Soft scales with HTL in high-density perturbation theory

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outline

1. Overview and HTLs
2. Organising the pressure
3. Soft Results in QCD
4. Mixed Results in QED

Starting point and assumptions

pQCD computations at finite density μ in equilibrium start from the pressure

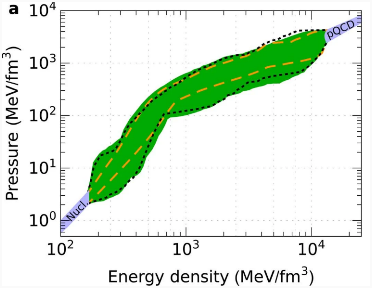
$$\Omega \sim \ln Z = \ln \int \mathcal{D}\overline{\psi}\psi\bar{c}cAe^{-S_{\mathrm{QCD}}}$$

sum of connected graphs $\rightarrow$ small-g_s expansion

- T and masses m_f ignored unless otherwise stated
- Pressure in Euclidean space, self-energy in Minkowski

Motivation from NSs and theory

Applications in neutron stars [eg. Annala et al., Nat Phys 16 908–910 (2020), PRL 120 172703 (2018), cf. also AV's plenary on Thursday and TG's & Oleg Komoltsev's talk on Wednesday] ...

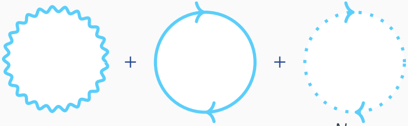


... but also interesting theory:

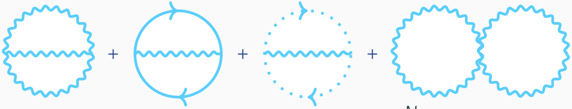
- First-principles access to μ-axis of the phase diagram
- No fundamental obstacles to perturbation theory
- (and lots of neat loop integrals)

First orders: Current pressure $\Omega = \Omega_0 + \Omega_1 g_s^2$

Let's start computing right away! LO and NLO textbook-level calculations, here at $d = 3$:



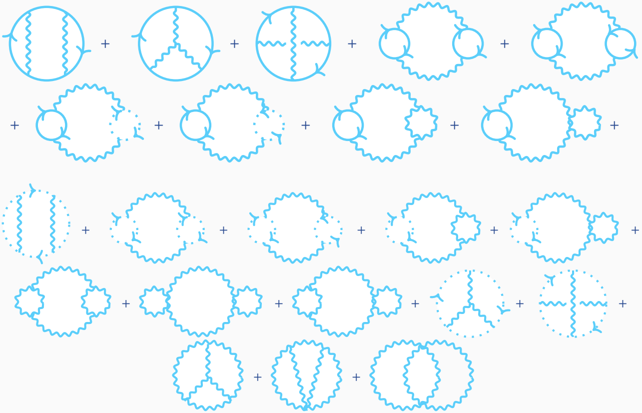
$$\Omega_0 = -\frac{\pi^2}{45}\left(d_A + \frac{7}{4}N_c N_f\right)T^4 - \frac{N_c}{12\pi^2}\sum_{i=1}^{N_f}\left(\mu_i^2 + 2T^2\right)\mu_i^2$$



$$g_s^2\Omega_1 = \frac{g_s^2}{144}d_A\left[\left(N_c + \frac{5}{4}N_f\right)T^4 + \frac{9}{4\pi^4}\sum_{i=1}^{N_f}\left(\mu_i^2 + 2\pi^2 T^2\right)\mu_i^2\right]$$

Three-loop diagrams: $\Omega = \Omega_0 + \Omega_1 g_s^2 + \Omega_{2,0} g_s^4 + 1/\varepsilon$!

These don't converge very well. Need at least NNLO, which a priori means "simply" computing these diagrams:



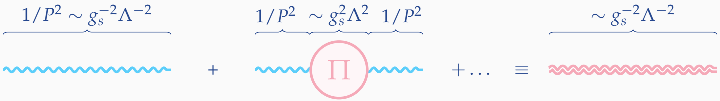
Post-renormalisation leftover divergences! (one graph at $T = 0$)

Soft gluons in a medium

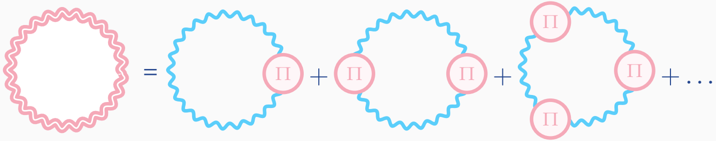
Gluon propagation is modified by in-medium constituents

Soft gluons with $P \sim g_s\mu$, (or T) have LO self-energies $\Pi \sim g_s^2\Lambda^2$

$\rightarrow$ need arbitrary many Π-insertions for g_s-expansion



$\therefore$ soft gluons must be resummed (enough at $T = 0$)



Suppressed when gluons are hard (and therefore IR-safe)

$\rightarrow$ Assume the loop momenta in self-energies are large compared to external gluon: A Hard Thermal Loop

Extremely dense overview of equally dense HTL

HTL is considerably simpler than full kinematics:

$$G^{\mu\nu}_{\mathrm{HTL}} = \frac{\mathbb{P}^{\mu\nu}_T}{P^2 + \Pi^{\mathrm{HTL}}_T} + \frac{\mathbb{P}^{\mu\nu}_L}{P^2 + \Pi^{\mathrm{HTL}}_L} + \xi \frac{P^\mu P^\nu}{P^4};$$

$$\Pi^{\mathrm{HTL}}_L \overset{d=3}{=} m_E^2 \frac{P^2}{p^2}\left(1 + \frac{ip_0}{2p} \ln\left(\frac{p_0 + ip}{p_0 - ip}\right)\right),$$

$$(d-1)\Pi^{\mathrm{HTL}}_T + \Pi^{\mathrm{HTL}}_L = m_E^2; \quad m_E^2 \overset{d=3}{=} g^2\left[N_f \frac{\mu^2}{2\pi^2} + (2N_c + N_f)\frac{T^2}{6}\right];$$

Adequately addresses IR-problems at finite μ. If applied to only soft contributions, beyond-HTL corrections are suppressed with powers of $g_s \rightarrow$ still pQCD without extra assumptions

For consistent EFT need also HTL vertices, no closed form

Cancelling divergences: $\Omega = \Omega_0 + \Omega_1 g_s^2 + \Omega_{2,1} g_s^4 \ln g_s + \Omega_{2,0} g_s^4$

First soft contribution at NNLO from the ring sum Ω_2^s

$$g_s^4 \Omega_2^s =$$

$$= -\frac{1}{2} \int_P \sum_{n\in\mathbb{N}_+} \frac{(-1)^n}{n} \mathrm{Tr}\left[\Pi^{\mathrm{HTL}} G_0\right]^n$$

$$\approx -\frac{d_A m_E^4}{(8\pi)^2} \left(\frac{m_E}{\bar{\Lambda}}\right)^{-2\varepsilon} \left[\frac{1}{2\varepsilon} + 1.17201\right]$$

By design: UV-divs of HTL cancel IR-divs of naïve theory

As a consequence: $\ln^k g_s$-terms in pressure, completely determined by one side of divergences

$$g_s^4 \Omega_2^s + g_s^4 \Omega_2^h = -\frac{d_A m_E^4}{(8\pi)^2} \left[\left(\frac{m_E}{\bar{\Lambda}}\right)^{-2\varepsilon} \frac{1}{2\varepsilon} - \left(\frac{\mu}{\bar{\Lambda}}\right)^{-2\varepsilon} \frac{1}{2\varepsilon}\right] + O(1)$$

$$= -\frac{d_A m_E^4}{(8\pi)^2} \ln \frac{m_E}{\mu} + O(g_s^4) = \Omega_{2,1} g_s^4 \ln g_s + O(g_s^4)$$

Organising the pressure into sectors

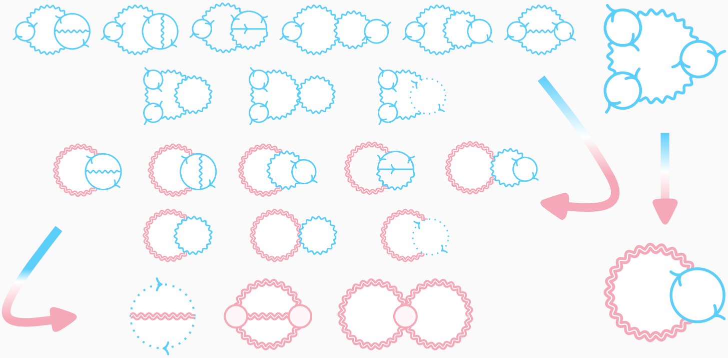
Since NNLO worked, let's see if we can go to N3LO. Now, need to organise soft contributions into different sectors based on the number of soft loop momenta

- Soft sector: Two soft momenta, HTL-resummed two-loop diagrams
- Mixed sector: One soft momenta, rings with higher-order HTL self-energies
- Hard sector: No soft momenta, four-loop unresummed diagrams

Introduced in PRD 104, 074015 (2021) with T. Gorda, A. Kurkela, R. Paatelainen, A. Vuorinen.

Matching divergences

Move between sectors by making gluon lines softer
(and HTL-resumming them in the process)



Shows which groups of graphs have mutually cancelling divergences, each naïve massless momentum ∼ (at most) one IR-divergence — also, handy for checking symmetry factors

$p_{\rm soft}^{\rm N3LO}$: $\Omega = \Omega_0 + \Omega_1 g_s^2 + \Omega_{2,1} g_s^4 \ln g_s + \Omega_{2,0} g_s^4 + \Omega_{3,2} g_s^6 \ln^2 g_s + \Omega_{3,1}^s g_s^6 \ln g_s + \Omega_{3,0}^s g_s^6$

N3LO soft sector graphically two-loop HTL pressure [HTLpt groundwork by Andersen et al., eg. PRD 66 085016 (2002)]. Computing without extra assumptions involves lots of work (analytic and numeric), but in the end

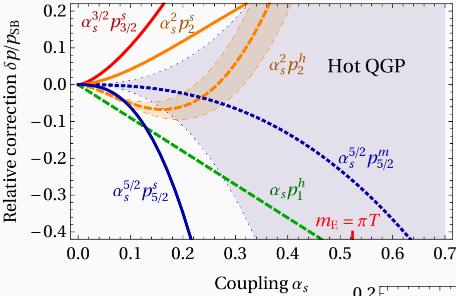
$$\Omega_3^s \approx -g_s^2 N_c \frac{d_A m_E^4}{4(4\pi)^3} \left(\frac{m_E}{\Lambda_h}\right)^{-4\varepsilon} \left[\frac{11/6\pi}{(2\varepsilon)^2} + \frac{1.50731(19)}{2\varepsilon} + 2.2125(9)\right]$$

Two soft lines means at most two divergences. Leading divergence correctly postdicts the double log

$$\Omega_{3,2} = -\frac{11}{48\pi} g_s^2 N_c \frac{d_A m_E^4}{(4\pi)^3} \ln^2 g_s$$

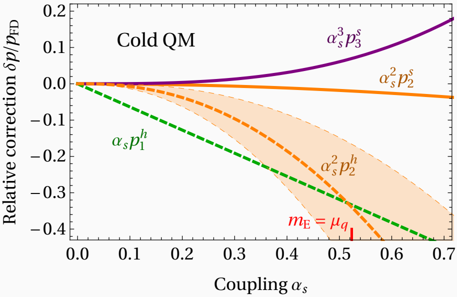
Computed in PRD 104, 074015 and PRL 127 162003 (2021) with T. Gorda, A. Kurkela, R. Paatelainen, A. Vuorinen [double log already in 2018 with cutoffs]

No soft breakdown at N3LO!



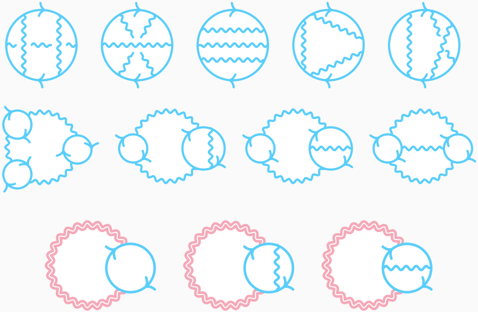
At large T, breakdown of pQCD conjectured to be due to soft (EFT) contributions flailing about

$T = 0$-pressure is much calmer $\rightarrow$ standard perturbation theory seems to be more sensible!



Simplifying things with QED

Again, soft sector worked, let's proceed to mixed sector. Need Π to $O(g^4, K^0)$ and $O(g^2, K^2)$. Simpler starting point: N_f-flavour QED: no vertices, vanishing soft sector, fewer diagrams.



Second and third lines in correspondance, focus on third.

Real-time formalism simplifies HTL-expansion, rr-propagators set loop momenta on shell.

Two-loop HTL ...

Two-loop diagrams generalise Carignano et al. PLB 801, 135193 (2020) at the HTL limit to finite density [$O(g^4, K^0)$-term]

$$\Pi^{\mathrm{NLO}}_{T,R,\mathrm{QED}} = -\frac{e^4}{8\pi^2}\left(T^2 + \frac{\mu^2}{\pi^2}\right)\frac{k_0}{2k}\ln\frac{k_0+k}{k^0-k}$$

$$\Pi^{\mathrm{NLO}}_{L,R,\mathrm{QED}} = -\frac{e^4}{8\pi^2}\left(T^2 + \frac{\mu^2}{\pi^2}\right)$$

$$-\frac{e^4}{4\pi^2}\frac{\mu^2}{\pi^2}\left(1-\frac{k_0^2}{k^2}\right)\left(1-\frac{k_0}{2k}\ln\frac{k_0+k}{k_0-k}\right)^2$$

Could (and needed to) do also in general d, not shown here.

Recently computed in 2204.11279 with T. Gorda, A. Kurkela, R. Paatelainen, K. Seppänen, P. Schicho, A. Vuorinen, J. Österman along with...

… and power-corrections

… similar power corrections $[O(g^2, K^2)$-term], which generalises Carignano et al. PLB 780, 308 (2018).

$$\Pi^{\mathrm{Pow}}_{T,R,\mathrm{QED}} = \frac{e^2}{4\pi^2}\frac{2K^2}{3}\left[-\frac{1}{2\varepsilon} - \frac{3}{4} + \left(1-\frac{K^2}{4k^2}\right)\left(1-\frac{k_0}{2k}\ln\frac{k_0+k}{k_0-k}\right)\right.$$
$$\left. - \mathrm{Li}_0^{(1)}\left(-e^{\frac{\mu}{T}}\right) - \mathrm{Li}_0^{(1)}\left(-e^{-\frac{\mu}{T}}\right) + \ln\frac{2e^{-\gamma_E}T}{\bar{\Lambda}}\right]$$

$$\Pi^{\mathrm{Pow}}_{L,R,\mathrm{QED}} = \frac{e^2}{4\pi^2}\frac{2K^2}{3}\left[-\frac{1}{2\varepsilon} - 1 + \left(1-\frac{K^2}{2k^2}\right)\left(1-\frac{k_0}{2k}\ln\frac{k_0+k}{k_0-k}\right)\right.$$
$$\left. - \mathrm{Li}_0^{(1)}\left(-e^{\frac{\mu}{T}}\right) - \mathrm{Li}_0^{(1)}\left(-e^{-\frac{\mu}{T}}\right) + \ln\frac{2e^{-\gamma_E}T}{\bar{\Lambda}}\right]$$

UV-divergence of vacuum Π shows up here, as does a funny nontrivial μ, T-independent matter part (exclusive to the $O[K^2]$ term in $d=3$)

$p^{\mathrm{N3LO}}_{\mathrm{soft,QED}}$: $\Omega_{\mathrm{QED}} = \Omega_0 + \Omega_1 g_s^2 + \Omega_{2,1} g_s^4 \ln g_s + \Omega_{2,0} g_s^4 + \Omega^m_{3,1} g_s^6 \ln g_s + \Omega^m_{3,0} g_s^6$

Use Πs to get QED mixed sector: 2204.11893 (same coauthors)

$$\Omega^m_{3,\mathrm{QED}} = -\frac{1}{2}\int_K \mathrm{Tr}\left[G_{\mathrm{HTL}}(K)\Pi^{\mathrm{Pow}}(K) + G_{\mathrm{HTL}}(K)\Pi^{\mathrm{NLO}}(K)\right]$$

$$\approx \frac{N_f e^2 m_E^{4-2\varepsilon}}{6(4\pi)^2\overline{\Lambda}^{2\varepsilon}}\left\{\frac{1}{2\varepsilon}\left[5 - \frac{66}{N_f} - \frac{\pi^2}{12}\left(7 - \frac{60}{N_f}\right) - 8\ln\left(\frac{\overline{\Lambda}}{2\mu}\right)\right]\right.$$

$$+ 1.05960 - \frac{1.03093}{N_f} + [1-\ln 2]\left[5 - \frac{66}{N_f} - \frac{\pi^2}{12}\left(7 - \frac{60}{N_f}\right)\right]$$

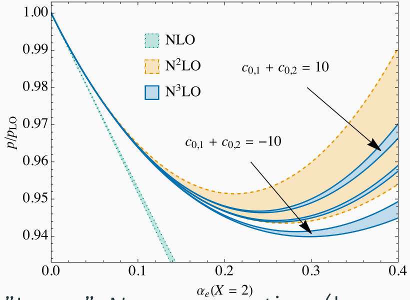
$$\left. + \left(13 - \frac{23\pi^2}{12} + \frac{20}{3}\ln 2 - \frac{32}{3}\ln^2 2 + 2\delta\right)\ln\left(\frac{\overline{\Lambda}}{2\mu}\right) - 4\ln^2\left(\frac{\overline{\Lambda}}{2\mu}\right)\right\}$$

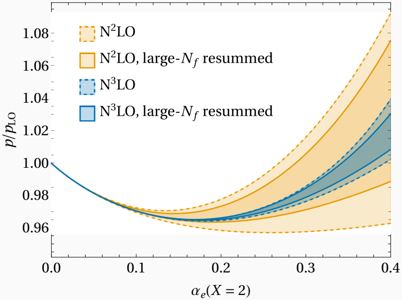
Gives $e^6 \ln e$, excellent accuracy (cf soft pressure)

In QED, could even explicitly separate IR-divergences of naïve diagrams (and get complete leading-N_f behaviour at N3LO as a fun bonus): Predicted cancellation!

Reducing uncertainty bands with the mixed sector

Hard N3LO coefficient unknown, but excellent reduction of the uncertainty bands for any even remotely sensible value (extremal $c_{\mathrm{hard}} = \pm 10$).

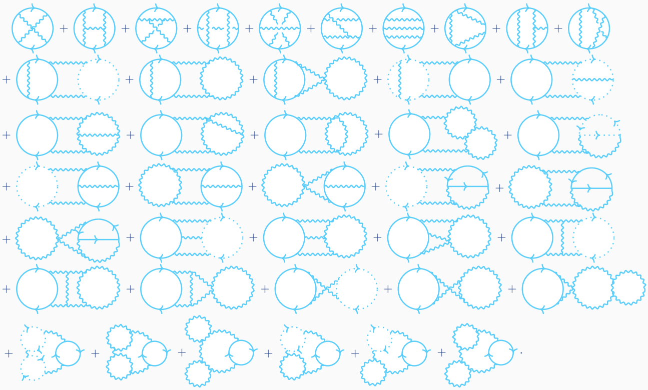




"Large"-N_f resummation (here $N_f = 3$) further shrinks bands — improved large-N_f coefficients of Ipp & Rebhan [JHEP 06 032 (2003)] from 3.18(5) and 3.4(3) to $(60 - 7\pi^2 + 96 \ln 2)/18$ and 3.36388(4).

The future looks bright!

Good progress in the QCD generalisation of the mixed sector and two-loop Π! Very preliminary results seem superficially similar to Abelian case, but it takes effort to get there.
maybe one day the IR-safe hard diagrams...



Summary

- HTL lets you do cold and dense pQCD without extra assumptions to seemingly any order
- Pressure naturally splits into sectors, N3LO soft sector $(\Omega^s_{3,2}, \Omega^s_{3,1}, \Omega^s_{3,0})$ is done (no sign of practical breakdown!)
- Two-loop photon self-energy and mixed sector $(\Omega^m_{3,1}, \Omega^m_{3,0})$ done in QED, QCD to follow (hopefully soonish).

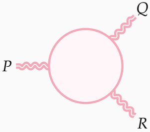
$$
\begin{aligned}
\Omega_{\mathrm{QCD}} \overset{T=0}{=} \; & \Omega_0 + \Omega_1 g_s^2 + \Omega_{2,1} g_s^4 \ln g_s + \Omega_{2,0} g_s^4 \\
& + \Omega^s_{3,2} g_s^6 \ln^2 g_s + (\Omega^s_{3,1} + \Omega^m_{3,1}) g_s^6 \ln g_s \\
& + (\Omega^s_{3,0} + \Omega^m_{3,0} + \Omega^h_{3,0}) g_s^6
\end{aligned}
$$

Bonus: Soft masses

More practical recent project: Soft quark masses. Massive integrals are tricky and always numerical, came up with a scheme for approximating them when masses are soft with T. Gorda. Very little loss of accuracy at $T, \mu > 0$. Now officially out in PRD 105 114005 (2022)!

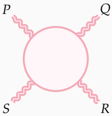
Using this to look at bulk quantities relevant for mergers and comparing with holographists (with T. Gorda, C. Hoyos, N. Jokela, A. Kurkela, R. Paatelainen, J. Remes, and A. Vuorinen)

HTL vertices [$V = (-i, v)$]:



$$= \delta\Gamma^{\alpha\beta\gamma}_{a,b,c}(P, Q, R)$$

$$= i g_s m_E^2 f_{abc} \int_{S^{d-1}} \Omega_v V^\alpha V^\beta V^\gamma \left(\frac{iQ_0}{P\cdot V Q\cdot V} - \frac{iR_0}{P\cdot V R\cdot V} \right)$$



$$\to \delta\Gamma^{\alpha\beta\gamma\delta}_{abcb}(P, Q, -P, -Q)$$

$$= -2 g_s^2 m_E^2 f_{eba} f^{eb}{}_c \int_{S^{d-1}} \Omega_v \frac{V^\alpha V^\beta V^\gamma V^\delta}{(P+Q)\cdot V (P-Q)\cdot V} \left(\frac{iQ_0}{Q\cdot V} - \frac{iP_0}{P\cdot V} \right)$$

...and $5, 6, \ldots$-point vertices at higher orders

No closed form, but symmetries and Wards help with computations

pressure

$$\Omega_{\rm QCD} = \Omega_0 + \Omega_1 g_s^2 + \Omega_{3/2} g_s^3 + \Omega_{2,1} g_s^4 \ln g_s + \Omega_{2,0} g_s^4 + \Omega_{5/2} g_s^5 + \Omega_{3,2} g_s^6 \ln^2 g_s + \Omega_{3,1} g_s^6 \ln g_s + \Omega_{3,0} g_s^6$$

Naïve loop expansion gives only even powers:

- $T > 0$: 3d EFTs (EQCD and MQCD) address Bose-enhanced thermal gluons. $\Omega_{3,1}$ known, $\Omega_{3,0}$ nonperturbative.
- $\mu > 0$: HTL addresses soft gluons. Up to $\Omega_{3,2}$ fully known [PRL 121 202701 (2018)]
- $\ln g_s$-enhancements from both $T > 0$ and $\mu > 0$; $\Omega_{2,0}$ and $\Omega_{3,0}$ have EFT and naïve contributions

This talk: Recent progress towards full N3LO using HTL