Transport, the Lattice, and Noise Reduction

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Outline

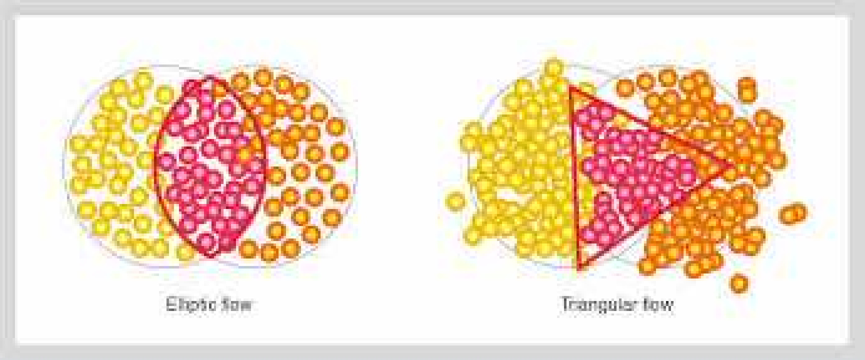
What are transport coefficients and how do we hope to compute them?

- Role of hydrodynamics in Heavy Ion Collisions
- What are shear and bulk viscosity?
- Kubo relations and Euclidean functions
- Lattice calculation: strategy
- Why signals are noisy
- Noise reduction: gradient flow
- Noise reduction: large separations
- Funny things we met along the way

Altenkort Eller Kaczmarek Mazur and GM, PRD 105 094505 arXiv:2112.02282 and in preparation

Space-anisotropy in Heavy-Ion Collision

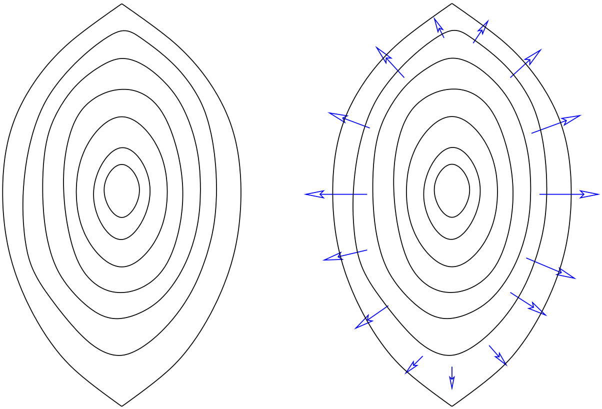
Colliding nuclei, as seen in transverse plane:



Geometry, and fluctuations, lead to *space-anisotropy* in initial conditions.

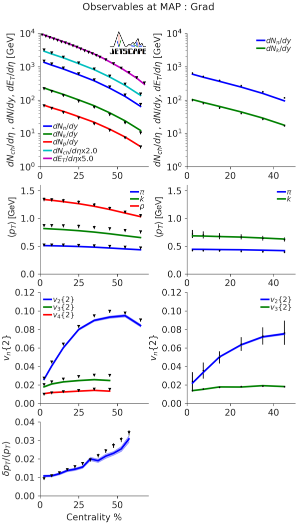
Effect: anisotropic flow

Consider how pressure varies in space, and apply $\partial_\mu T^{\mu\nu} = 0$: space anisotropy becomes flow anisotropy.



Simplest treatment: Ideal Hydro. Assume
$T^{\mu\nu} = T^{\mu\nu}_{\rm ideal} = (\epsilon + P) u^\mu u^\nu + P g^{\mu\nu}$

Viscous Hydro



Ideal hydro gives *rough but imprecise* fit to data. Overestimates flow, especially high moments.

Better: include viscous corrections

$$T^{\mu\nu} = T^{\mu\nu}_{\rm ideal} - \eta\sigma^{\mu\nu} - \zeta\Delta^{\mu\nu}\nabla_\alpha u^\alpha \,,$$

$$\Delta^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu \quad \text{rest-frame projector}$$

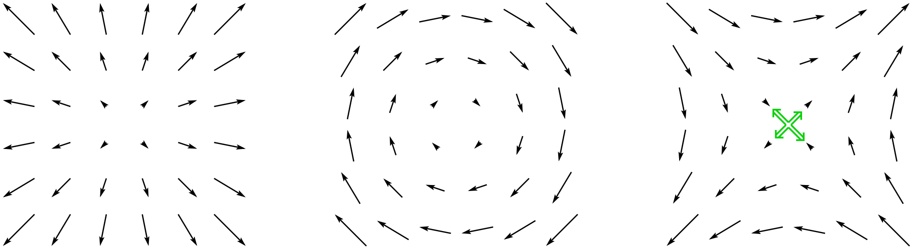
$$\sigma^{\mu\nu} = \Delta^\mu_\alpha \Delta^\nu_\beta \left(\nabla^\alpha u^\beta + \nabla^\beta u^\alpha - \frac{2}{3}\Delta^{\alpha\beta}\nabla_\gamma u^\gamma \right)$$

Fit data (JetScape arXiv:2011.01430)

Works well at small **nonzero** η/s, ζ/s

What are shear and bulk viscosity?

$\partial_i u_j$ decomposed into $\ell = 0, 1, 2$ spherical harmonics:



Divergence　　Vorticity　　Shear flow

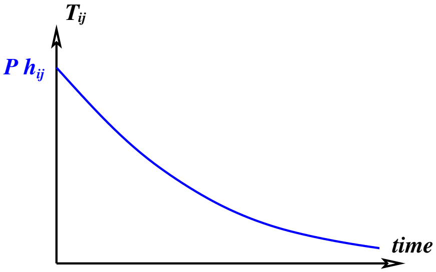
$T_{\mu\nu}$ is symmetric so $\ell = 0, 2$ can appear linearly:

- Divergent flow $\partial_\alpha u^\alpha > 0$: pressure reduced by $\zeta \, \vec{\nabla} \cdot \vec{u}$
- Shear flow: stress higher in in-going direction, lower in out-going direction by $\eta \, |\nabla u|$

Shear viscosity, time, geometry

Suppose $\partial_i u_j dt \equiv h_{ij}$ of flow occurs in time dt
Equivalent to distorting geometry, $g_{ij} \to g_{ij} + h_{ij}$
Instantaneous change in $T^{ij} = P g^{ij}$ is $\delta T^{ij} = -P h_{ij}$

Then T^{ij} relaxes back to equilibrium over time:
$T^{ij}(t) = T^{ij}(0) r(t)$
Many disturbances: $T^{ij}(t) = \int_{-\infty}^{t} dt' \ -P \partial_i u_j(t') r(t-t')$
Slower relaxation: larger disturbance
Shear viscosity: area under curve
$\eta = -T^{ij}/\partial_i u_j = P \int_0^\infty r(t) dt$



Is viscosity nonequilibrium, or equilibrium?

Yes!

- Fluid randomly fluctuates: T_{ij} can be "accidentally" nonzero
- Mean-squared fluctuation: $\int_V d^3x d^3y \langle T_{ij}(x) T_{ij}(y) \rangle = PTV$
- Fluctuations relax just like in non-equilibrium: $\int_V d^3x \langle T_{ij}(x,t) T_{ij}(y,0) \rangle = PTr(t)$
- Integrate over time: $\int_0^\infty PT\, r(t) = \eta T$ or

$$G^{>}(\omega = 0, \vec{k} = 0) \equiv \int d^3x dt \langle T_{xy}(x,t) T_{xy}(0,0) \rangle = 2\eta T$$

Equilibrium physics + fluctuations tell us about viscosity

More formally

Nonequilibrium picture:

$$\eta = \lim_{\omega \to 0} i\partial_\omega G_R^{T_{xy}T_{xy}}(\omega, \vec{k}=0) \qquad \text{or}$$

$$G_R^{T_{xy}T_{xy}}(\omega, \vec{k}=0) = P - i\omega\eta + \ldots$$

$$\sigma(\omega, \vec{k}=0) = 2\omega\eta + \ldots$$

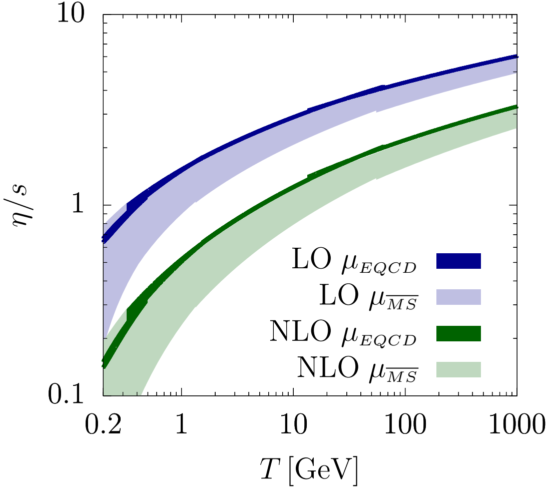
Equilibrium picture:

$$G^{>}_{T_{xy}T_{xy}}(\omega=0, \vec{k}=0) = 2\eta T$$

Either case: $\omega \to 0$ or integral over all unequal times.
Bulk viscosity: same but T_{ii} (connected $T_{ii}T_{jj}$ correlator)

Shear involves equilibrium but **unequal time**

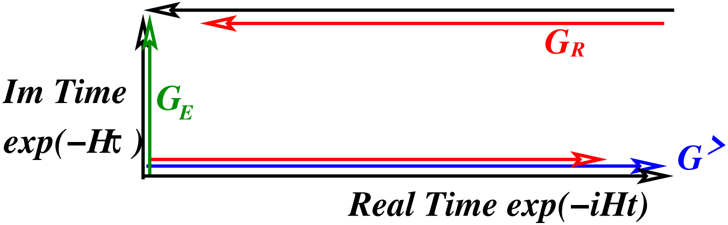
We could try! Maybe it will work down to a few times T_c?



Nope! Even at 100 GeV, convergence is poor. Ghiglieri GM Teaney PRL 121 052302 arXiv:1805.02663

Lattice methods?

QCD at physically achievable temperatures is **strongly coupled.**
Only **nonperturbative** (theory) methods reliable. **Lattice?**



Lattice studies Euclidean time, analytic continuation of Minkowski.

Relation :
$$G_E(\tau) = \int_{-\infty}^{\infty} \frac{d\omega}{4\pi} \sigma(\omega) \frac{\cosh(\omega(\tau - \beta/2))}{\sinh(\omega\beta/2)}$$

Lattice: the idea

Directly perform path integral

$$Z = \int \mathcal{D}(A_\mu, \overline{\psi}, \psi) \ \exp(-S(A, \bar{\psi}, \psi))$$

$$\langle \mathcal{O} \rangle = Z^{-1} \int \mathcal{D}(A_\mu, \overline{\psi}, \psi) \ \exp(-S(A, \bar{\psi}, \psi)) \ \mathcal{O}(A, \bar{\psi}, \psi)$$

by **sampling** A_μ from the ensemble $e^{-S}\mathcal{D}A$ and **estimating** $\langle \mathcal{O} \rangle$ as its sample average.

Get $\langle \mathcal{O} \rangle$ with *statistical errors* caused by limited sampling. How big are errors, and are they a problem?

Errors are a problem

Generic problem on the lattice: mass of pion

$$\langle \overline{u}\gamma^5 d(x)\, \overline{d}\gamma^5 u(0)\rangle \propto x^{-2} e^{-m_\pi x}$$

Measure at several separations x, fit to find m_π.
$O(1)$ errors lead to $O(1)$ mistakes (at worst).
Percent errors lead to percent mass estimates.

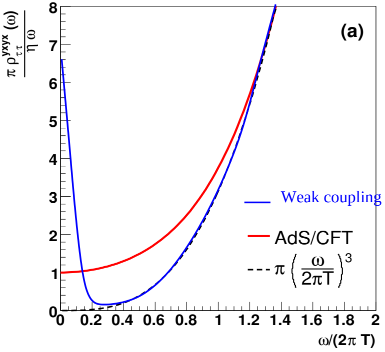
For us:

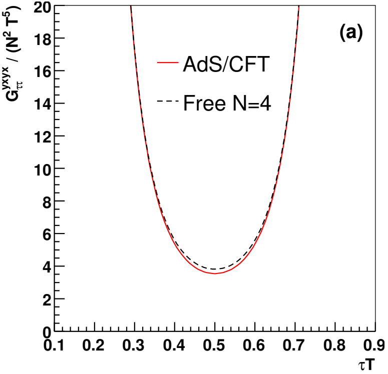
$$G_E(\tau) = \int_{-\infty}^{\infty} \frac{d\omega}{4\pi} \sigma(\omega) \frac{\cosh(\omega(\tau - \beta/2))}{\sinh(\omega\beta/2)}$$

We are trying to get σ from G_E – not vice versa. *inverse problem*
Demands very high precision for a chance to work eg, Ghiglieri et al arXiv:1604.07544

Errors are a problem

Compare weak-coupled QCD to strong-coupled $\mathcal{N} = 4$ SYM:





Very different (IR) spectral functions have very similar Euclidean function Teaney PRD 74 045025 arXiv:hep-ph/0602044 we will need precision to tell these apart!

Quenched approximation??

CRC-TR 211

All results for lattice η, ζ are in "quenched approximation"
Same thing as "Pure-Glue QCD" (ignore all quarks)

High temperature: quantitatively off. Wrong number of DOF, wrong Casimirs and spin factors.

Low temperature $T \sim T_c$: qualitatively off:

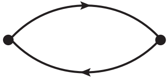
- Crossover transition $\rightarrow$ 1'st order transition
- Wrong transition temperature ~ 230MeV not 155MeV
- Wrong physics: *should be* transition from strongly-rescattering quarks$+$gluons to strongly-rescattering light mesons
  Instead, transition from strongly-rescattering gluons to dilute, heavy glueballs

We need unquenched results!

Config by config fluctuations

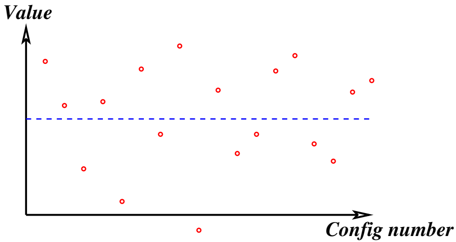
Why are correlation functions noisy, and how do we understand noise level?
Think about pion again: we want

$$\langle \overline{d}\gamma^5 u(x)\ \overline{u}\gamma^5 d(0)\rangle = \langle \mathrm{Tr}\, S(0,x)\gamma^5 S(x,0)\gamma^5 \rangle$$



Two fermion propagators going between 0 and x.

Value on each config is different. Fluctuations about a mean value.

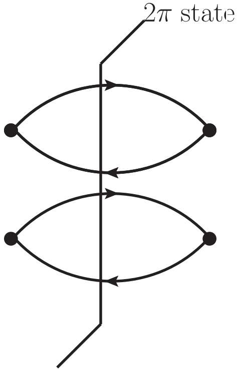


How do I understand *size of fluctuations*??

Size of fluctuations

Let's compute the *mean-square fluctuations* in observable

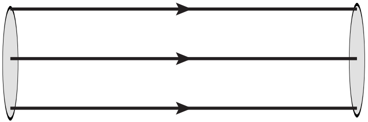
$$\sigma^2_{\mathcal{O}} \equiv \langle \mathcal{O}^\dagger \mathcal{O} \rangle - \left| \langle \mathcal{O} \rangle \right|^2$$
$$\equiv \langle \bar{d}\gamma^5 u(x) \bar{u}\gamma^5 d(0) \; \bar{u}\gamma^5 d(x) \bar{d}\gamma^5 u(0) \rangle - \dots$$



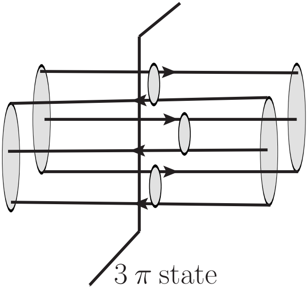
Four fermionic propagators.
Lightest state: 2π state.
$\langle \mathcal{O}^\dagger \mathcal{O} \rangle \propto \exp(-2m_\pi x) \sim \langle \mathcal{O} \rangle^2$
One pair of points gives $O(1)$ errors
One lattice gives $\sim V m_\pi^4$ independent measurements.
Signal-to-noise $(S/N)^2 \sim N_{\rm configs} V m_\pi^4$.
Rapidly get great statistics.

The proton problem

Consider the same thing for a proton:
$\langle \mathcal{O} \rangle = \langle \epsilon_{ijk} q_i q_j q_k(x)\ \epsilon_{lmn} \overline{q}_l \overline{q}_m \overline{q}_n(0) \rangle$
Create proton at 0, propagate to x:
$\langle \mathcal{O} \rangle \propto \exp(-m_p x)$



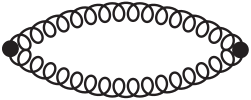
Consider the square:
$\langle qqq\overline{qqq}(x)\ qqq\overline{qqq}(0) \rangle$
Six propagators.
But that *Can be* 3π,
$\propto \exp(-3m_\pi x)$



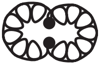
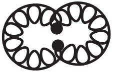
Signal-to-noise?
$\langle \mathcal{O}^\dagger \mathcal{O} \rangle \propto e^{-3m_\pi x}$
but $\langle \mathcal{O} \rangle^2 \propto e^{-2m_p x}$
Noise exponentially
larger than signal.

Stress tensors are even worse

Correlator of two stress tensors separated by distance x:

$T_{xy} \propto F_{x\mu} F_{y\mu}$. $\langle T_{xy}(x) T_{xy}(0) \rangle \sim$  $\propto x^{-8}$

Gluon propagator *not explicit object*, but obtained by computing operator correlation in A_μ backgrounds. "Disconnected"
Square, to determine noise??

$\langle T_{xy}(x) T_{xy}(x) T_{xy}(0) T_{xy}(0) \rangle \sim$   $\propto a^{-16}$

is x-independent! Signal-to-noise $(S/N)^2 \propto (a/x)^{16}$

Stress tensors: what went wrong?

For pions, propagator is explicit. For $F_{x\mu}F_{y\nu}$ it is not.
$F_{x\mu}F_{y\mu}$ is pointlike local operator. Couples to all 4-momenta
Long-distance correlators dominated by $p \sim x^{-1}$, but fluctuations, noise dominated by $p \sim a^{-1}$ the most UV scale.

UV gluons give noise, but no signal. I need to apply some *form factor* to remove coupling to very UV gluons.

There's no gauge invariant way to do that, is there?
Actually there is! "Gradient Flow" Lüscher JHEP08 071, arXiv:1006.4518

Gradient flow

- Sample gauge fields as usual: $\int \mathcal{D}A_\mu \exp\left(-\int d^4x\, F_{\mu\nu}F_{\mu\nu}/4g^2\right)$
- Apply a "gradient-flow" transformation on the gauge fields: Define $B_\mu(\tau_f=0)=A_\mu$, and

$$\frac{dB_\mu(x,\tau_f)}{d\tau_f} = -\frac{\delta}{\delta B_\mu(x)} \int d^4y \frac{1}{2} F^B_{\mu\nu} F^B_{\mu\nu}(y)$$

  same as gradient-flow descent under field-strength-squared action
- Transformation is gauge invariant and easy to do on lattice
- Build observables out of $B_\mu(\tau_f)$, not A_μ

This defines a new set of "fluffy" operators. At lowest order,

$$B_\mu(\tau_f,p) = e^{-\tau_f p^2} A_\mu(p)\,, \qquad G_{\mu\nu}(p) = \frac{g_{\mu\nu} e^{-2\tau_f p^2}}{p^2}$$

Application: scale setting

Consider squared field strength after gradient flow:

$$\left\langle \frac{F^a_{\mu\nu}F^a_{\mu\nu}}{2} \right\rangle_{\tau_f} \simeq \frac{g^2}{2} \int \frac{d^4p}{(2\pi)^4} \delta_{aa}(p^2 g_{\mu\nu} - p_\mu p_\nu) \frac{g_{\mu\nu} e^{-2\tau_f p^2}}{p^2}$$

$$\simeq \frac{3g^2(N_c^2-1)}{2} \int \frac{d^4p}{(2\pi)^4} e^{-2\tau_f p^2}$$

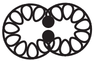
$$\simeq g^2 \times \frac{3(N_c^2-1)}{128\pi^2\tau_f^2}$$

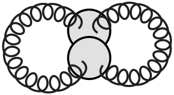
gives a way to measure g^2 as a function of scale!

- Renormalization of (bosonic) operators is automatic
- Coupling to UV cut off at scale $\mu \sim 1/\sqrt{4\tau_f}$
- Fact that we are on lattice becomes irrelevant!
- Lattice artifacts exist but are order $a^2/\pi^2\tau_f$.

Application: noise reduction

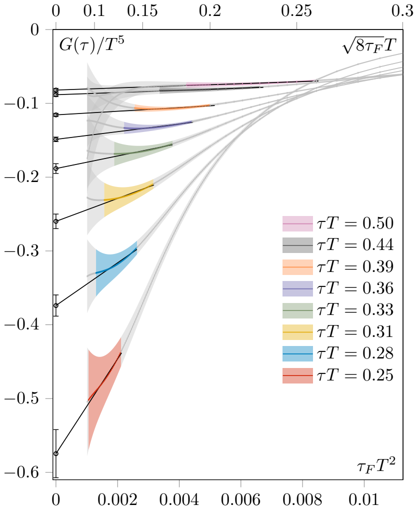
Local pointlike T_{xy} operators become fuzzy blobs:

$\langle T_{xy}(x)T_{xy}(x)\rangle \sim$  $\sim a^{-8}$

$\langle T_{xy}(x,\tau_f)T_{xy}(x,\tau_f)\rangle \sim$  $\sim \tau_f^{-4}$

Lattice-spacing divergences replaced by flow-scale: $(S/N)^2 \sim \tau_f^8/x^{16}$

It works – but causes new problems



Topological density correlator
$\int d^3x \langle q(x,\tau) q(0,0)\rangle_{\tau_f}$
Noise falls as we increase τ_f
But the *answer* also changes.
Flow-smeared operator $=$
desired operator $+$ OPE of
high-dimension op's

$$\mathcal{O}_{\tau_f} = \mathcal{O} + \sum_{D>4} \tau_f^{\frac{D-4}{2}} Z_D \mathcal{O}_D$$

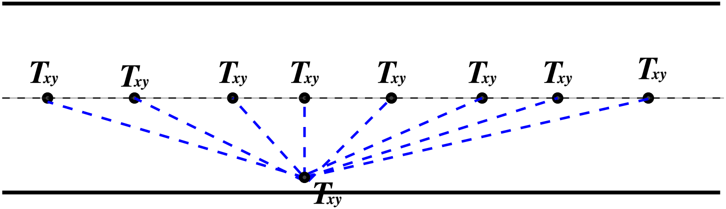
$D = 6$: Corrections $\propto \tau_f/\tau^2$.
Must extrapolate to $\tau_f = 0$

Noise and large space separations

Shear, bulk viscosity defined as $\vec{k} = 0$ limits:

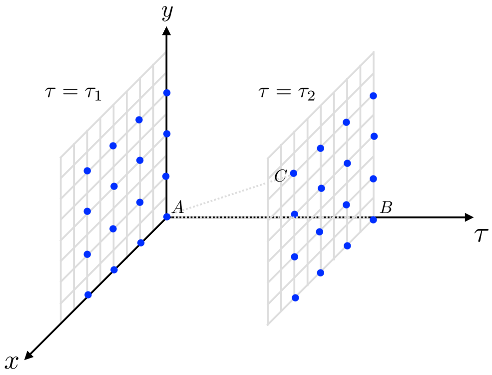
$$\eta = \lim_{\omega \to 0} i\partial_\omega \int d^3x dt e^{i\omega t} G_R(x,t)$$

What about the d^3x part? Lattice: $G_E(\tau) = \int d^3x \langle T_{xy}(x,\tau) T_{xy}(0,0) \rangle$



Need correlator between $T_{xy}(0,0)$ and T_{xy} at each space-separation.
Free-thy estimate: correlator $\propto \frac{1}{(x^2+\tau^2)^4}$.
But noise is same at all distances

Problem: need Δx-differential information at reasonable cost.

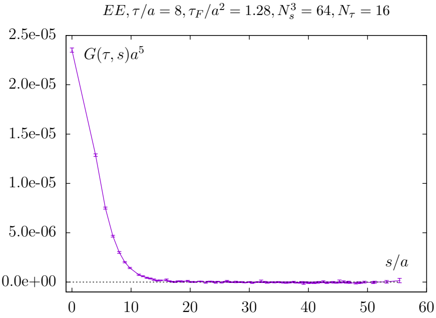


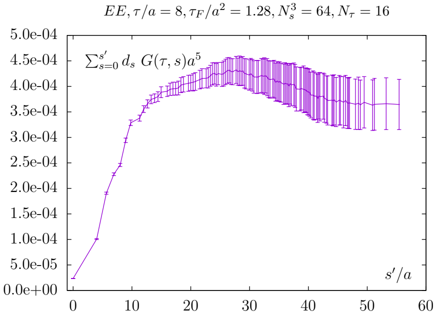
Correlating every $\langle T_{xy}(x,\tau) T_{xy}(y,\tau') \rangle$: is V^2 operations $\to$ too expensive.
averaging $T_{xy}(x,\tau)$ over *blocks*, finding block-block correlations is feasible

- find all block-block correlators as function of block-separation
- integrate over small block separations, but
- fit larger block-separations to functional form, use to replace large-separation, noisy data

Correlator as function of separation

Look at correlation as function of r, and its integral:

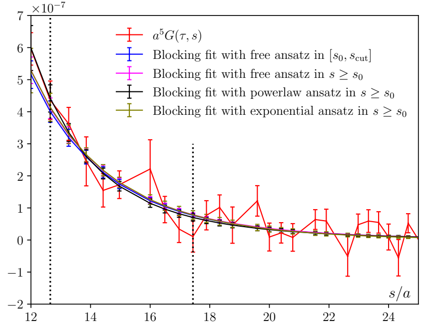


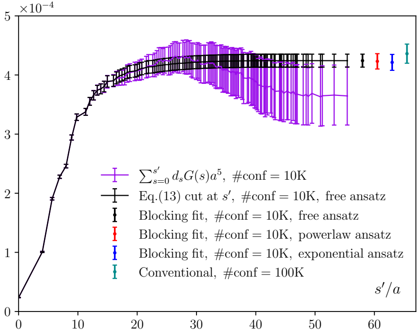


correlation dominated by smaller separations,
noise dominated by large separations.
This is why many people use very small boxes ($L = 2\beta$).

Solution: Fit the tail

Fit tail region with $10 > S/N > 1$, use at $S/N < 1$:





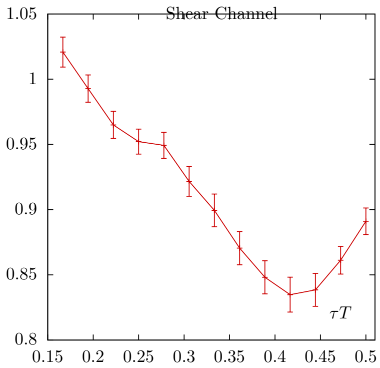
Errors removed without changing result:
Answer consistent with result with $10\times$ more data

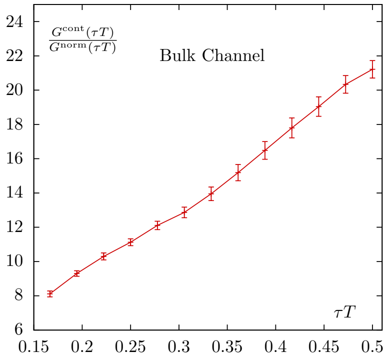
Shear and Bulk Viscosity

- Evaluate correlators of $T_{ij} - \delta_{ij} T_{kk}/3$ and of $T_{\mu\mu}$
- Need accurate operator normalization
  - Traceless: Measure $\langle T_{xx} - T_{00} \rangle = sT$, use measured sT values to determine right normalization nonperturbatively Giusti Pepe 1503.07042
  - Trace: NNLO gradient-flowed perturbation theory Suzuki Takaura 2102.02174
- Shear must be summed over all 5 independent spin-2 components $T_{xy}, T_{xz}, T_{yz}, T_{xx} - T_{yy}, T_{xx} + T_{yy} - 2T_{zz}$.
- Continuum limit *followed by* $\tau_F \to 0$ limit

Shear + Bulk Euclidean correlator

Pure-glue QCD at $T = 1.5\,T_c$ normalized to free behavior
Both $a \to 0$ and $\tau_F \to 0$ limits already taken





Bulk is missing g^4-type multiplicative factor.
"Standard" fits (LO or NLO UV + IR structure) do *not* provide a good fit!

Unquenching Challenges

Unquenching challenges we have *not* solved:

- Full $T_{\mu\nu}$ has bosonic $F_{\mu\alpha}F_{\nu\alpha}$ and fermionic $\bar{\psi}(\gamma_\mu \partial_\nu + \ldots)\psi$ parts.
- Each part has independent, τ_f-dependent Z-factor

$$T_{xy} = \frac{Z_b}{g^2} G_{x\mu} G_{y\mu} + \frac{Z_f}{4} \bar{\psi} \left(\gamma_x \partial_y + \gamma_y \partial_x - \gamma_x \overleftarrow{\partial}_y - \gamma_y \overleftarrow{\partial}_x \right) \psi$$

- Need to determine Z-factors nonperturbatively
- Noise reduction, gradient flow with fermionic operators?
  How best to use gradient flow + fermions

Still plenty to do.

Summary

- Shear + bulk viscosity: interesting physics about nonequilibrium corrections in hydrodynamics
- Calculating viscosities in QCD is hard:
  - Need nonperturbative (lattice) treatment
  - Lattice is Euclidean-time, transport is real-time
- Lattice calculations confront severe noise problems.
  - Reduce fluctuations with gradient flow
  - Avoid fluctuations where there is no signal with careful treatment of transverse separations

I think we can get good signal-to-noise in unquenched theory.
Then we will see whether we can overcome analytical continuation....

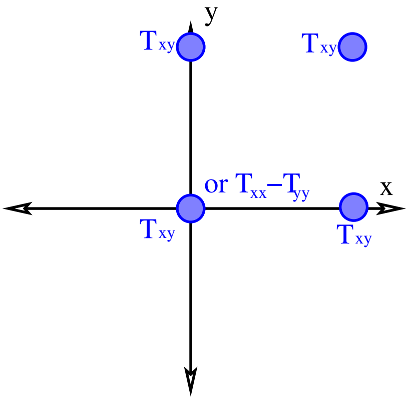
Thank the Organizers

We would all like to thank the organizers for such an interesting meeting!

SEWM 2024 in Frankfurt!

Owe Philipsen will be lead organizer. See you all there!!!

Don't just compute $\langle T_{xy} T_{xy} \rangle$!



Consider correlator of T_{xy} along x or y axis: correlator strictly negative (reflection positivity). But in $x = y$ direction, positive. The $T_{xx} - T_{yy}$ correlator is positive along x-axis but negative in $x = y$ direction.

Different block-distances are in different directions!
To get rotationally invariant $G(r)$, need to sum over all 5 spin-2 T_{ij} correlators.

If you don't use all T components

Here is how the r-dependent correlators will look if we use only the T_{xy}, T_{xz}, T_{yz} correlators: