Absence of inhomogeneous phases in $2 + 1$-dimensional Four-fermion models

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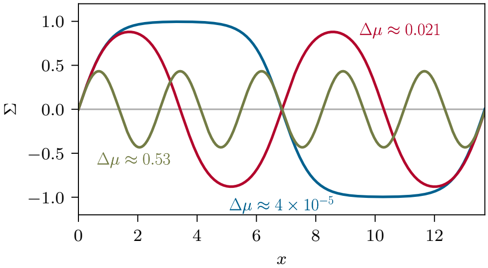
Strong and Electro-Weak Matter 2022, Paris

June 21, 2022

Introduction

- Simple strong-interaction models feature so-called inhomogeneous, chiral phase
- Chiral condensate breaks translational invariance spontaneously $\langle\bar{\psi}\psi\rangle = f(\mathbf{x})$
- Indications for such phases and related phenomena found in QCD[1]



$1+1$-dimensional Gross-Neveu model in the mean-field approximation[2,3]

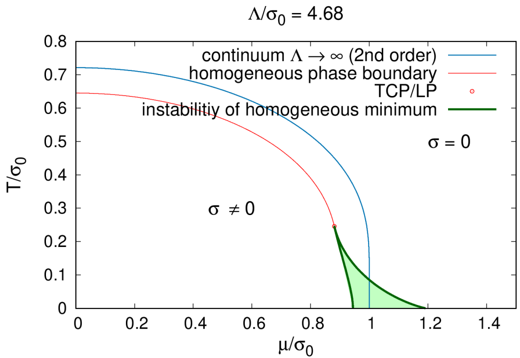
[1] W.-j. Fu, J. M. Pawlowski, F. Rennecke, *Phys. Rev. D* **2020**, *101*, 054032.
[2] M. Thies, K. Urlichs, *Phys. Rev. D* **2003**, *67*, 125015.
[3] A. Koenigstein et al., **2021**.

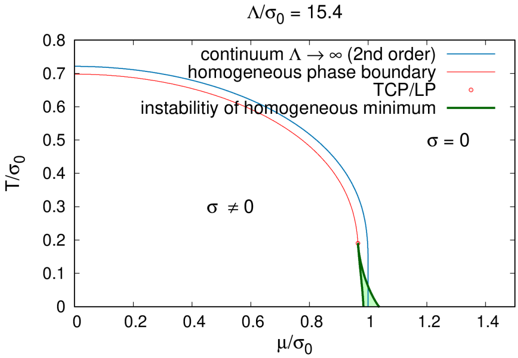
Introduction

- In mean-field models inhomogeneous phases are:
  - Established in $1+1$ dimensions
  - Also found in $3+1$ dimensions[4], but the results are questionable ...
    - In the renormalizable Quark-Meson model the action gets unbounded when renormalizing[5]
    - In non-renormalizable NJL model the results depend on the regularizations scheme $\Rightarrow$ see Laurin Pannullos talk today
- In $2+1$ dimensions we refer to Refs.[6][7][8]
- In short: Inhomogeneous phases are found at finite regulator values depending on regularization scheme, but vanish when $\Lambda \to \infty$

[4] M. Buballa, S. Carignano, *Prog. Part. Nucl. Phys.* **2015**, *81*, 39–96.
[5] S. Carignano, M. Buballa, B.-J. Schaefer, *Phys. Rev. D* **2014**, *90*, 014033.
[6] M. Buballa, L. Kurth, M. Wagner, M. Winstel, *Phys. Rev. D* **2021**, *103*, 034503.
[7] R. Narayanan, *Phys. Rev. D* **2020**, *101*, 096001.
[8] L. Pannullo, M. Wagner, M. Winstel, *Symmetry* **2022**, *14*, 265.

Motivation: 2+1-dimensional GN model





- Studied $2+1$-dim. Gross-Neveu model, which is a Four-fermion model with scalar $(\bar{\psi}\psi)^2$ channel

$\Rightarrow$ Can this result be transferred to more involved models in 2+1 dimensions?

A more general Four-fermion model?

- In principle, stability analysis applies to every kind of interaction-channel
- 4×4 Dirac Algebra allows for 16 possible bilinears, i.e.

$$\{\gamma_A\}_{A=1,\ldots,16} = \{\mathbb{1}, \mathrm{i}\gamma_4, \mathrm{i}\gamma_5, \gamma_{45} \equiv \mathrm{i}\gamma_4\gamma_5, \gamma_\mu, \tfrac{\mathrm{i}}{2}[\gamma_\mu, \gamma_\nu], \mathrm{i}\gamma_\mu\gamma_4, \mathrm{i}\gamma_\mu\gamma_5, \}$$

- Vector interactions do not exhibit chiral condensation in the mean-field approximation[9]
  $\Rightarrow$ Inhomogeneous chiral phases are not expected

[9] G. Parisi, *Nucl. Phys. B* **1975**, *100*, 368–388.

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$$\{\gamma_B\}_{B=1,\dots,16} = \{\mathbb{1}, \mathrm{i}\gamma_4, \mathrm{i}\gamma_5, \gamma_{45}, \vec{\tau}, \mathrm{i}\vec{\tau}\gamma_4, \mathrm{i}\vec{\tau}\gamma_5, \vec{\tau}\gamma_{45}\}$$

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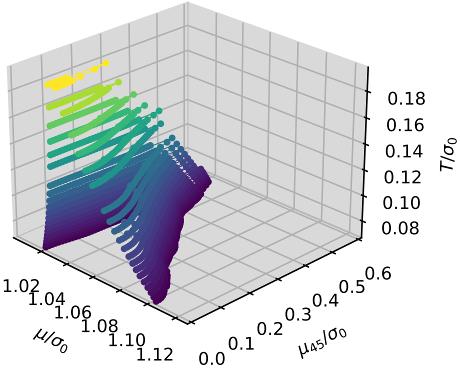
- All kind of chemical potentials can be included, i.e. $\mu_B, \mu_{45}, \mu_5, \mu_4$ with corresponding structures in Dirac space, but also isospin potential μ_I

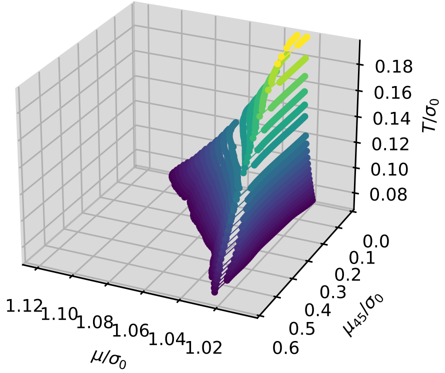
[9] G. Parisi, *Nucl. Phys. B* **1975**, *100*, 368–388.

Chiral/Isospin imbalance in the Gross-Neveu model

- Stability analysis on the lattice of GN model with μ_{45} or μ_I
- Obtain instability region of $\sigma = \bar{\sigma}$ for one of two discretizations

$$a\sigma_0 = 0.2327,\ L\sigma_0 = 23.27$$





A more general Four-Fermion model

- Bosonization of Four-Fermion models leads to the action

$$S_{\mathrm{FF}}[\bar\psi,\psi,\vec\phi] = \int \mathrm{d}^3x \, \left(N_{\mathrm{f}} \, \frac{\vec\phi(\mathbf{x})\cdot\vec\phi(\mathbf{x})}{2\lambda} \; + \; \bar\psi(x)\, Q\, \psi(x) \right)$$

$$Q = \partial\!\!\!/ + \gamma_0\mu + M + \sum_j c_j\phi_j(\mathbf{x})$$

- Equivalent action after integrating fermions out

$$S_{\mathrm{eff}}[\vec\phi]/N_{\mathrm{f}} = \int \mathrm{d}^3x \; \frac{\vec\phi(\mathbf{x})\cdot\vec\phi(\mathbf{x})}{2\lambda} \; - \; \mathrm{Tr}\ln Q$$

- M contains all type of allowed mass terms, set $M = 0$ for chiral limit

$$\langle\phi_j\rangle \sim \langle\,\bar\psi c_j\psi\,\rangle$$

A more general Four-Fermion model

- Bosonization of Four-Fermion models leads to the action $c_j \in \{\gamma_B\}_{B=1,\dots,16}$

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$$\langle \phi_j \rangle \sim \langle \bar{\psi} c_j \psi \rangle$$

The stability analysis

- Analyze the stability of the homogeneous ground state $\vec{\phi} = \bar{\vec{\phi}}$

$$\phi_j = \bar{\phi}_j + \delta\phi_j(\mathbf{x})$$

- Mean-field approximation $\Rightarrow$ Find homogeneous ground state via optimization
- Compute corrections to the action due to perturbation $\delta\phi_j(\mathbf{q})$
- Second order corrections determine whether action is lowered by the perturbation

The stability analysis in the GN model

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$$S_{\text{eff}}[\sigma]/N_{\text{f}} = \frac{1}{2\lambda}\int \mathrm{d}^3x\,(\bar{\sigma}+\delta\sigma(\mathbf{x}))^2 - \mathrm{Tr}(\ln(\underbrace{\not{\partial}+\gamma_0\mu+\bar{\sigma}}_{\equiv\bar{Q}}+\delta\sigma(\mathbf{x})))$$

- Expand $S_{\text{eff}}[\bar{\sigma}+\delta\sigma]$ in powers of $\delta\sigma$ yields

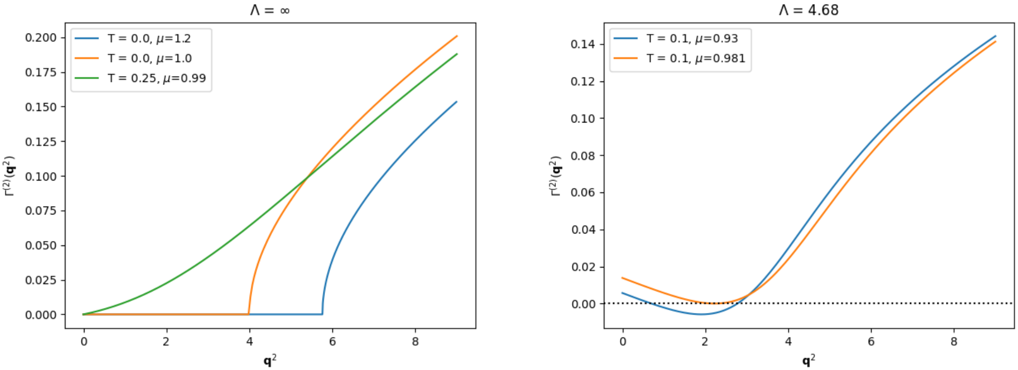
$$S_{\text{eff}}^{(2)}/N_{\text{f}} = \frac{\beta}{2\lambda}\int d^2x\,\left(\delta\sigma(\mathbf{x})\right)^2 + \frac{1}{2}\mathrm{Tr}\left(\bar{\mathrm{Q}}^{-1}\delta\sigma\bar{\mathrm{Q}}^{-1}\delta\sigma\right)$$

- Evaluating traces and fourier transform gives

$$S_{\text{eff}}^{(2)}/N_{\text{f}} = \frac{1}{2}\beta\int\frac{d^2q}{(2\pi)^2}|\delta\tilde{\sigma}(\mathbf{q})|^2\Gamma^{(2)}(\mathbf{q}^2)$$

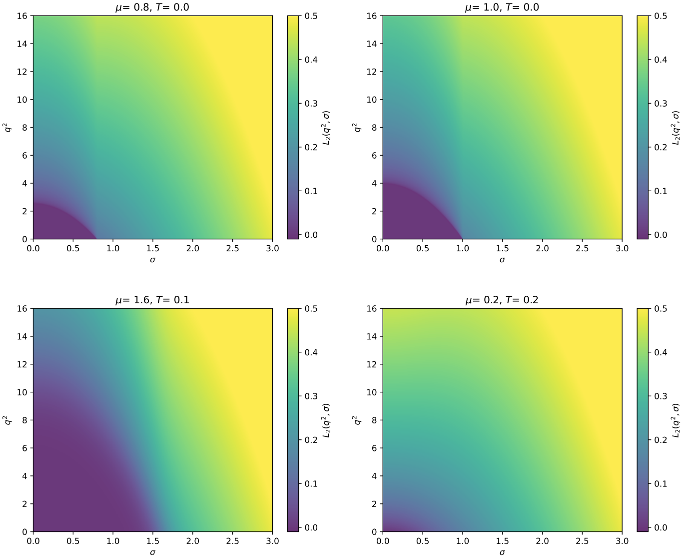
$$\Gamma^{(2)}(\mathbf{q}^2) = \frac{1}{\lambda} - \ell_1 \underbrace{-\frac{1}{2}(\mathbf{q}^2+4\bar{\sigma}^2)\ell_2(\mathbf{q}^2)}_{L_2(\mathbf{q}^2)}$$

- $L_2(\mathbf{q}^2)$ is monotonically increasing $\forall\mu,T,\bar{\sigma}$



- $T = 0, \bar{\sigma} = 0:\; L_2 \sim \Theta(\frac{q^2}{4} - \mu^2)\, q \arctan\left(\sqrt{\frac{q^2}{4} - \mu^2}/\mu\right)$

Momentum dependence of $\Gamma^{(2)}$ - Explore $\mu, T, \bar{\sigma}$

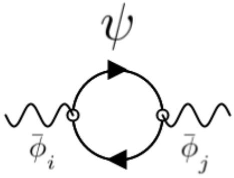


Analyzing the two-point function for other models

- The stability analysis can be applied to all Four-fermion channels

$$S_{\text{eff}}^{(2)}/N_f = \frac{\beta}{2\lambda}\int d^2q\,|\delta\phi_j(\mathbf{q})|^2 \; + \frac{\beta}{2}\sum_{i,j}\delta\phi_i^*(\mathbf{q}))\delta\phi_j(\mathbf{q})\Gamma^F_{\phi_i\phi_j}(\mathbf{q}^2))$$

$$\Gamma^F_{\phi_i\phi_j} = \int \mathrm{d}^3p\,\mathrm{tr}\left(c_i\tilde{\bar{Q}}^{-1}(p+q)c_j\tilde{\bar{Q}}^{-1}(p)\right)$$



- $\Gamma_{\phi_i\phi_j}(\mathbf{q})$ gives fermionic contribution to curvature

⇒ Strategy: Identify $L_2(\mathbf{q})$, as found for the GN model, for more complex models

- Potential terms and standard kinetic terms will not change the monotonic behavior of $\Gamma^{(2)}$

First example: Gross-Neveu-Yukawa model

$$S[\bar{\psi}, \psi, \sigma] = \int \mathrm{d}^3x \left(\frac{1}{2g^2}\sigma^2 + \frac{1}{2}(\partial\sigma)^2 + \frac{\lambda}{4}\sigma^4 + \bar{\psi}_f \left(\gamma_\nu \partial_\nu + \gamma^0 \mu + h\sigma \right) \psi_f \right)$$

- By redefinitions of fields and couplings

$$\frac{S_{\text{eff}}[\sigma]}{N_f} = \int \mathrm{d}^3x \left(\frac{m_0^2}{2}\sigma^2 + \frac{\gamma}{2}(\partial_\nu\sigma)(\partial_\nu\sigma) + \frac{\kappa}{4}\sigma^4 \right) - \ln\left(\mathrm{Det}(\partial\!\!\!/ + \gamma^0\mu + \sigma) \right)$$

- The two point function yields

$$\Gamma^{(2)} = m_0^2 - \ell_1 + \underbrace{L_2(\mathbf{q}^2)}_{\text{known from GN model!}} + \frac{\gamma}{2}\mathbf{q}^2 + \frac{3\kappa}{2}\bar{\sigma}^2$$

- No inhomogeneous phases - as long as m_0^2, γ, κ are chosen such that action is bounded

Second example: $U(2N_f)$ chiral symmetry

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- Study full chiral symmetry group $U(2N_f)$

$$S_{\bar{\psi},\psi,\sigma,\eta_4,\eta_5} = \int \mathrm{d}^3x \Big[N_\mathrm{f} \frac{\sigma^2+\eta_4^2+\eta_5^2}{2\lambda} + \bar{\psi}_f \left(\partial\!\!\!/ + \gamma^0\mu + \sigma + \mathrm{i}\gamma_4\eta_4 + \mathrm{i}\gamma_5\eta_5 \right) \psi_f \Big]$$

- Two-point function as $\bar{\eta}_4 = \bar{\eta}_5 = 0$ through chiral rotation

$$S^{(2)}_{\mathrm{eff}}/N_f = \frac{\beta}{2} \int \frac{d^2q}{(2\pi)^2} \sum_{\phi\in\{\sigma,\eta_4,\eta_5\}} |\delta\tilde{\phi}(\mathbf{q})|^2 \Gamma^{(2)}_{\phi}(\mathbf{q}^2)$$

$$\Gamma^{(2)}_{\sigma} = \frac{1}{\lambda} - \ell_1 + L_2(\mathbf{q}^2)$$

$$\Gamma^{(2)}_{\eta_4} = \Gamma^{(2)}_{\eta_5} = \frac{1}{\lambda} - \ell_1 - \frac{1}{2}\mathbf{q}^2\ell_2(\mathbf{q}^2)$$

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$$\Gamma_\sigma^{(2)} = \frac{1}{\lambda} - \ell_1 + L_2(\mathbf{q}^2)$$

$$\Gamma_{\eta_4}^{(2)} = \Gamma_{\eta_5}^{(2)} = \frac{1}{\lambda} - \ell_1 - \frac{1}{2}\mathbf{q}^2 \ell_2(\mathbf{q}^2)$$

- The two-point functions are again monotonically increasing with $|\mathbf{q}|$

$U(2N_f)$ chiral symmetry + Parity breaking

- Study full chiral (flavor) symmetry group $U(2N_f)$ + possibility of parity breaking

$$S = \int \mathrm{d}^3x \left(N_\mathrm{f} \frac{\sigma^2 + \eta_4^2 + \eta_5^2}{2\lambda} + N_\mathrm{f} \frac{\eta_{45}}{2\lambda_{45}} + \bar{\psi}_f \left(\not{\partial} + \gamma_0 \mu + \sigma + \mathrm{i}\gamma_4\eta_4 + \mathrm{i}\gamma_5\eta_5 + \mathrm{i}\gamma_{45}\eta_{45} \right) \psi_f \right)$$

- Two-point function as $\bar{\eta}_{45} = 0$ when $\lambda_{45} = \lambda$

$$S_{\mathrm{eff}}^{(2)}/N_f = \frac{\beta}{2} \int \frac{d^2q}{(2\pi)^2} \sum_{\phi \in \{\sigma, \eta_4, \eta_5, \eta_{45}\}} |\delta\tilde{\phi}(\mathbf{q})|^2 \Gamma_\phi^{(2)}(\mathbf{q}^2)$$

$$\Gamma_{\eta_{45}}^{(2)} = \Gamma_\sigma^{(2)} = \frac{1}{\lambda} - \ell_1 + L_2(\mathbf{q}^2)$$

$$\Gamma_{\eta_4}^{(2)} = \Gamma_{\eta_5}^{(2)} = \frac{1}{\lambda} - \ell_1 - \frac{1}{2}\mathbf{q}^2 \ell_2(\mathbf{q}^2)$$

- When $\lambda_{45} \neq \lambda$: Offdiagonal terms come into play!

$U(2N_f)$ chiral symmetry + Parity breaking

- Study full chiral (flavor) symmetry group $U(2N_f)$ + possibility of parity breaking
- When $\lambda_{45} \neq \lambda$: Offdiagonal terms come into play!
- $\phi_\pm \sim (\alpha\sigma \pm \beta\eta_{45})$

$$S_{\text{eff}}^{(2)}/N_f = \frac{\beta}{2}\int \frac{d^2q}{(2\pi)^2} \sum_{\phi\in\{\eta_4,\eta_5,\phi_+,\phi_i\}} |\delta\tilde{\phi}(\mathbf{q})|^2 \Gamma_\phi^{(2)}(\mathbf{q}^2)$$

$$\Gamma_{\eta_4}^{(2)} = \Gamma_{\eta_5}^{(2)} = \frac{1}{\lambda} - \ell_1 - \frac{1}{2}\mathbf{q}^2\ell_2(\mathbf{q}^2)$$

$$\Gamma_{\phi_\pm}^{(2)} = \frac{1}{2\lambda} + \frac{1}{2\lambda_{45}} - \ell_1 - \frac{1}{2}\left(\mathbf{q}^2 + 4(\bar{\sigma}^2 + \bar{\eta}_{45}^2) + C_\pm(\bar{\sigma}, \bar{\eta}_{45})\right)\ell_2(\mathbf{q}^2)$$

- This is still montonically increasing

The full model

- $c_j \in \{\mathbb{1}, \mathrm{i}\gamma_4, \mathrm{i}\gamma_5, \gamma_{45}, \vec{\tau}, \mathrm{i}\vec{\tau}\gamma_4, \mathrm{i}\vec{\tau}\gamma_5, \vec{\tau}\gamma_{45}\}$ with corresponding scalars ϕ_j

$$S = \int \mathrm{d}^3x \left[N_\mathrm{f} \sum_i \frac{\phi_i^2}{2\lambda} + \bar{\psi}_f \left(\partial\!\!\!/ + \gamma_0 \mu + \sum_j c_j \phi_j \right) \psi_f \right]$$

- 16×16 matrix in field space has to be diagonalized to compute $S_{\mathrm{eff}}^{(2)}$
- In principle: Possible, but roots of high order polynomials occur $\Rightarrow$ Not solvable

Conclusions

- Studied inhomogeneous phases in Four-Fermion and Yukawa models in $2+1$-dimension
- No inhomogeneous condensation via stability analysis in the renormalized limit
- Strong regularization scheme dependence at finite regulator values
- Inhomogeneous condensates with energy barrier towards homogeneous ground state still possible
  - No evidence found in Gross-Neveu model and extensions via optimization on the lattice

Ongoing studies regarding inhomogeneous phases

- Regularization scheme dependence in $3+1$ dimensions
- Scalar lattice field theory - negative wave function renormalization & inhomogeneous order parameters with bosonic fluctuations

Appendix

Symmetries and chiral imbalance

- Discrete chiral symmetries (4×4 Representation of Euclidean Dirac algebra)

$$\psi_f \to \gamma_4 \psi_f \,, \quad \bar{\psi}_f \to -\bar{\psi}_f \gamma_4 \,, \quad \sigma \to -\sigma,$$
$$\psi_f \to \gamma_5 \psi_f \,, \quad \bar{\psi}_f \to -\bar{\psi}_f \gamma_5 \,, \quad \sigma \to -\sigma$$

- $\gamma_{45} = \mathrm{i}\gamma_4\gamma_5$ generates continuous (chiral) symmetry
- Dirac-Operator is block-diagonal

$$Q[\mu, \mu_{45}, \sigma] = \begin{pmatrix} Q^{(2,+)}[\mu + \mu_{45}, \sigma] & 0 \\ 0 & Q^{(2,-)}[\mu - \mu_{45}, \sigma] \end{pmatrix}$$

- Dirac operators build from irreducible fermion representation

$$Q^{(2,\pm)}[\mu, \sigma] = \pm\tau_2(\partial_0 + \mu) \pm \tau_3\partial_1 \pm \tau_1\partial_2 + \sigma$$

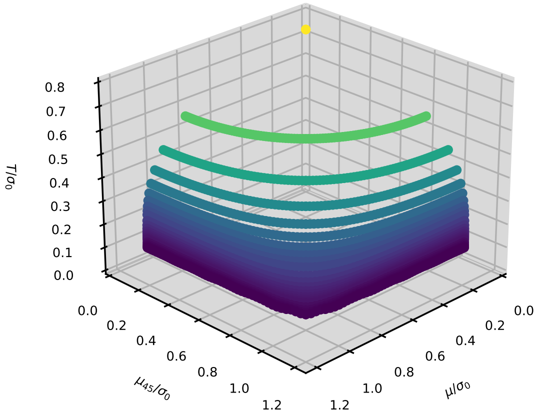
- $\mu \neq 0, 0 \leqslant \mu_{45} \leqslant \mu$ increases chiral imbalance, i.e. difference between $\mu_L = \mu + \mu_{45}$ for upper 2 comp. and $\mu_R = \mu - \mu_{45}$ for lower 2 comp.
- What are the effects on the respective (in-)homogeneous phases?

$\Rightarrow$ Study with two different lattice regularizations using naive fermions and different coupling to σ

Homogeneous phase diagram, $\sigma(\mathbf{x}) = \text{const.}$

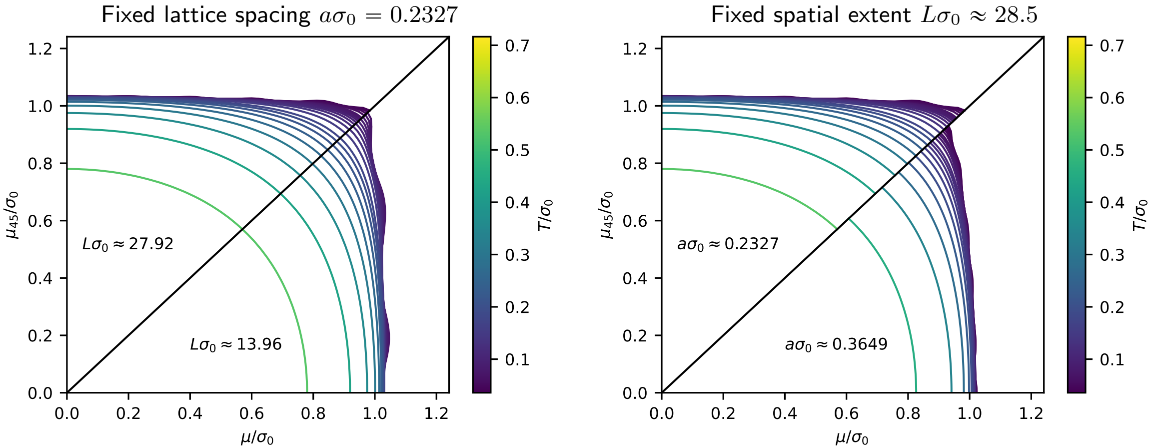
- $\sigma(\mathbf{x}) = \bar{\sigma} = \text{const.}$, Minimization of lattice action, identical for both discretization
- Theoretically observed symmetry $\mu_{45} \leftrightarrow \mu$ & $\mu \to -\mu$ & $\mu_{45} \to -\mu_{45}$

$$a\sigma_0 = 0.2327, L\sigma_0 = 27.92$$



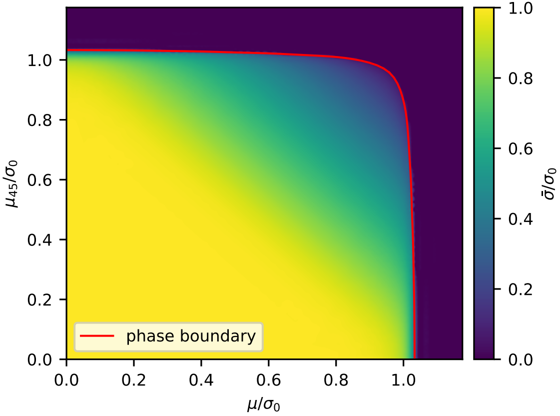
Homogeneous phase diagram, $\sigma(\mathbf{x}) = \text{const.}$

- Results for $\mu_{45} = 0$ are already quite close to continuum results[10]



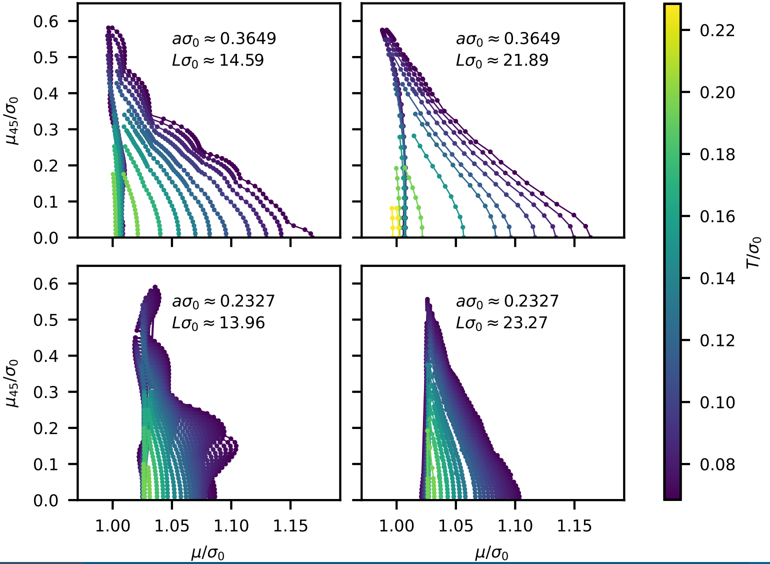
[10] K. Klimenko, *Z. Phys. C* **1988**, *37*, 457.

The homogeneous order parameter $\bar{\sigma}$



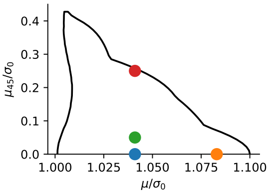
- Order parameter at $T/\sigma_0 = 0.0716$ with $a\sigma = 0.2327, L\sigma_0 = 27.92$
- Plateau for $\mu_L/\sigma_0 = \mu/\sigma_0 + \mu_{45}/\sigma_0 \leqslant 1.0$, where $\bar{\sigma} \approx \sigma_0$, then continuous decrease of $\bar{\sigma}$
- Competition of $|\mu_L/\sigma_0| > 1.0$ and $|\mu_R/\sigma| < 1.0$ leads to continuous decrease

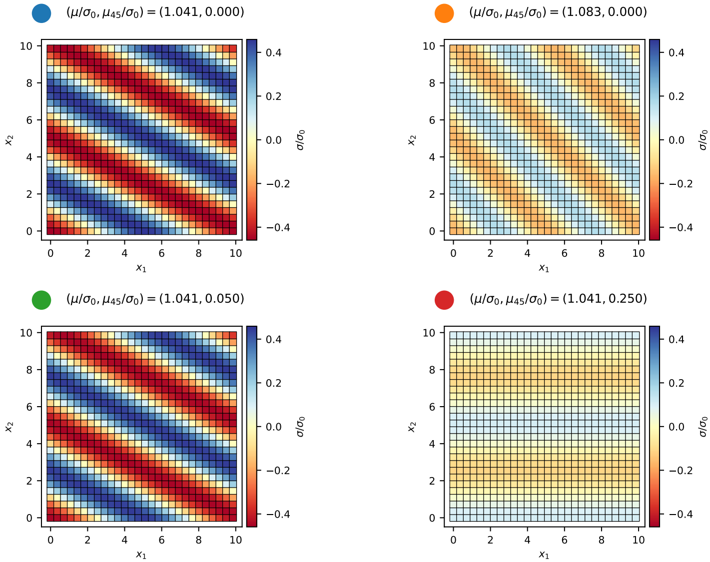
Stability analysis on multiple lattices



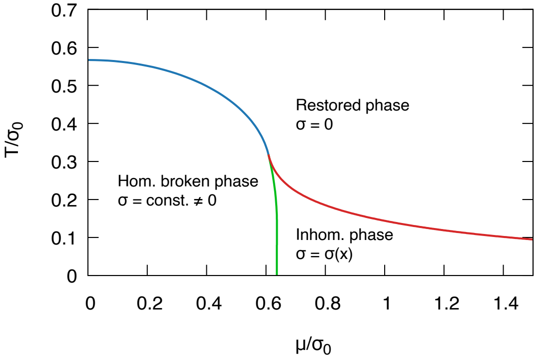
Within instability region

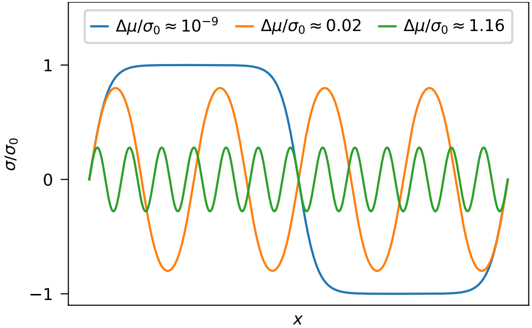
$a\sigma_0 \approx 0.3649,$
$L\sigma_0 = 10.22,$
$T/\sigma_0 = 0.114$





Phase diagram of $1+1$-dim. GN in the large-N_f limit





Chiral condensate with $\mu = \mu_c + \Delta\mu$ at $T = 0$

Chirality operator

- Definition of irreducible representation of fermions in 2+1 dimensions
  Dirac matrices as Pauli matrices

$$\gamma^0 = \sigma_1,\ \gamma^1 = \sigma_2,\ \gamma^2 = \sigma_3 \tag{2}$$

$$\gamma^0 = -\sigma_1\ \gamma^1 = -\sigma_2,\ \gamma^2 = -\sigma_3 \tag{3}$$

$\Rightarrow$ Non-trivial γ_5 not available

- Which symmetry is spontaneously broken by the condensate ?

Parity in 2+1 dimensions

- Parity as inversion of all spatial coordinates equivalent to rotation

$$(x_0, x_1, x_2)^T \xrightarrow{P} (x_0, x_1, -x_2)^T$$
$$\psi \xrightarrow{P} -\mathrm{i}\gamma_2\psi$$
$$\bar{\psi} \xrightarrow{P} -\mathrm{i}\bar{\psi}\gamma_2$$

- Obtain $\sigma \xrightarrow{P} -\sigma$
- Non-vanishing σ indicates spontaneous breaking of parity

Chiral symmetry transformation

- Use four component spinors via combination of two inequivalent irreducible spinors ($\tau_i \equiv$ Pauli matrices in isospin space)

$$\gamma_\nu = \tau_3 \otimes \sigma_{\nu+1}, \quad \gamma_4 = \tau_1 \otimes \mathbb{1},$$
$$\gamma_5 = -\tau_2 \otimes \mathbb{1}, \quad \gamma_{45} = i\gamma_4\gamma_5 = \text{diag}(\mathbb{1}, -\mathbb{1})$$

- Parity to be defined via tensor product with τ_1
- Mass term $\propto \bar{\psi}\psi$ now invariant under parity

Chiral transformations

- Symmetries of free massless fermions in 2+1 dimensions $(U(2N_f))$

$$\psi_f \to e^{i\theta\Gamma}\psi_f \qquad \Gamma \in \{\mathbb{1}, \gamma_{45}, \gamma_4, \gamma_5\}$$

- For the Gross-Neveu model only a subgroup is realized

$$\psi_f \to \gamma_5\psi_f, \ \bar{\psi}_f \to -\bar{\psi}_f\gamma_5 \tag{5}$$

$$\psi_f \to \gamma_4\psi_f, \ \bar{\psi} \to -\bar{\psi}_f\gamma_4 \tag{6}$$

- Together with this discrete transformation we have continuous symmetries

$$\psi_f \to e^{i\phi\gamma_{45}}\psi_f, \ \bar{\psi}_f \to \bar{\psi}_f e^{-i\phi\gamma_{45}} \tag{7}$$

$$\psi_f \to e^{i\alpha}\psi_f, \ \bar{\psi}_f \to \bar{\psi}_f e^{-i\alpha} \tag{8}$$

- Combination of (7) with (5) reproduces (6)