Exact thermal equilibrium distributions for massive and massless fermions with rotation and acceleration

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Based on:

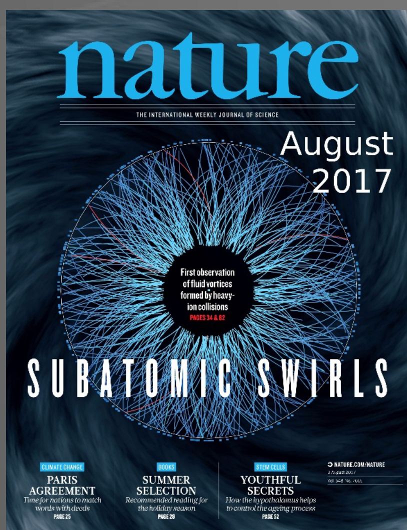
F. Becattini, MB, A. Palermo, JHEP 02 (2021) 101, arXiv:2007.08249
A. Palermo, MB, F. Becattini, JHEP 10 (2021) 077, arXiv:2106.08340

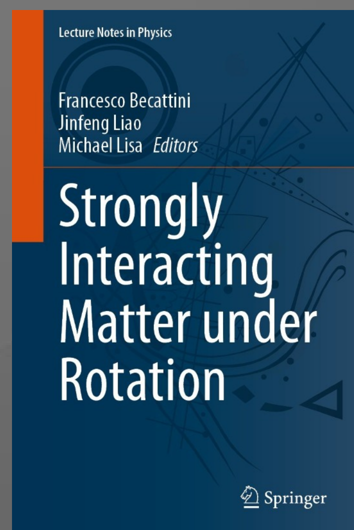
Motivations

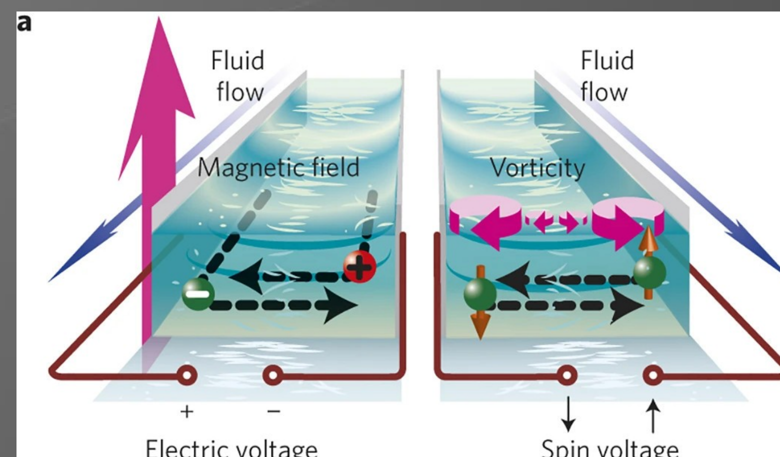
Equilibrium quantum phenomenon: (thermal) vorticity in a fluid induces spin polarization.

Spin polarization measurements in heavy-ion collisions has opened the possibility for new phenomenological investigations of spin physics in relativistic fluids.

It is a rapidily evolving field both for theory and experiments.







Takahashi et al, Nature Physics 12, 52–56 (2016)

Exact thermal properties with vorticity for free fields where not known.

Find all quantum corrections in relativistic fluids at global equilibrium.

Exact **distribution functions** and **spin vector** at global equilibrium!

Other approaches

- QFT in rotating medium

  A. Vilenkin, Phys. Rev. D 21 (1980) 2260
- Solution of Dirac Eq. in rotating coordinates

  V.E. Ambrus and E. Winstanley, Phys. Lett. B 734 (2014) 296

  V.E. Ambrus and E. Winstanley, Lect. Notes Phys. 987 (2021)

  V.E. Ambrus, Ph.D. Thesis
- Conical singularity

  G.Y. Prokhorov and V.I. Zakharov, Phys. Rev. D 100 (2019) 12, 125009

  G.Y. Prokhorov, O.V. Teryaev and V.I. Zakharov, JHEP 03 (2020) 137

Global equilibrium

Density operator at global equilibrium:

$$\widehat{\rho} = \frac{1}{Z} \exp\left[-b_\mu \widehat{P}^\mu + \frac{1}{2} \varpi_{\mu\nu} \widehat{J}^{\mu\nu} \right] \qquad \langle \widehat{O} \rangle = \mathrm{Tr}\left[\widehat{\rho}\, \widehat{O} \right]$$

The four-temperature vector is a Killing vector. The thermal vorticity ϖ is a constant antisymmetric tensor:

$$\beta^\mu(x) = \frac{u^\mu}{T} = b^\mu + \varpi^{\mu\nu} x_\nu$$

At global equilibrium:

$$\frac{A^\mu}{T} = \varpi^{\mu\nu} u_\nu$$

Acceleration

$$\frac{\omega^\mu}{T} = -\frac{1}{2} \epsilon^{\mu\nu\rho\sigma} \varpi_{\nu\rho} u_\sigma$$

Angular velocity

Factorization of statistical operator

The generators of the Poincaré group appear in the density operator!
Analytic continuation of the thermal vorticity: $\varpi \mapsto -i\phi$

$$\widehat{\rho} = \frac{1}{Z} \exp\left[-b_\mu \widehat{P}^\mu - \frac{i}{2}\phi_{\mu\nu}\widehat{J}^{\mu\nu}\right]$$

P $\mapsto$ translations
J $\mapsto$ Lorentz transformations

Factorization of the density operator:

$$\widehat{\rho} = \frac{1}{Z} \exp\left[-\widetilde{b}_\mu(\phi)\widehat{P}^\mu\right] \exp\left[-i\frac{\phi_{\mu\nu}}{2}\widehat{J}^{\mu\nu}\right] \equiv \frac{1}{Z} \exp\left[-\widetilde{b}_\mu(\phi)\widehat{P}^\mu\right]\widehat{\Lambda}$$

$$\widetilde{b}^\mu(\varpi) = \sum_{k=0}^{\infty} \frac{1}{(k+1)!} \underbrace{(\phi^\mu{}_{\alpha_1}\phi^{\alpha_1}{}_{\alpha_2}\dots\phi^{\alpha_{k-1}}{}_{\alpha_k})}_{k \text{ times}} b^{\alpha_k}$$

We can use group theory to calculate thermal expectation values!

Any thermal expectation value in a free quantum field theory is obtained from:

$$\langle \widehat{a}_s^\dagger(p)\widehat{a}_t(p')\rangle = \frac{1}{Z}\mathrm{tr}\left[\mathrm{e}^{-\tilde{b}_\mu(\phi)\widehat{P}^\mu}\widehat{\Lambda}\,\widehat{a}_s^\dagger(p)\widehat{a}_t(p')\right]$$

$$[\widehat{a}_s^\dagger(p), \widehat{a}_t(p')]_\pm = 2\varepsilon\delta^3(p-p')\delta_{st}$$

Using Poincaré transformation rules and (anti)commutation relations (particle with spin S):

$$\widehat{\Lambda}\,\widehat{a}_s^\dagger(p)\,\widehat{\Lambda}^\dagger = \sum_r D^S(W(\Lambda,p))_{rs}\widehat{a}_r^\dagger(\Lambda p)$$

We obtain

$$\langle \widehat{a}_s^\dagger(p)\widehat{a}_t(p')\rangle = (-1)^{2S}\sum_r D^S(W(\Lambda,p))_{rs}\mathrm{e}^{-\widetilde{b}\cdot\Lambda p}\langle \widehat{a}_r^\dagger(\Lambda p)\widehat{a}_t(p')\rangle +$$

$$+2\varepsilon\,\mathrm{e}^{-\widetilde{b}\cdot\Lambda p}D^S(W(\Lambda,p))_{ts}\delta^3(\Lambda p-p')$$

$\mathrm{D(W)}$ is the “Wigner rotation” in the S-spin representation.

Iterative solution

We find a solution by iteration:

I
$$\langle \widehat{a}_s^\dagger(p)\widehat{a}_t(p')\rangle \sim 2\varepsilon\, \mathrm{e}^{-\widetilde{b}\cdot\Lambda p} D^S(W(\Lambda,p))_{ts}\delta^3(\Lambda p - p')$$

II
$$\langle \widehat{a}_s^\dagger(p)\widehat{a}_t(p')\rangle \sim 2\varepsilon\, (-1)^{2S} D^S(W(\Lambda^2,p))_{ts}\mathrm{e}^{-\widetilde{b}\cdot(\Lambda p+\Lambda^2 p)}\delta^3(\Lambda^2 p - p') + \\ +2\varepsilon\, \mathrm{e}^{-\widetilde{b}\cdot\Lambda p} D^S(W(\Lambda,p))_{ts}\delta^3(\Lambda p - p')$$

∞
$$\langle \widehat{a}_s^\dagger(p)\widehat{a}_t(p')\rangle = 2\varepsilon' \sum_{n=1}^{\infty}(-1)^{2S(n+1)}\delta^3(\Lambda^n p - p') D^S(W(\Lambda^n,p))_{ts}\mathrm{e}^{-\widetilde{b}\cdot\sum_{k=1}^{n}\Lambda^k p}$$

For vanishing vorticity (i.e. $\Lambda=\mathrm{I}$):

$$\langle \widehat{a}_s^\dagger(p)\widehat{a}_t(p')\rangle = 2\varepsilon' \sum_{n=1}^{\infty}(-1)^{2S(n+1)}\delta^3(p - p')\delta_{ts}\, \mathrm{e}^{-nb\cdot p} = \frac{2\varepsilon\, \delta^3(p-p')\delta_{ts}}{\mathrm{e}^{b\cdot p} + (-1)^{2S+1}}$$

Wigner function with thermal vorticity

The Wigner for free fermions:

$$W(x,k) = -\frac{1}{(2\pi)^4}\int \mathrm{d}^4y \; \mathrm{e}^{-ik\cdot y}\langle :\Psi(x-y/2)\overline{\Psi}(x+y/2):\rangle$$

Exact Wigner function for free fermions at global equilibrium:

$$W(x,k) = \frac{1}{(2\pi)^3}\int \frac{\mathrm{d}^3\mathrm{p}}{2\varepsilon}\sum_{n=1}^{\infty}(-1)^{n+1}\mathrm{e}^{-n\tilde{\beta}(in\phi)\cdot p}\times$$
$$\left[\mathrm{e}^{-in\frac{\phi:\Sigma}{2}}(m+\not{p})\delta^4\left(k-\frac{\Lambda^n p+p}{2}\right)+(m-\not{p})\mathrm{e}^{in\frac{\phi:\Sigma}{2}}\delta^4\left(k+\frac{\Lambda^n p+p}{2}\right)\right]$$

where $\Lambda = \mathrm{e}^{-i\frac{\phi}{2}:J}$

Solves the Wigner equation! Full summation of the $\hbar$ expansion.

Can be used to compute exact expectation values!

Analytic distillation

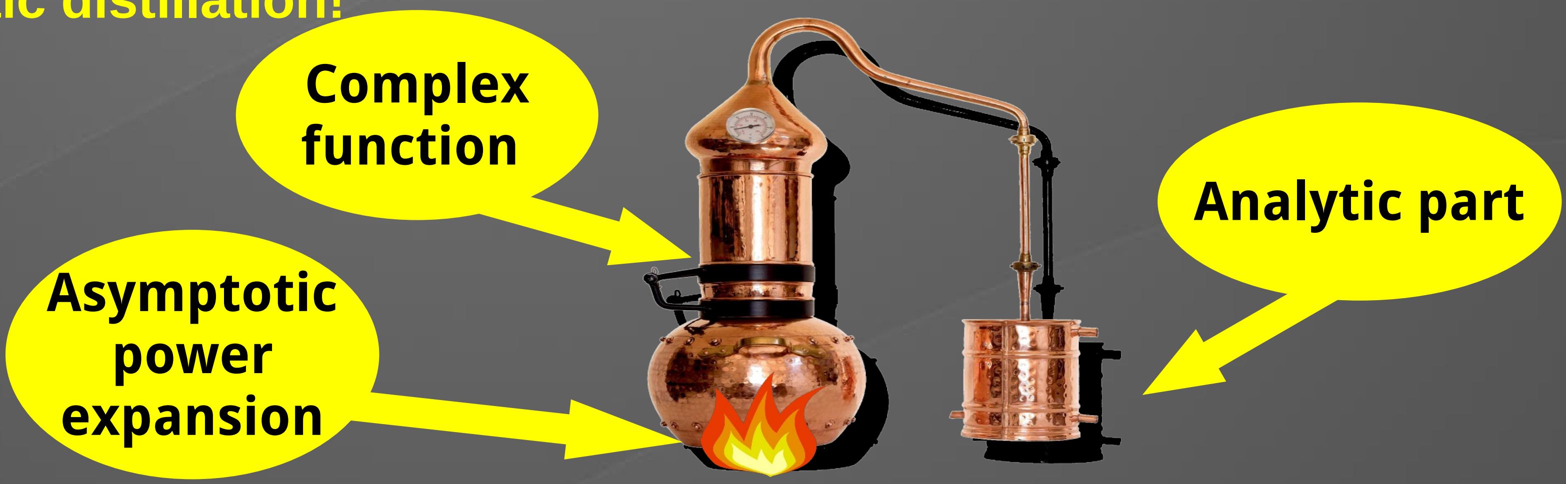
Energy density for *massless* fermions, equilibrium with acceleration ($\phi = ia/T$)

$$\rho = \langle \widehat{T}^{00} \rangle = \frac{3T^4}{8\pi^2} \sum_{n=1}^{\infty} (-1)^{n+1} \phi^4 \frac{\sinh n\phi}{\sinh^5 (n\phi/2)}$$

The series is finite as long as ϕ is real. For real thermal vorticity it diverges!

The series includes terms which are non analytic at $\phi = 0$.

Analytic distillation!



Constitutive eq.s Accelaration

The series boils down to polynomials: $\alpha^\mu = \frac{A^\mu}{T} \qquad w^\mu = \frac{\omega^\mu}{T}$

$$\rho = \frac{7\pi^2}{60\beta^4} - \frac{\alpha^2}{24\beta^4} - \frac{17\alpha^4}{960\pi^2\beta^4}$$

Expectation values vanish at the Unruh temperature $T_U = \sqrt{-A \cdot A}/2\pi$ in accordance with the Unruh effect [F. Becattini, Phys. Rev. D 97 (2018) 085013]

Defines new class of polynomials!
For scalar field these polynomials are connected with Ramanujian polynomials [MB, arXiv:2102.08676]

Case of pure rotation and acceleration: the results found are in agreement with previous derivation both perturbative and exact.

New results

Global equilibrium with both acceleration and rotation
Axial current:

$$j_A^\mu = T^2\left(\frac{1}{6}-\frac{w^2}{24\pi^2}-\frac{\alpha^2}{8\pi^2}\right)\frac{w^\mu}{\sqrt{\beta^2}}$$

Stress-energy tensor:

$$T_B^{\mu\nu}(x) = \rho\, u^\mu u^\nu - p\,\Delta^{\mu\nu} + W\, w^\mu w^\nu + A\,\alpha^\mu\alpha^\nu + G^l\, l^\mu l^\nu + G\,(l^\mu u^\nu + l^\nu u^\mu) + \mathbb{A}\,(\alpha^\mu u^\nu + \alpha^\nu u^\mu)$$
$$+ G^\alpha\,(l^\mu\alpha^\nu + l^\nu\alpha^\mu) + \mathbb{W}\,(w^\mu u^\nu + w^\nu u^\mu) + A^w\,(\alpha^\mu w^\nu + \alpha^\nu w^\mu) + G^w\,(l^\mu w^\nu + l^\nu w^\mu)\,.$$

$$\rho = T^4\left(\frac{7\pi^2}{60}-\frac{\alpha^2}{24}-\frac{w^2}{8}-\frac{17\alpha^4}{960\pi^2}+\frac{w^4}{64\pi^2}+\frac{23\alpha^2 w^2}{1440\pi^2}+\frac{11(\alpha\cdot w)^2}{720\pi^2}\right)$$

$$p = \frac{7\pi^2}{180\beta^4}-\frac{\alpha^2}{72\beta^4}-\frac{w^2}{24\beta^4}-\frac{17\alpha^4}{2880\pi^2\beta^4}+\frac{w^4}{192\pi^2\beta^4}+\frac{(\alpha\cdot w)^2}{96\pi^2\beta^4},$$

$$G^l = -\frac{11}{160\pi^2\beta^4},\quad G = \frac{1}{18\beta^4}-\frac{31\alpha^2}{360\pi^2\beta^4}-\frac{13w^2}{120\pi^2\beta^4},$$

$$W = -\frac{61\alpha^2}{1440\pi^2\beta^4},\ A = -\frac{61w^2}{1440\pi^2\beta^4},$$

$$A^w = \frac{61\alpha\cdot w}{1440\pi^2\beta^4},\qquad \mathbb{A} = \mathbb{W} = G^\alpha = G^w = 0.$$

Spin vector

Spin vector of massive particles:

$$S^\mu(p) = \frac{1}{2}\frac{\int \mathrm{d}\Sigma\cdot p\ \mathrm{tr}\left[\gamma^\mu\gamma_5 W_+(x,p)\right]}{\int \mathrm{d}\Sigma\cdot p\ \mathrm{tr}\left(W_+(x,p)\right)}$$

Exact spin vector at global equilibrium:

$$S^\mu(p) = \frac{1}{2m}\frac{\sum_{n=1}^\infty (-1)^{n+1}\delta^3(\Lambda^n p - p)\mathrm{e}^{-n\widetilde{b}(in\phi)\cdot p}\mathrm{tr}\left[\gamma^\mu\gamma_5 \mathrm{e}^{-in\frac{\phi:\Sigma}{2}}\not{p}\right]}{\sum_{n=1}^\infty (-1)^{n+1}\delta^3(\Lambda^n p - p)\mathrm{e}^{-n\widetilde{b}(in\phi)\cdot p}\mathrm{tr}\left[\mathrm{e}^{-in\frac{\phi:\Sigma}{2}}\right]}$$

Reproduces the known formula in the literature!
[F. Becattini, V. Chandra, L. Del Zanna, E. Grossi, Ann. Phys. 338:32 (2013)]

However, the resummation of all the series is difficult because of the delta functions.

Kinetic theory and distribution function

Wigner equation, a constraint:

$$\left[\gamma\cdot\left(p+\mathrm{i}\frac{\hbar}{2}\partial\right)-m\right]W_{\alpha\beta}=\hbar\,\mathcal{C}_{\alpha\beta}$$

Holds regardless of the density operator!

Equilibrium problem

Equilibrium form of the Wigner function in the presence of vorticity is an *ansatz*.
Kinetic theory starts assuming a distribution function.

$$f(x,p)=\frac{1}{\mathrm{e}^{-\beta\cdot p+|\beta|\lambda\hat{p}\cdot\vec{\omega}}+1}$$

[M.A. Stephanov and Y. Yin, PRL 109 (2012) 162001
J.-Y. Chen, D.T. Son, M.A. Stephanov, H.-U. Yee and Y. Yin, PRL 113 (2014) 182302]

Current and distribution funciton

The current of particles is (massive or massless):

$$j_+^\mu(x) = \frac{1}{(2\pi)^3}\sum_\lambda \int \frac{\mathrm{d}^3p}{2\varepsilon}\frac{\mathrm{d}^3p'}{2\varepsilon'}\,\mathrm{e}^{i(p'-p)\cdot x}\langle \widehat{a}_\lambda^\dagger(p')\widehat{a}_\lambda(p)\rangle\,\bar{u}_\lambda(p')\gamma^\mu u_\lambda(p)$$

This is not parallel to p^μ like in classical kinetic theory

$$j^\mu(x) = \int \mathrm{d}^4k\ \mathrm{tr}\left[\gamma^\mu W(x,k)\right] \neq \int \frac{\mathrm{d}^3p}{\varepsilon}\,p^\mu f(x,p)$$

In the massless case $\bar{u}_\lambda(p')\gamma^\mu u_\lambda(p)$ is a light-like complex vector.
It is decomposed using **two light-like non-orthogonal vectors.**

$$\bar{u}_\lambda(p')\gamma^\mu u_\lambda(p) = \frac{\bar{u}_\lambda(p')\not{q}u_\lambda(p)}{q\cdot p}p^\mu + \underbrace{\frac{\bar{u}_\lambda(p')\not{p}u_\lambda(p)}{q\cdot p}}_{=0}q^\mu + N^\mu(p,q)$$

Distribution function

Define the distribution function from:

$$j_+^\mu(x) = \sum_\lambda \int \frac{\mathrm{d}^3 p}{\varepsilon} p^\mu f_\lambda(x,p) + N_\lambda^\mu(x,p)$$

$$f_\lambda(x,p)_{(q)} \equiv \frac{1}{(2\pi)^3} \int \frac{\mathrm{d}^3 p'}{2\varepsilon'} \mathrm{e}^{i(p'-p)\cdot x} \langle \widehat{a}_\lambda^\dagger(p') \widehat{a}_\lambda(p) \rangle \frac{\bar{u}_\lambda(p') \not{q} u_\lambda(p)}{2q\cdot p}$$

Exact distribution function at global equilibrium:

$$f_\lambda(x,p)_{(q)} = \frac{1}{(2\pi)^3} \frac{1}{2p\cdot q} \sum_{n=1} (-1)^{n+1} \mathrm{e}^{-n\widetilde{\beta}(in\phi)\cdot p} \mathrm{tr}\left(\frac{I + 2\lambda\gamma_5}{2} \mathrm{e}^{-in\frac{\phi:\Sigma}{2}} \not{p}\not{q} \right)$$

It differs from the usual ansatz of the chiral kinetic theory.

Conclusions & Outlook

Exact solutions at general global equilibrium with thermal vorticity.

New operation on complex functions: the **analytic distillation.**

Expressions for the **spin polarization vector** and the **chiral distribution function** to all order in thermal vorticity.

Look to the future:

- Detailed study of the chiral distribution function
- Massive fermions

Thank you for the attention!

Analytic distillation

Definition. *Let $f(z)$ be a function on a domain D of the complex plane and $z_0 \in \bar{D}$ a point where the function may not be analytic. Suppose that asymptotic*[2] *power series of $f(z)$ in $z - z_0$ exist in subsets $D_i \subset D$ such that $\cup_i D_i = D$:*

$$f(z) \sim \sum_n a_n^{(i)} (z - z_0)^n$$

where n can take integer negative values. If the series formed with the common coefficients in the various subsets restricted to $n \geq 0$ has a positive radius of convergence, the analytic function defined by this power series is called analytic distillate of $f(z)$ in z_0 and it is denoted by $\mathrm{dist}_{z_0} f(z)$.

[D. Zagier, Appendix in E. Zeidler, "QFT I" book]

Theorem 1. *Let $F(z)$ be a C^∞ complex valued function in a domain of the complex plane and suppose that F has the following asymptotic power series in $z = 0$*

$$F(z) \sim \sum_{k=-M}^{\infty} A_k z^k$$

with M a positive integer and let F be $o(1/|z|)$ when $|z| \to +\infty$ in the real axis. Let $G(z)$ be the function defined by the series:

$$G(z) = \sum_{n=1}^{\infty} (-1)^{n+1} F(nz). \tag{6.3}$$

The asymptotic power series of $G(z)$ for $|z| \to 0^+$, is given by:

$$G(z) \sim \sum_{n=-M}^{\infty} A_n \eta(-n) z^n, \tag{6.4}$$

where η is the Dirichlet function:

$$\eta(s) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^s} = \left(1 - 2^{1-s}\right) \zeta(s).$$