Inhomogeneous phases in the 3+1-dimensional mean-field Nambu-Jona-Lasinio model on the lattice

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Motivation

- Several low-energy effective models exhibit a chiral inhomogeneous phase (IP) i.e. a space-dependent chiral condensate

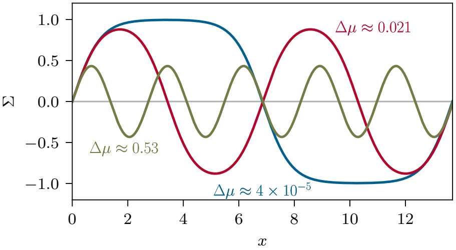
[M. Buballa, S. Carignano, *Prog. Part. Nucl. Phys.*, **81**, 39–96 (2015), arXiv: 1406.1367]

- Indications for such phases and related moat regimes in QCD

[R. D. Pisarski, F. Rennecke, *Phys. Rev. Lett.*, **127**, 152302 (2021), arXiv: 2103.06890]

[W.-j. Fu *et al.*, *Phys. Rev. D*,. **101**, 054032 (2020), arXiv: 1909.02991]

- Some of these models suffer from non-renormalizability and an inherent regulator dependence
- Vast majority of investigations in mean-field



[M. Thies, K. Urlichs, *Phys. Rev.*, **D67**, 125015 (2003), arXiv: hep-th/0302092]

[A. Koenigstein *et al.*, (2021), arXiv: 2112.07024]

Motivation

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- Long term goal: Investigate inhomogeneous phases in $3+1$-dimensional (Four-Fermi) models beyond mean-field via lattice Monte-Carlo simulations.

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- Long term goal: Investigate inhomogeneous phases in $3+1$-dimensional (Four-Fermi) models beyond mean-field via lattice Monte-Carlo simulations.
- But first: Are IPs a consistent feature when applying different regularization schemes in $3+1$-dimensional Four-Fermi models? Is the lattice a suitable regularization in these models?
- Start with stability analysis of the $3+1$-dimensional mean-field Nambu-Jona-Lasinio (NJL) model on the lattice

$3+1$-dimensional mean-field NJL model

$$S = \int \mathrm{d}^3 x \int_0^\beta \left\{ \bar{\psi}(\not{\partial} + \gamma_0 \mu + m_0)\psi + G\left[\left(\bar{\psi}\psi\right)^2 - \left(\bar{\psi}\gamma_5 \boldsymbol{\tau}\psi\right)^2 \right] \right\}$$

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↓ Bosonization, integration over fermions

$$S_{\mathrm{eff}} = \int \mathrm{d}^3x \int_0^\beta \mathrm{d}\tau \, \frac{\sigma^2 + \boldsymbol{\pi}^2}{4G} - \ln \mathrm{Det}(\not{\partial} + \gamma_0\mu + m_0 + \sigma + \mathrm{i}\gamma_5\,\boldsymbol{\tau}\cdot\boldsymbol{\pi})$$

$3+1$-dimensional mean-field NJL model

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- Mean-field approximation: only field configurations that minimize S_{eff} contribute
  $\Rightarrow$ path-integral is reduced to a minimization problem

$3+1$-dimensional mean-field NJL model

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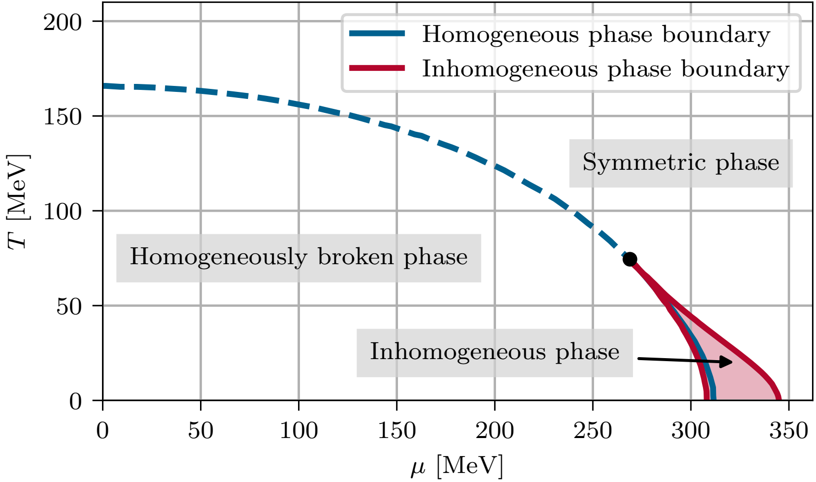
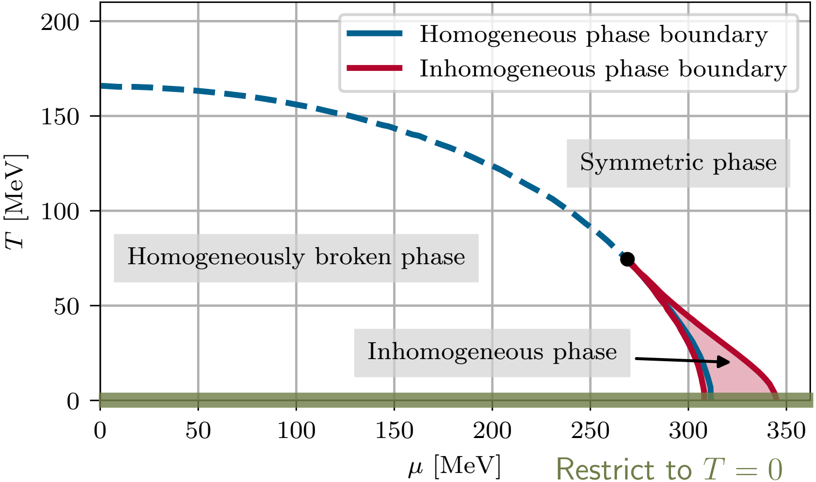
- 3 parameters: the coupling G, the bare fermion mass m_0 and the regulator Λ
- Parameters are tuned such that certain observable assume physically motivated values [S. P. Klevansky, *Rev. Mod. Phys.*, **64**, 649–708 (1992)]
  - Pion mass $m_\pi = 0$ (chiral limit) $\Rightarrow m_0 = 0$
  - Pion decay constant $f_\pi = 92.4\,\text{MeV}$
  - Constituent quark mass M_0 in the range of 200 MeV – 500 MeV
    $\Rightarrow$ will be used to control the value of the regulator

Phase diagram of the $3+1$-d mean-field NJL model



[D. Nickel, *Phys. Rev. D*,. **80**, 074025 (2009), arXiv: 0906.5295]

Phase diagram of the $3 + 1$-d mean-field NJL model



Restrict to $T = 0$

[D. Nickel, *Phys. Rev. D*,. **80**, 074025 (2009), arXiv: 0906.5295]

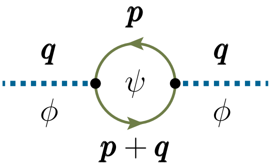
How to detect an inhomogeneous phase?

- How to detect an inhomogeneous phase in the $3+1$-dimensional NJL model?
- Two choices:
  - Use explicit ansatz for the chiral condensate / minimize on the lattice
    $\Rightarrow$ Difficult, sometimes impossible and often numerically expensive
  - Analyze stability of the homogeneous minimum
    $\Rightarrow$ Flexible and cheap $\Rightarrow$ better suited for our investigation

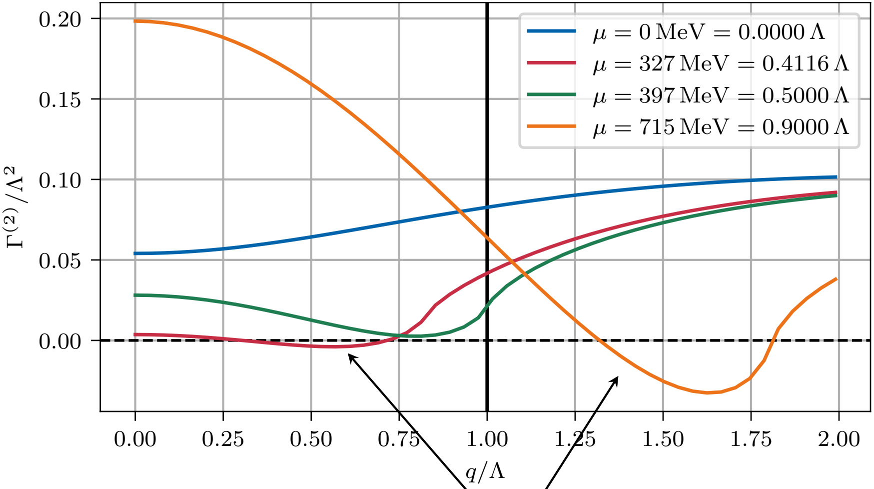
Stability analysis

- Investigate curvature of the effective action under inhomogeneous perturbations $\delta\tilde{\phi}(\boldsymbol{q})$ at the homogeneous minimum
- Curvature in direction $\delta\tilde{\phi}(\boldsymbol{q})$ is given by the bosonic two-point function $\Gamma_{\phi}^{(2)}(\boldsymbol{q})$
- Simple quantity in the mean-field approximation

$$\Gamma_{\phi}^{(2)}(\boldsymbol{q}) = \frac{1}{2G} +$$



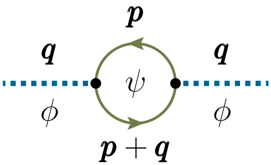
Example two-point function



signals instability towards
inhomogeneous condensate

Regularization of the loop integral

$$\Gamma^{(2)}_{\phi}(\boldsymbol{q}) = \frac{1}{2G} +$$



- The loop integral is UV-divergent and needs to be regularized
- Possible regularization scheme choices
  - Most common: Pauli-Villars

    e.g. [D. Nickel, *Phys. Rev. D*,. **80**, 074025 (2009), arXiv: 0906.5295] [M. Buballa *et al.*, *The Eur. Phys. J. Special Top.*, **229**, 3371–3385 (2020), arXiv: 2006.02133]
  - Momentum cutoff might have some conceptual problems
  - Successful applications of energy cutoffs and dim. regularization when using ansätze

    [P. Adhikari, J. O. Andersen, *Phys. Rev. D*,. **95**, 036009 (2017), arXiv: 1608.01097] [D. Ebert *et al.*, *Phys. Rev. D*,. **84**, 025004 (2011), arXiv: 1102.4079]
- How do lattice regularizations fit in?

Lattice regularizations

Investigate three different lattice regularizations

- Two variations of naive fermions
- SLAC fermions
  - Linear dispersion relation, thus conceptually very similar to a sharp cutoff in the continuum

Lattice regularizations

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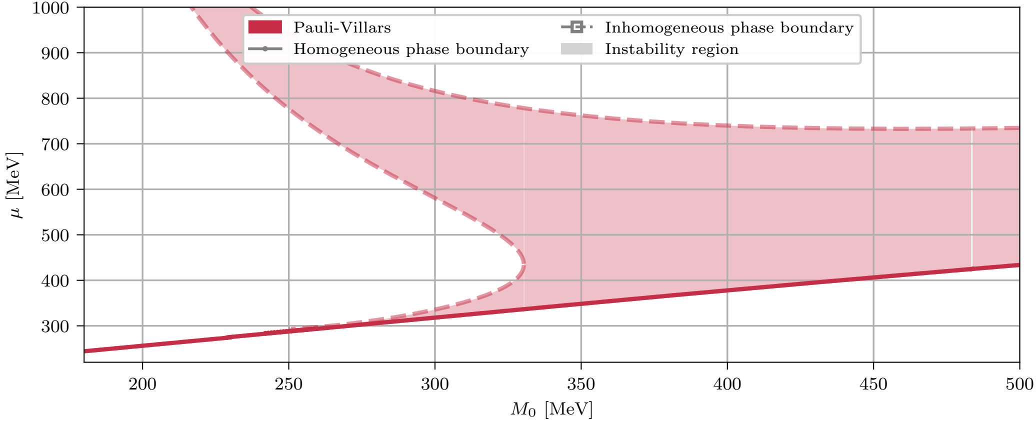
Investigate three different lattice regularizations

- Two variations of naive fermions
- SLAC fermions $\Leftarrow$ Will be the focus of this talk.
  - Linear dispersion relation, thus conceptually very similar to a sharp cutoff in the continuum

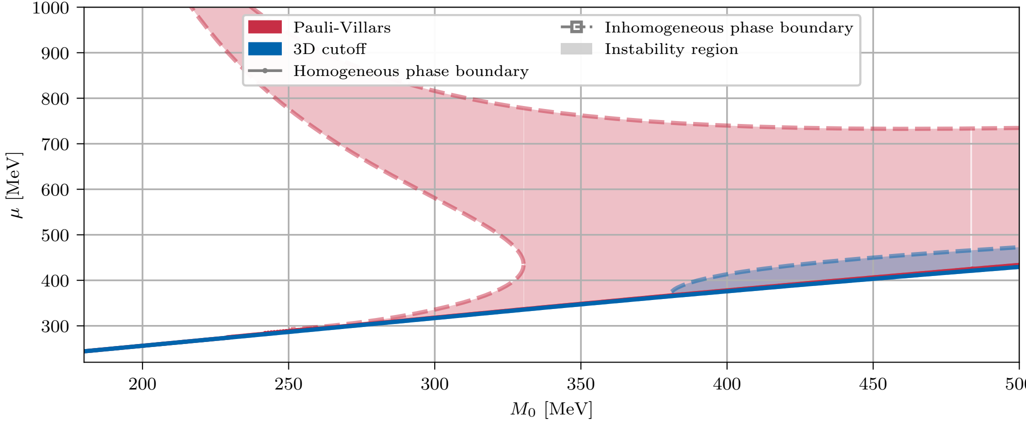
Results

Quark mass scan



- Instability detection restricted to $\mu \leq \Lambda$

Quark mass scan



Quark mass scan

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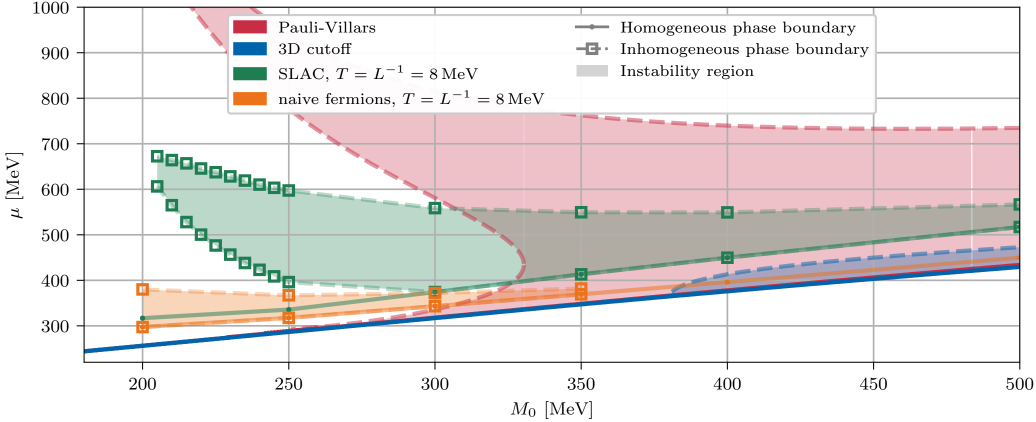
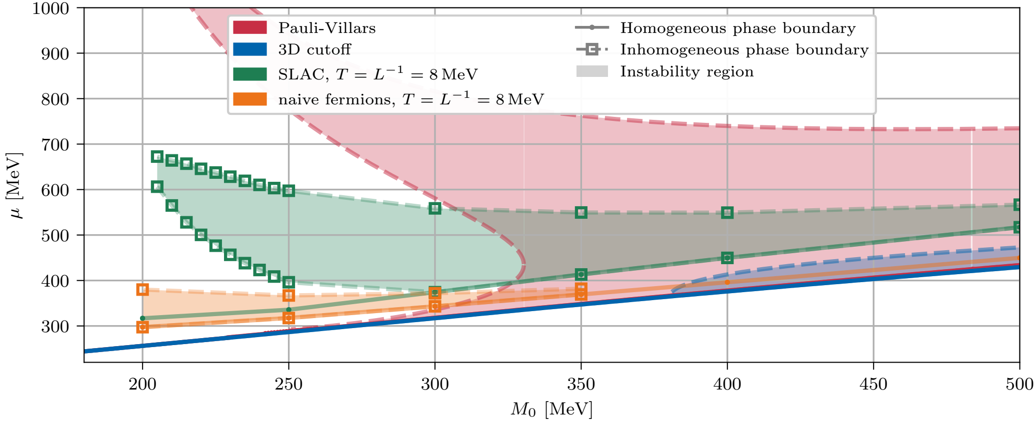


- The lattice results are calculated at $T = 8\,\mathrm{MeV}$ and $L^{-3} = (8\,\mathrm{MeV})^3$
- One of the naive discretizations does not exhibit an instability

Quark mass scan

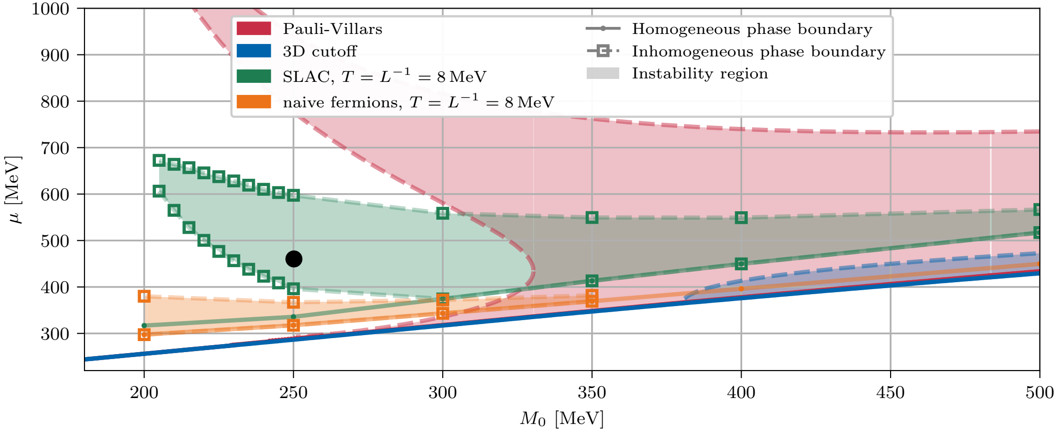


- No single point where all regularizations show an instability

Inhomogeneous field configurations

- Energetically preferred chiral condensate can be computed on the lattice
- Restrict to one-dimensional modulations due to cost

Inhomogeneous field configurations



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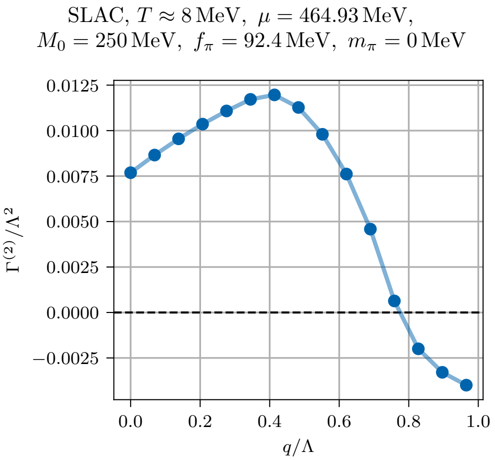
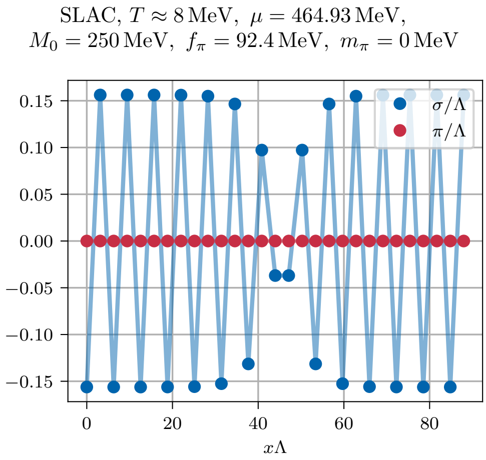
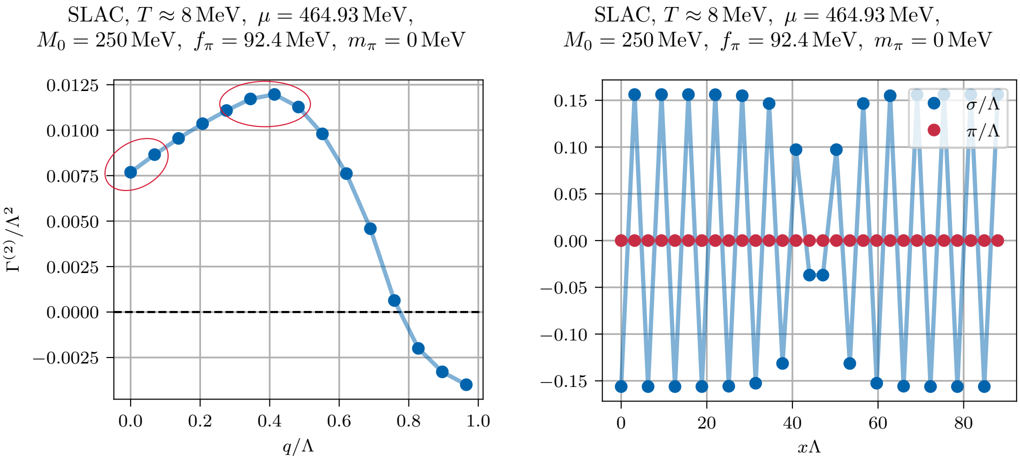




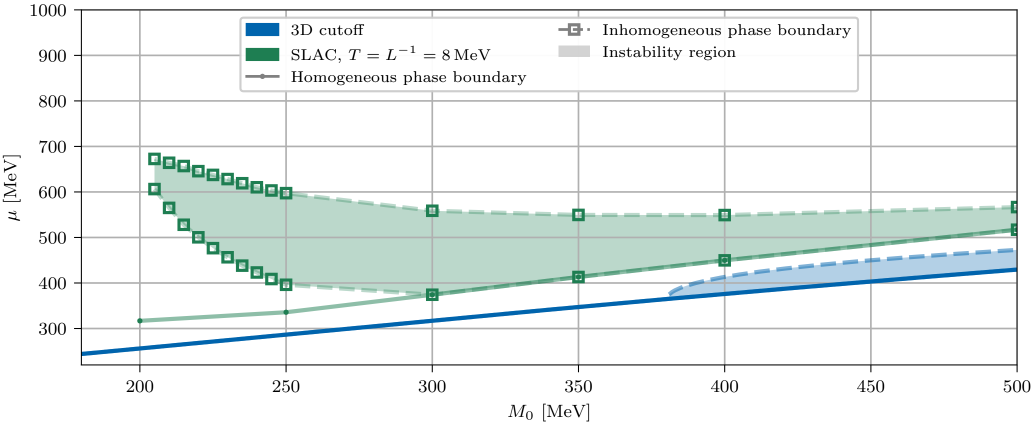
- Artifactual modulation of the condensate

Inhomogeneous field configurations



- Artifactual modulation of the condensate
- Suspicious and problematic features of two-point function

SLAC vs. 3D momentum cutoff



- Why does SLAC show such a different behavior than the momentum cutoff?

SLAC vs. 3D momentum cutoff

- Why does SLAC show such a different behavior than the momentum cutoff?
- Differences between the regularizations:
  - I The lattice results are at finite temperature and volume
  - II The SLAC fermions regulate the temporal direction – the momentum cutoff does not
  - III The cutoff region of SLAC fermions is cubic and that of the 3D momentum cutoff is spherical.
- Analyze influence of the differences

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Lattice in the infinite volume at $T = 0$

- In conventional lattice field theory infinite volume and $T = 0$ only via extrapolation
- In the stability analysis we only need to compute the 1-loop diagram via momentum sums/integrals
- continue momentum sums to integrals $\Rightarrow$ exact infinite volume and $T = 0$ on the lattice

SLAC in infinite volume vs. 3D momentum cutoff

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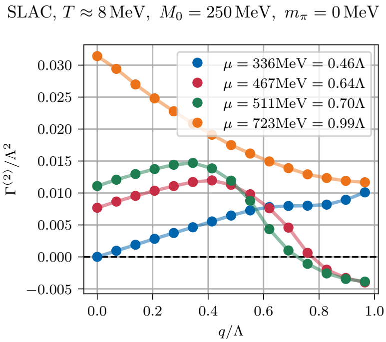
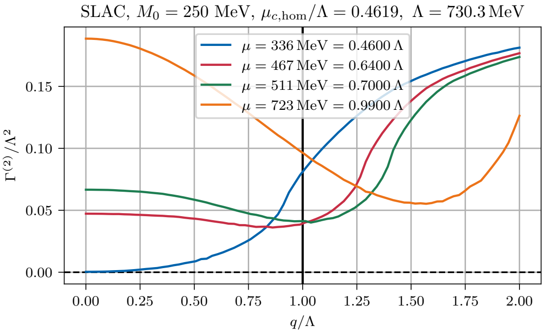
SLAC in finite volume



SLAC in infinite volume



SLAC in infinite volume vs. 3D momentum cutoff

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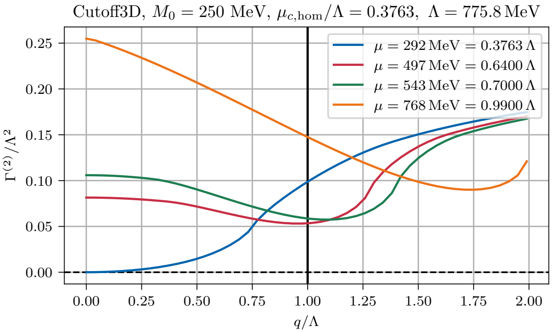
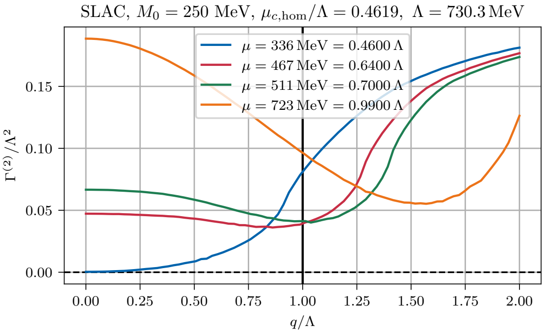
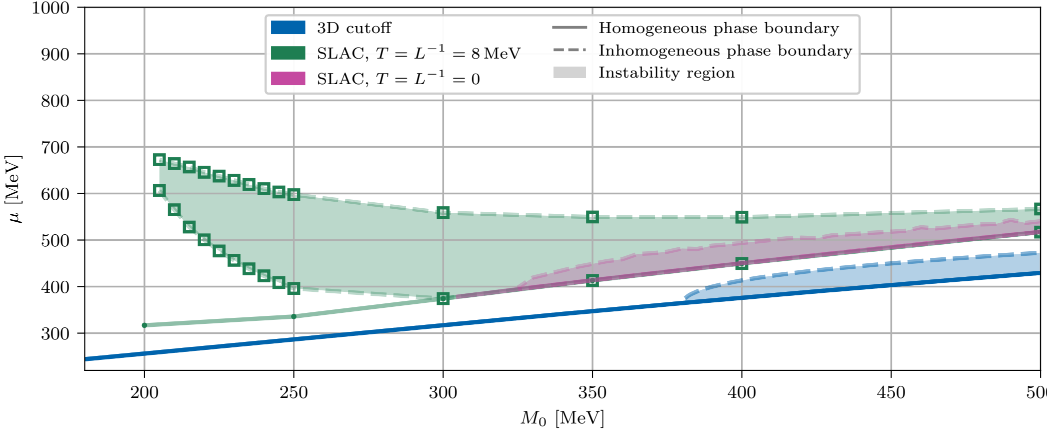
3D Cutoff



SLAC in infinite volume



SLAC vs. 3D momentum cutoff



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SLAC vs. 3D momentum cutoff

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  - I The lattice results are at finite temperature and volume
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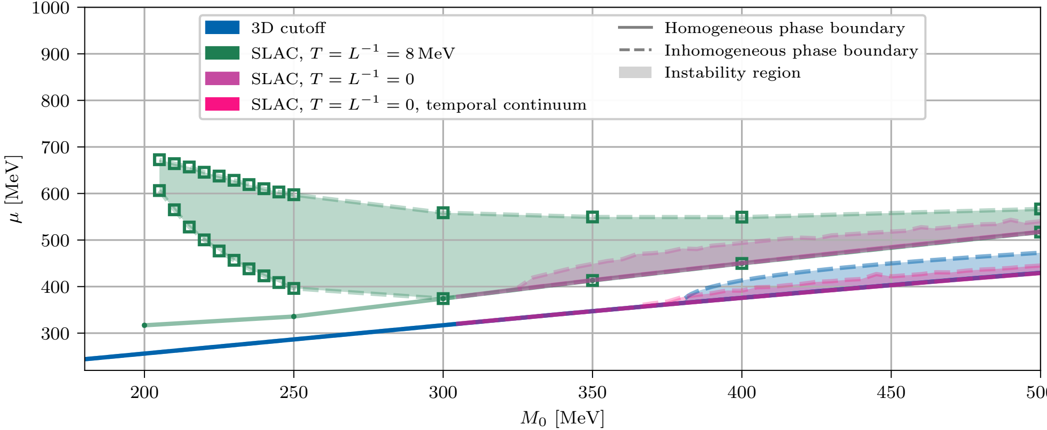
- Perform continuum limit in temporal direction

SLAC vs. 3D momentum cutoff



Conclusions and Outlook

Conclusions:

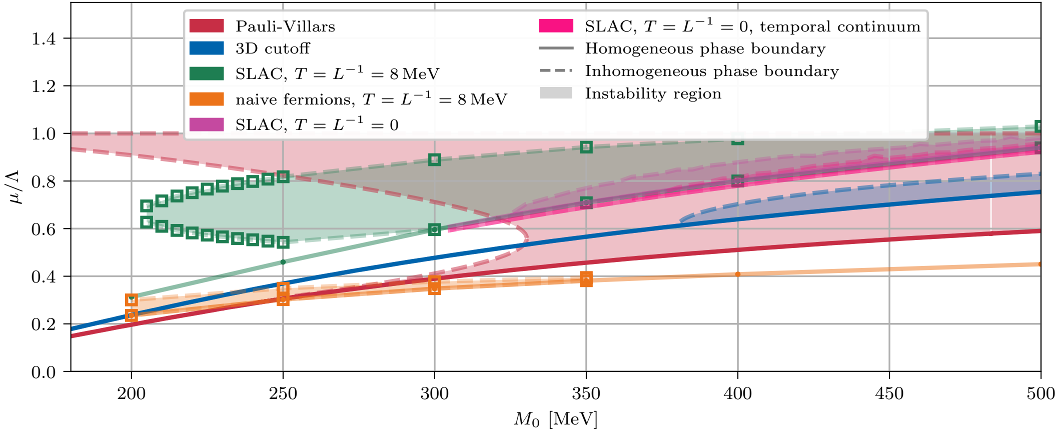
- No single point in $\mu - M_0$ plane where all regularizations show an instability
- A lattice investigation of inhomogeneous phases using SLAC fermions in the $3+1$-dimensional NJL model is ...
  - at best not straightforward and expensive
  - at worst not possible due to conceptual problems
- Some problems not discussed, e.g., baryon density saturation

Outlook:

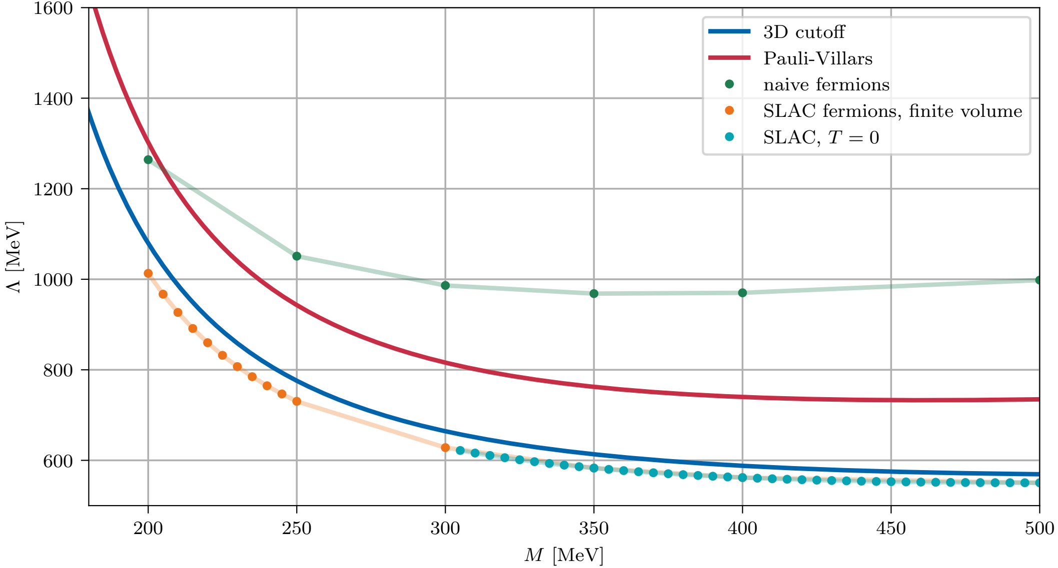
- Explore impact of RG consistency
- Understand limitations and problems of cutoff regularizations
- Redo infinite volume, $T = 0$ investigation with naive/staggered fermions

Appendix

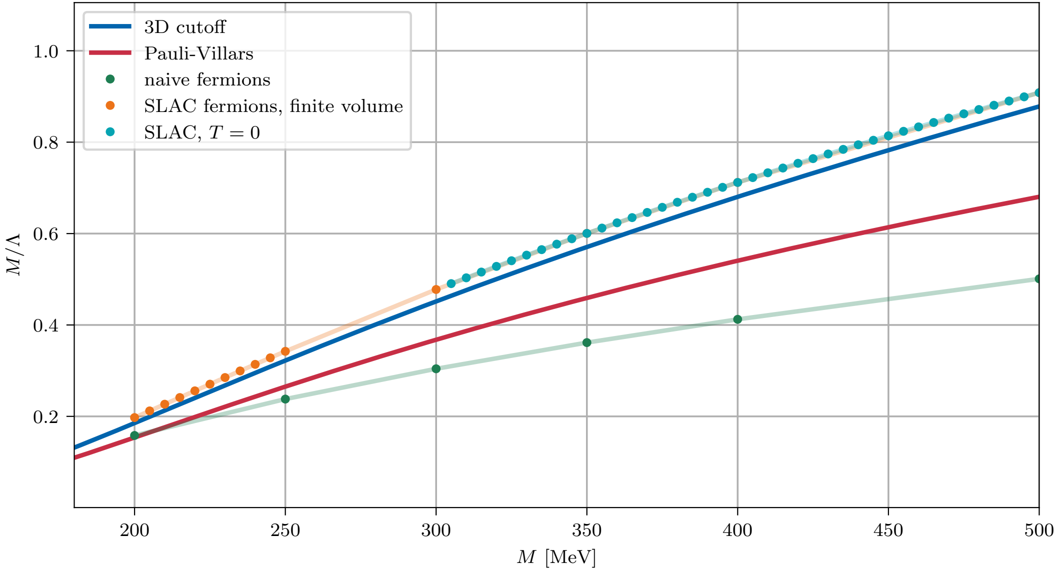
Quark mass scan - Cutoff units



Quark mass vs Cutoff



Quark mass vs Quark mass in Cutoff



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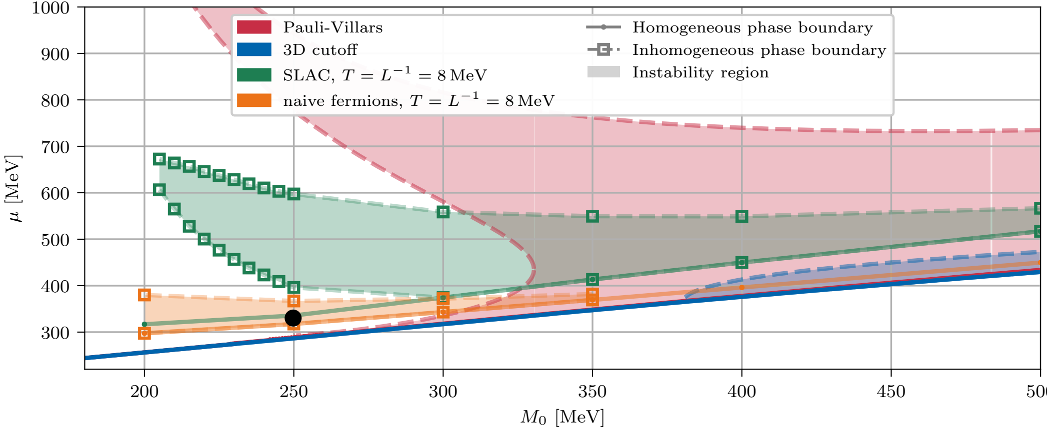

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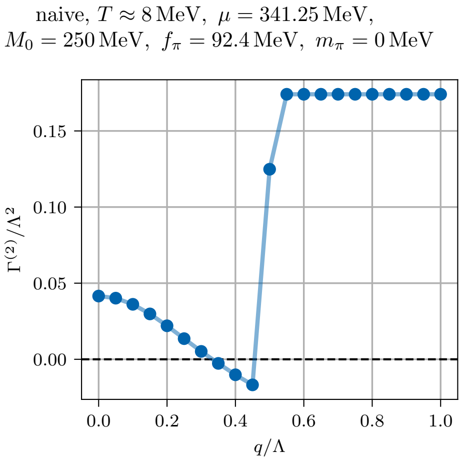
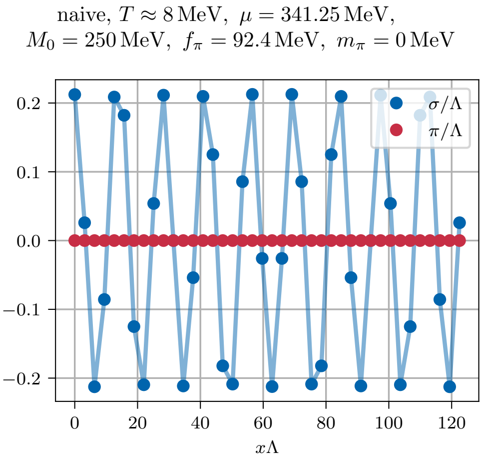


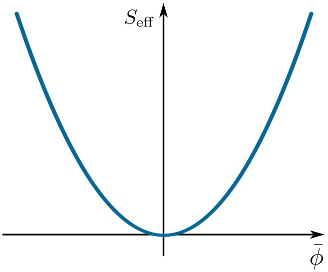


Stability analysis

- Homogeneous fields

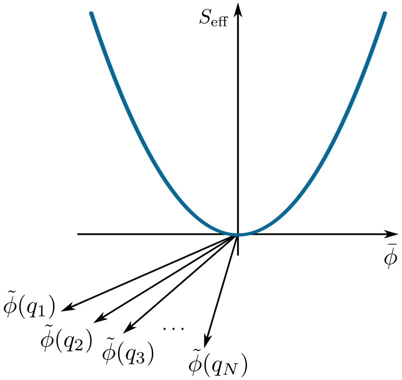
$$\phi(x) = \bar{\phi}$$

- Minimum is easy to obtain.



- In general fields have full space dependence

$$\phi(x) = \bar{\phi} + \phi_s(x) = \bar{\phi} + \sum\!\!\!\!\!\!\!\int_j \; \tilde{\phi}_s(q_j)\, \mathrm{e}^{\mathrm{i}xq_j}$$



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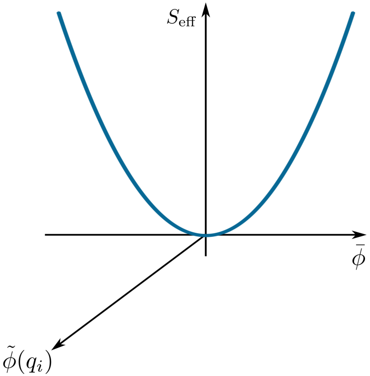

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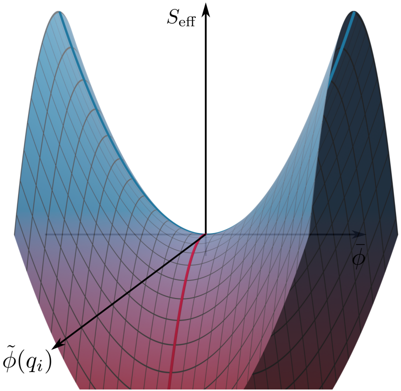
$$\begin{aligned}\phi(x) &= \bar{\phi} + \phi_s(x) \\ &= \bar{\phi} + \sum\!\!\!\!\!\!\int_j \tilde{\phi}_s(q_j)\, \mathrm{e}^{\mathrm{i}xq_j}\end{aligned}$$

- Former homogeneous minimum might only be saddle point
- Full dependence of S_{eff} on $\phi(x)$ extremely difficult or impossible



Stability analysis

- Consider only inhomogeneous perturbations

$$\phi(x) = \bar{\phi} + \delta\phi_s(x)$$
$$= \bar{\phi} + \sum\!\!\!\!\!\!\int_j \delta\tilde{\phi}_s(q_j)\, \mathrm{e}^{\mathrm{i}xq_j}$$

- investigate curvature at homogeneous minimum