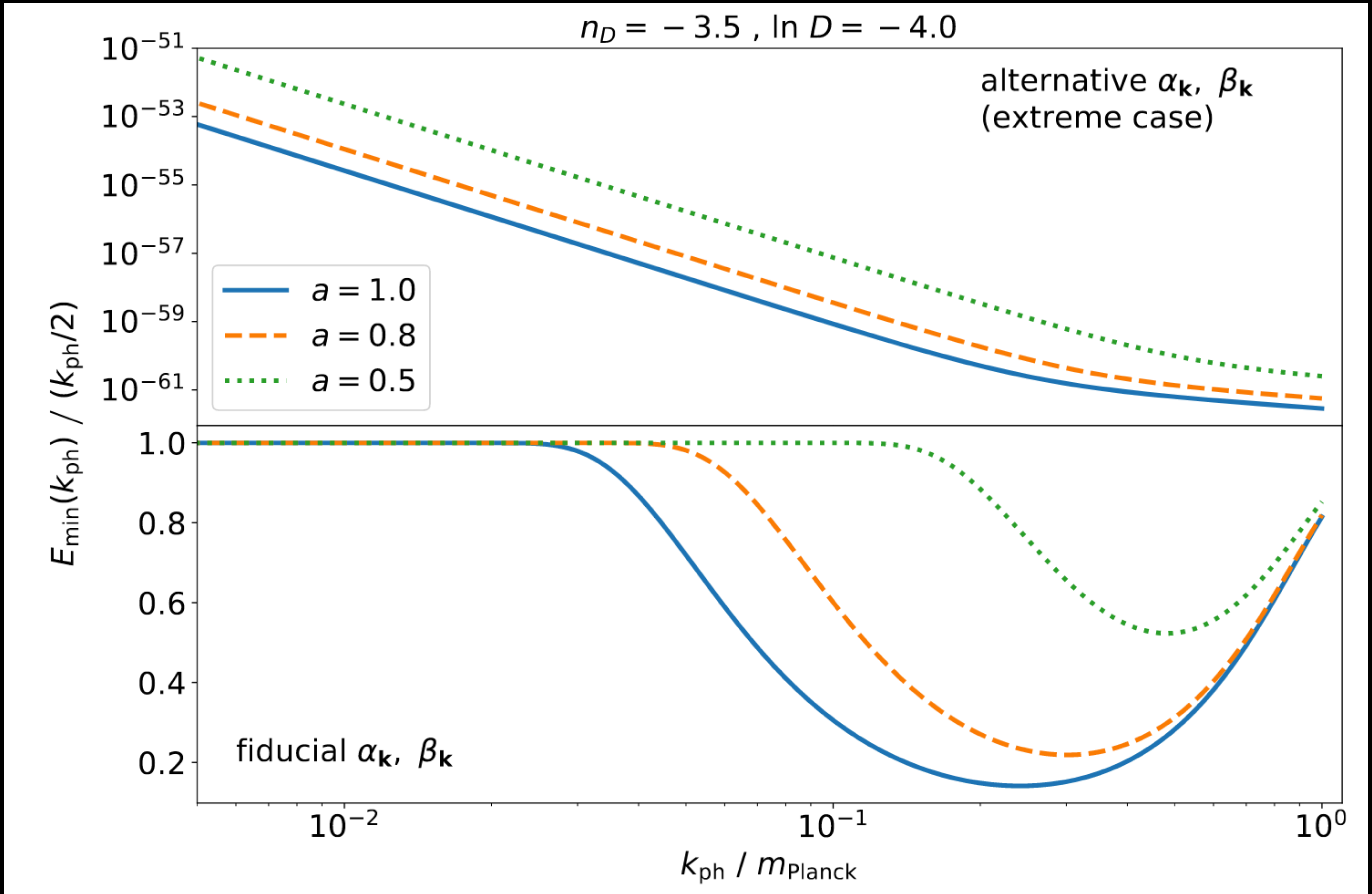
1. Vacuum energy can be dynamical
2. Suppression compared to its infinite-dimensional counterpart
3. Can decay between two constant epochs

$$\left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G}{3}\rho(t)$$

"Friedmann Equation"

Energy Density

Suppression of Vacuum Energy



Possible implications for the "cosmological constant" problem

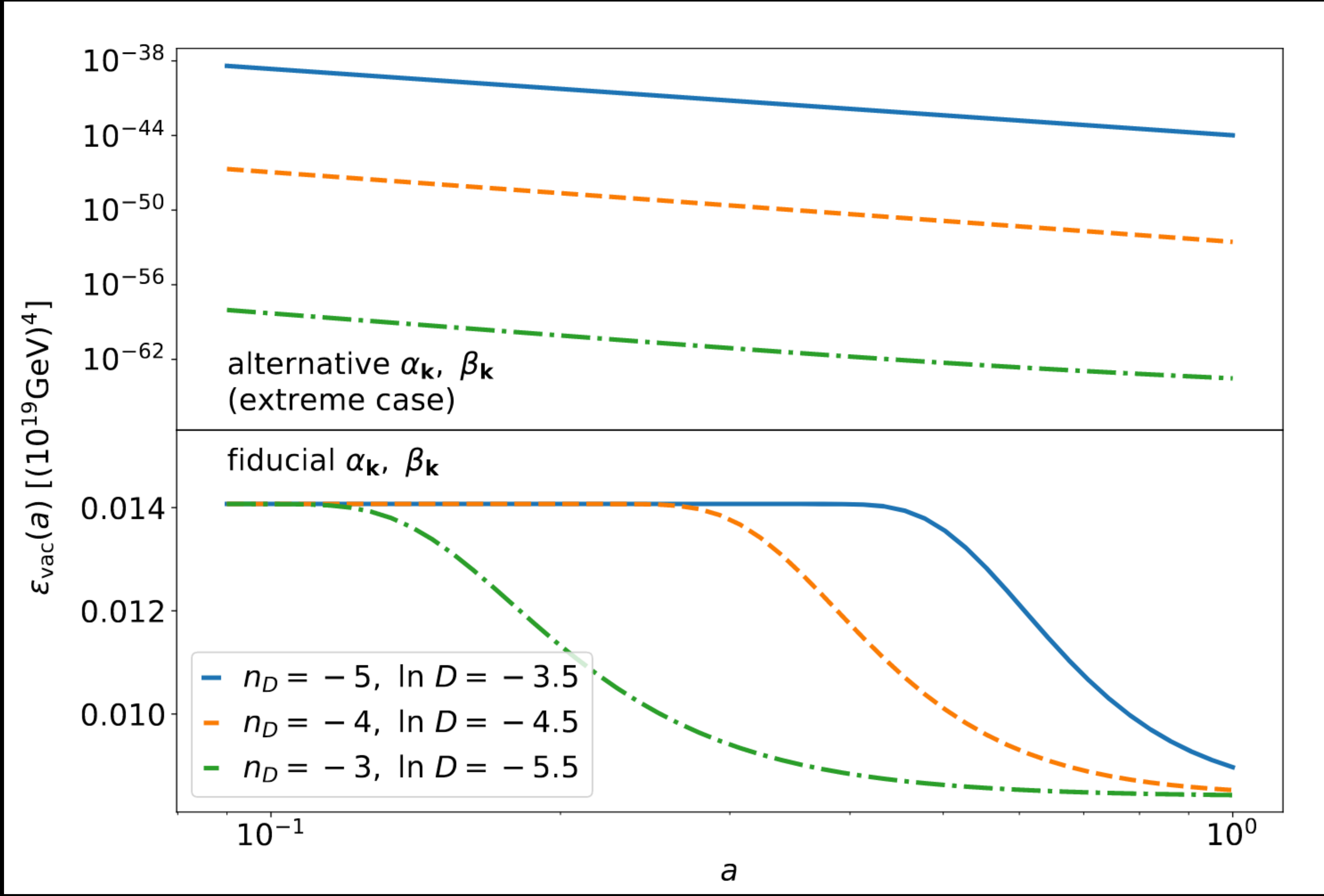
Also see, Cao, Chatwin-Davies & Singh 2019, arXiv:1905.11199

Dynamical Vacuum Energy

Time-dependence

- Implications for various cosmological epochs, and transitions

- Can decay between two constant epochs



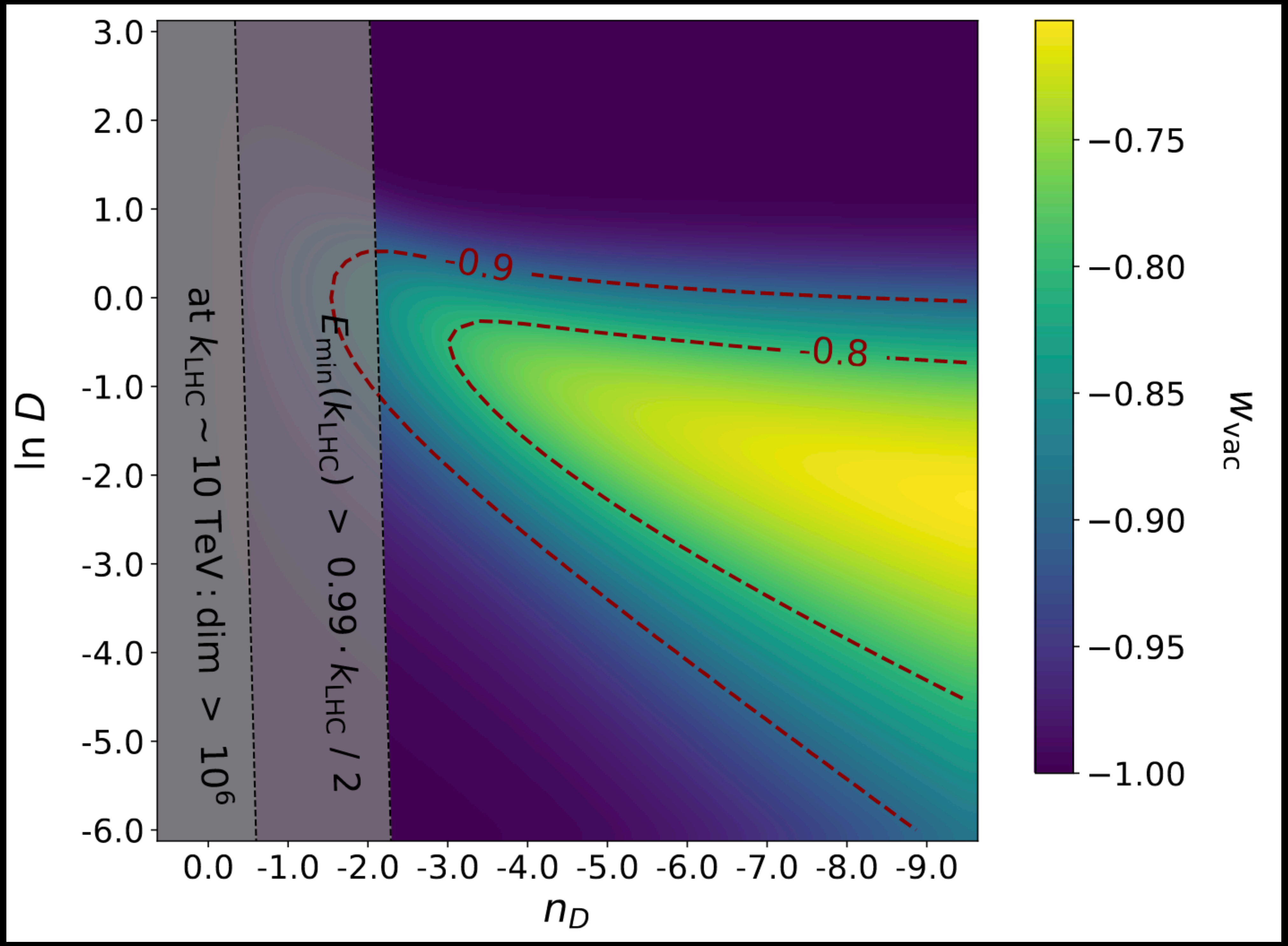
Vacuum Equation of State

$$w_{\rm vac} = -\left(1 + \frac{1}{3}\frac{d\ln\epsilon_{\rm vac}}{d\ln a}\right)$$

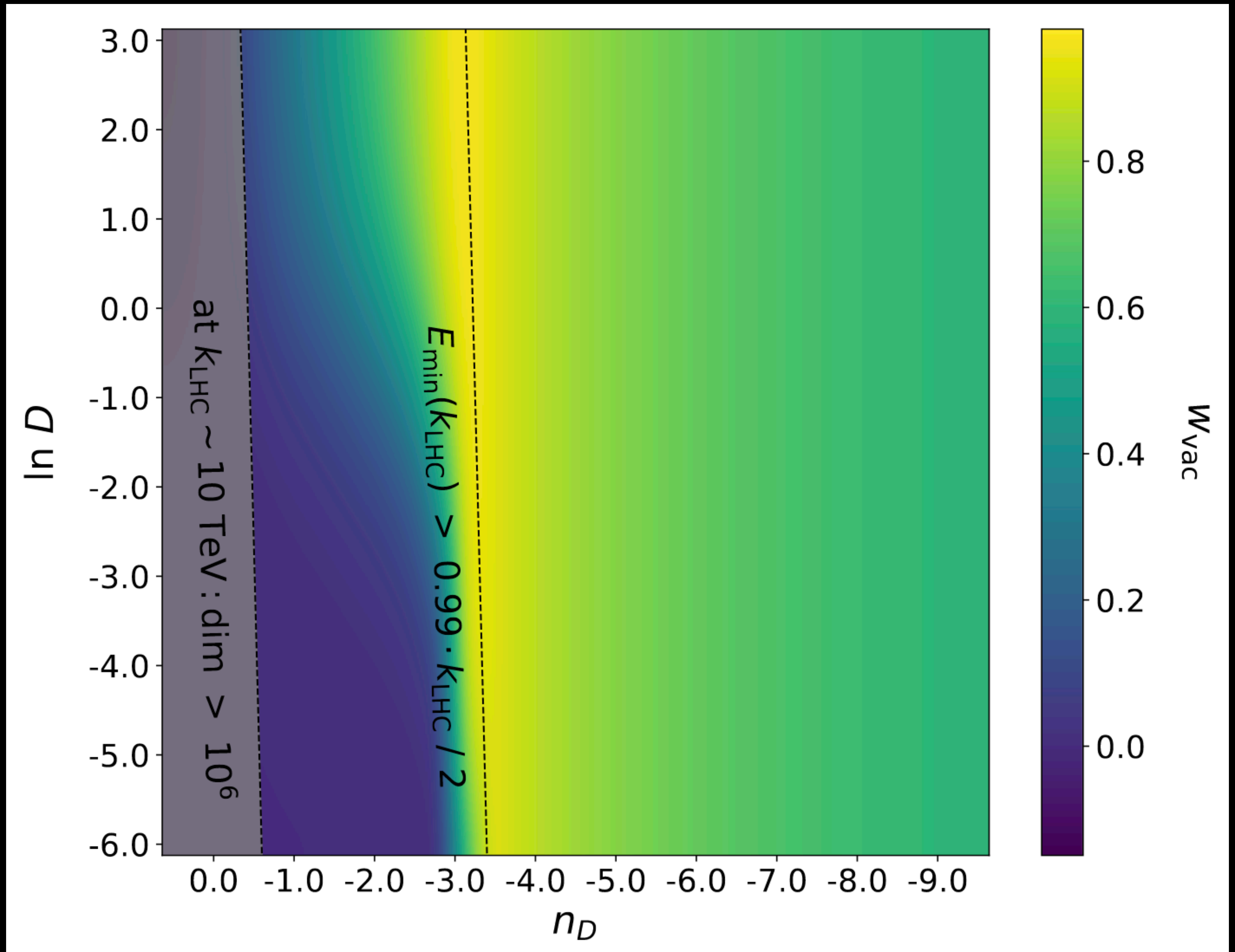
$w_{\rm vac} = -1$ "Cosmological Constant"

$w_{\rm vac} = 1/3$ Radiation era

$w_{\rm vac} = 0$ Matter era



Fiducial eigenvalue spacing



Alternate eigenvalue spacing

Exclude regions of parameter space based on known LHC scale particle physics

...okay, to recap,

Fiducial Choice ⟷ Alternate Choice

a range of possible effects from first-principle, finite-dimensional QM!

1. Finite-dimensional quantum mechanics applied to expanding cosmology
2. Precise expressions for ground state energy of the finite-dim oscillator
3. Sub-volume scaling of degrees of freedom, connections to holography
4. Parametrized models of dimension and eigenvalue scalings
5. Dynamical vacuum energy, can be suppressed, and decays between two constant epochs

A possibly fruitful point for exploring implications for cosmological physics

Public Code Available At:

https://github.com/OliverFHD/GPUniverse

We welcome you try it out, play with these tools!

In Hindsight

- Local finite-dimensionality from quantum gravity, along with an intrinsic finite-dimensional construction as a new paradigm
- Predictive consequences for (quantum) gravity motivated by first principle ideas.
- Corrections to Heisenberg Uncertainty Principle, Feynman Diagrams, Cassimir Effect, etc.
- Dynamical Dark Energy, connections with cosmological observations, such as to constrain the dimension the Hilbert space of the universe
- Connections with inflation, reheating, epoch transitions, etc. in cosmology