Adiabatic hydrodynamization in the ‘bottom-up’ thermalization scenario

Strong and Electroweak Matter
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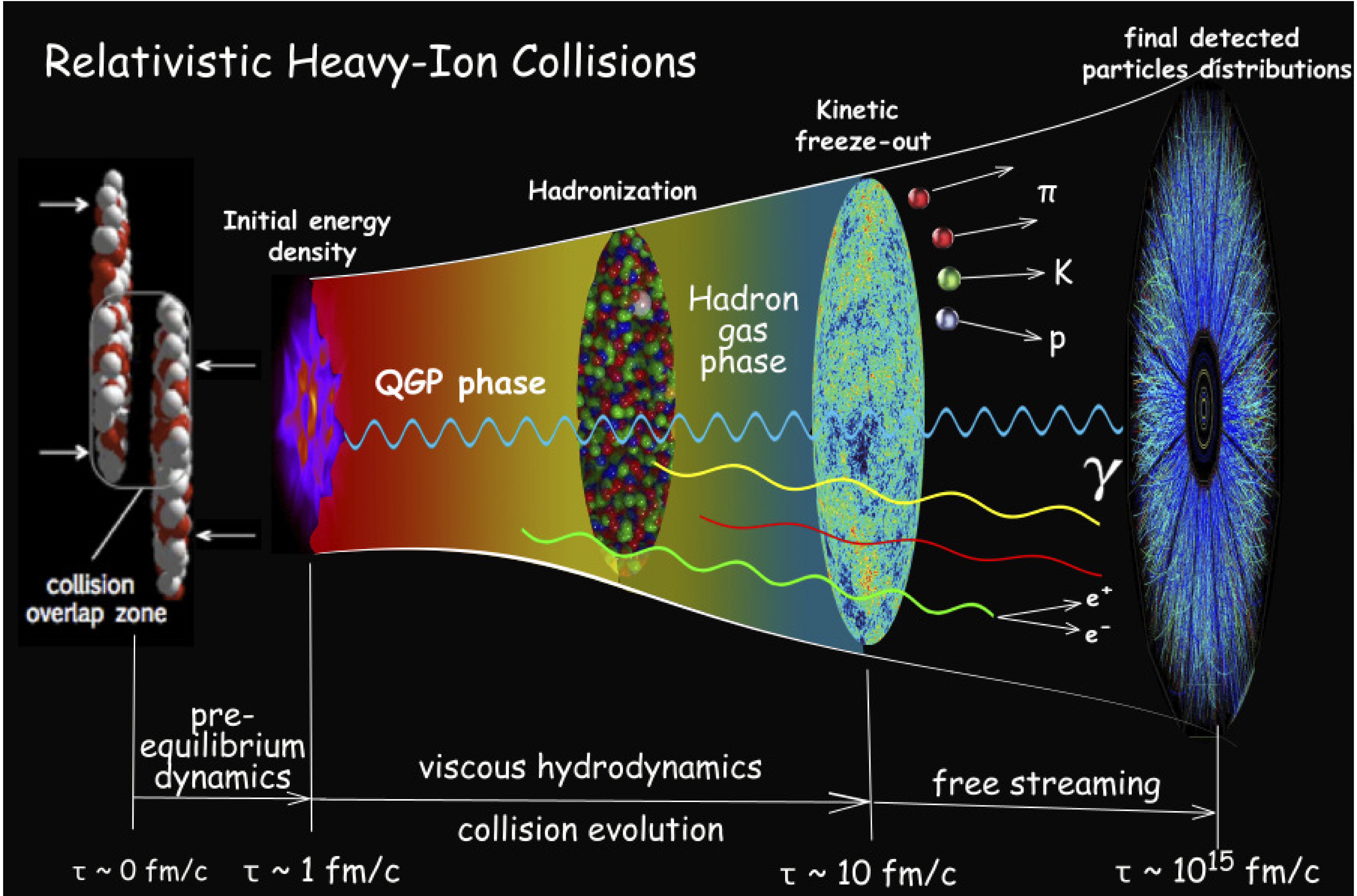
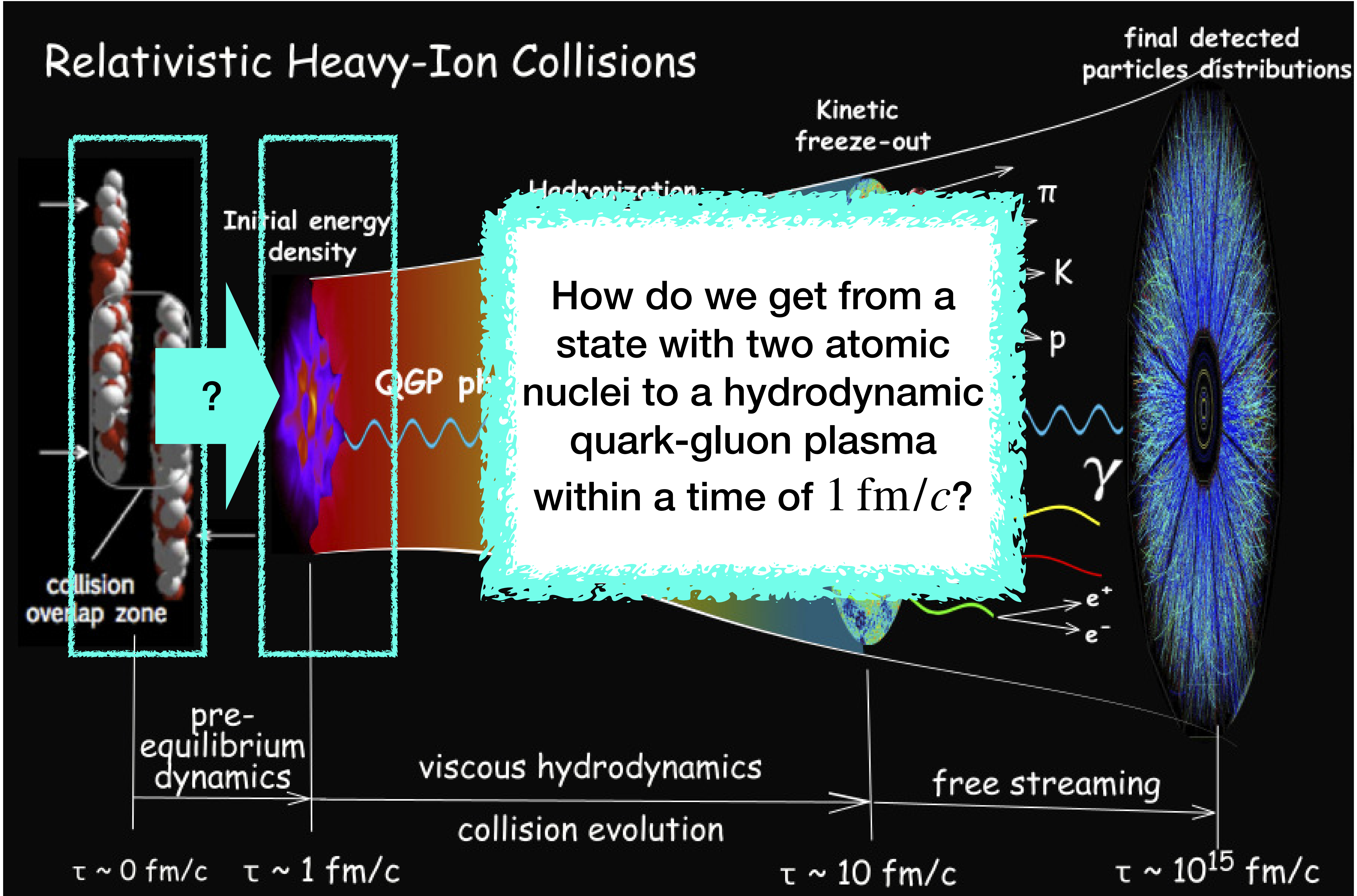
in collaboration with Jasmine Brewer (CERN) and Yi Yin (IMP-CAS)
based on *JHEP 05 (2022) 145* [arXiv:2203.02427 [hep-ph]]

Typical time evolution of the gluon occupation number in a weakly-coupled Bjorken-expanding plasma
$f(p_z;\tau)$
p_z
$C(\tau_S)$
$C(\tau)$
τ
ground state dominance
$C(\tau)$
$\gamma = -\frac{d\log C}{d\log\tau}$
Fixed Point: γ = const.
time-dependent scaling
τ_I
$\tau_S = \tau_{\text{Redu}}$
τ_{FP}
universal scaling
τ
τ_{Hydro}

Introduction: Hydrodynamization



credit: Paul Sorensen and Chun Shen



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Out of equilibrium attractors

far and close to equilibrium

- Many theories describing the pre-hydrodynamic stage exhibit so-called “attractor” solutions. These solutions have been sought, found, and intensively studied over the past decade.

- The nature of the attractors can be different in different models [1]:

[1] A. Kurkela, W. van der Schee, U. A. Wiedemann, B. Wu “What attracts to attractors?” Phys. Rev. Lett. 124, 102301 (2020)

Adiabatic hydrodynamization (AH)

Adiabatic hydrodynamization

as proposed by Brewer, Yan, and Yin [2]

- Idea: the essential feature of an attractor is a reduction in the number of quantities needed to describe the system.
- Brewer, Yan and Yin [2] conjectured that this is due to an emergent timescale $\tau_{\mathrm{Redu}} \ll \tau_{\mathrm{Hydro}}$ after which a set of "pre-hydrodynamic" slow modes (that gradually evolve into hydrodynamic modes) govern the system.
- Their proposal: try to understand the emergence of τ_{Redu} (at which only slow modes remain) using the machinery of the adiabatic approximation in quantum mechanics.

[2] J. Brewer, L. Yan, Y. Yin "Adiabatic hydrodynamization in rapidly-expanding quark-gluon plasma" Phys. Lett. B 816, 136189 (2021)

Adiabatic hydrodynamization

adiabatic theorem and notion of adiabaticity

- Consider a system whose evolution is given by

$$\partial_\tau |\psi\rangle = -H(\tau)|\psi\rangle,$$

where $H(\tau)$ has eigenstates/eigenvalues $\{|n(\tau)\rangle, E_n(\tau)\}_{n=0}^{\infty}$:

$$H(\tau)|n(\tau)\rangle = E_n(\tau)|n(\tau)\rangle.$$

- Then, one may write the system's evolution as

$$|\psi\rangle = \sum_{n=0}^{\infty} a_n(\tau) e^{-\int^{\tau} E_n(\tau')d\tau'} |n(\tau)\rangle.$$

- Adiabaticity is the degree to which transitions between different instantaneous eigenstates are suppressed:

$$\text{Adiabaticity for the } n\text{-th eigenstate} \iff \frac{\dot{a}_n}{a_n} \ll |E_n - E_m|, \text{ for } n \neq m.$$

- When this is the case, provided there is an "energy" gap between the ground state and the excited states, one has

$$\begin{aligned} |\psi\rangle &= \sum_{n=0}^{\infty} a_n(\tau) e^{-\int^{\tau} E_n(\tau')d\tau'} |n(\tau)\rangle \\ &\approx a_0\, e^{-\int^{\tau} E_0(\tau')d\tau'} |0(\tau)\rangle\,, \end{aligned}$$

that is to say, the dynamics of the system collapses onto a single form.

$\implies$ Reduction in the number of variables needed to describe the system.

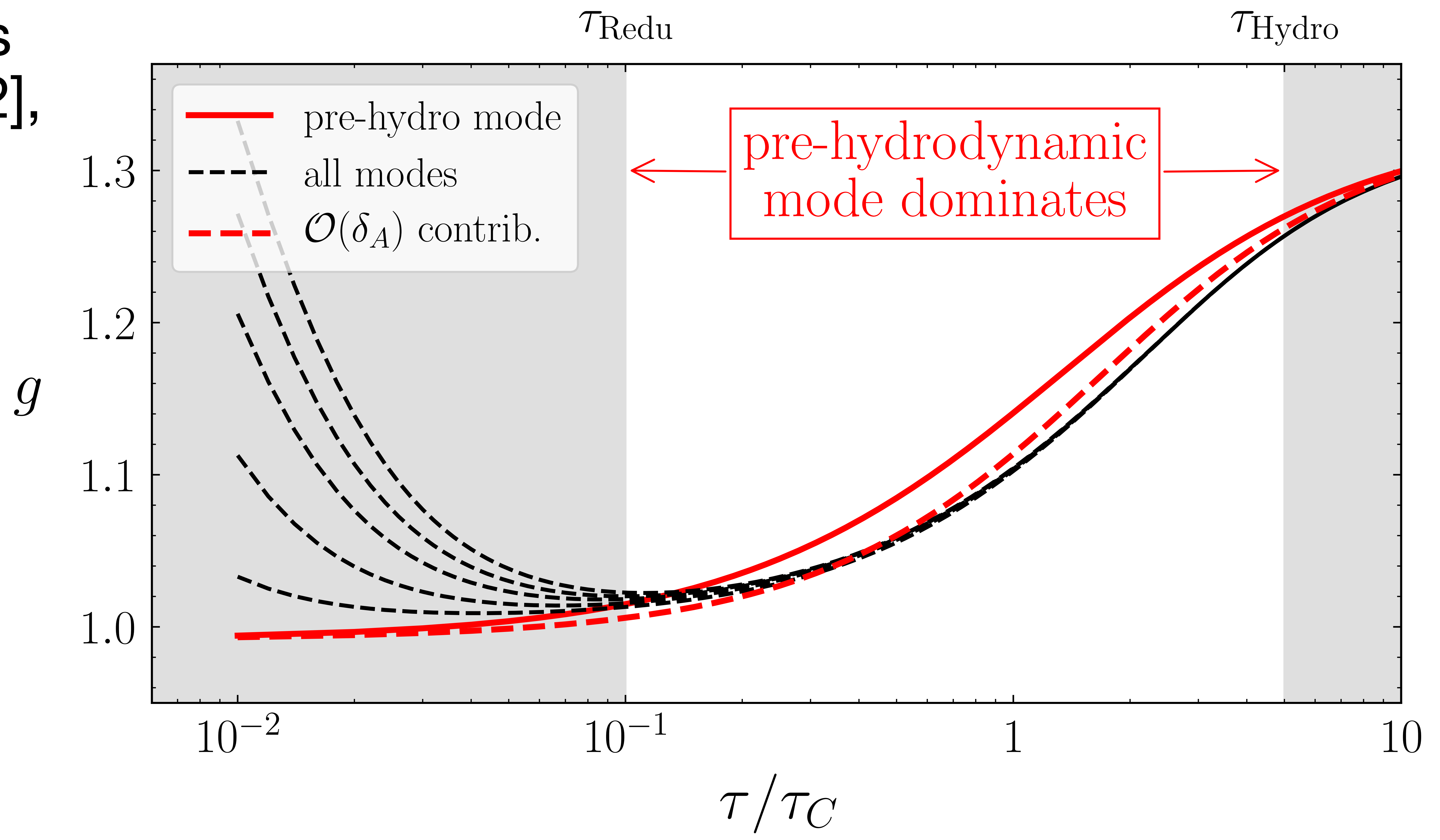
Adiabatic hydrodynamization

Brewer, Yan, and Yin's RTA analysis

- The first exploration of this hypothesis was made in [2], studying an RTA kinetic theory in a Bjorken-expanding plasma:

$$\partial_\tau f(\mathbf{p},\tau) - \frac{p_z}{\tau}\partial_{p_z} f(\mathbf{p},\tau) = -\frac{f(\mathbf{p},\tau) - f_{\rm eq}(\mathbf{p};T(\tau))}{\tau_C}$$

$$g(\tau) = \partial_{\ln\tau} \ln \epsilon(\tau)$$



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‘Bottom-up’ thermalization

‘Bottom-up’ thermalization

as formulated by Baier, Mueller, Schiff, and Son [3]

In the BMSS scenario (in weakly-coupled QCD), thermalization proceeds as

1. Over-occupied hard gluons $f_g \gg 1$ at very early times $1 \ll Q_s\tau \ll \alpha_s^{-3/2}$
2. Hard gluons become under-occupied $f_g \ll 1$, when $\alpha_s^{-3/2} \ll Q_s\tau \ll \alpha_s^{-5/2}$
3. Thermalization of the soft sector after $\alpha_s^{-5/2} \ll Q_s\tau$

[3] R. Baier, A. H. Mueller, D. Schiff, D. T. Son, "‘Bottom-up’ thermalization in heavy ion collisions" Phys. Lett. B 502, 51-58 (2001)

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Specifically, stage 1 predicts that

$$\gamma \equiv -\frac{1}{2}\frac{\partial_{\ln\tau}\langle p_z^2\rangle}{\langle p_z^2\rangle} = \frac{1}{3}\,, \quad \beta \equiv -\frac{1}{2}\frac{\partial_{\ln\tau}\langle p_\perp^2\rangle}{\langle p_\perp^2\rangle} = 0\,.$$

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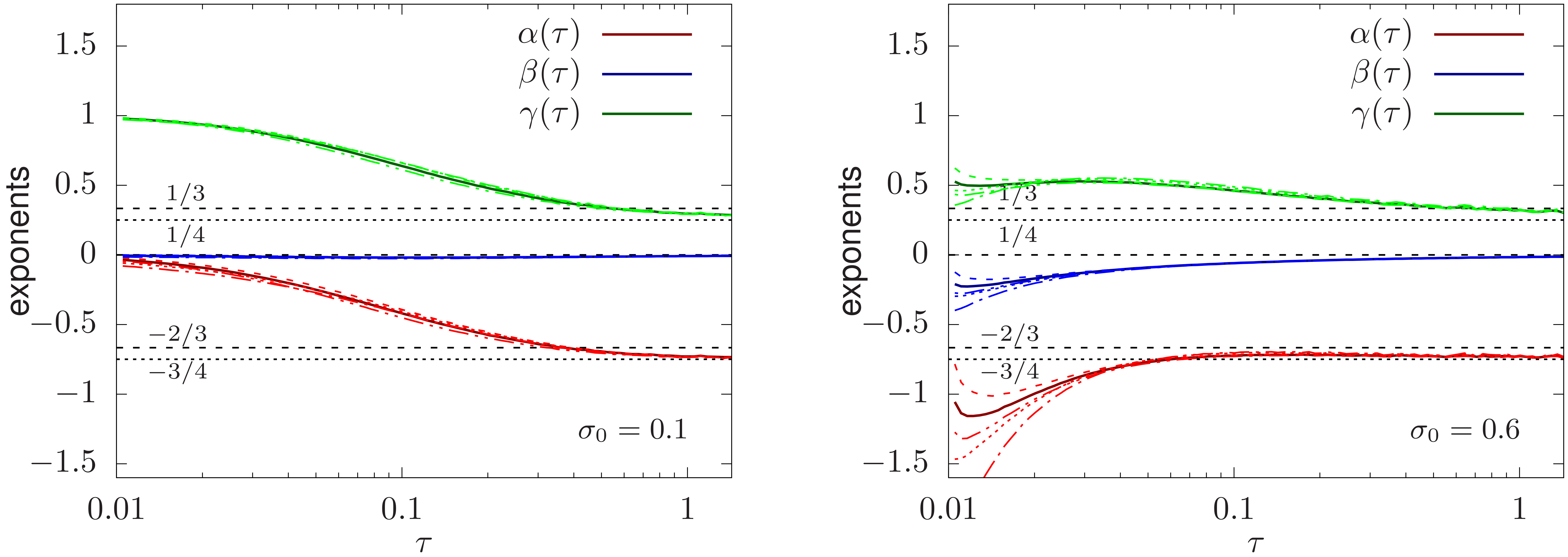
Evidence for AH in QCD effective kinetic theory

A. Mazeliauskas, J. Berges [4]

- After a transient time, [4] observed that the distribution function took a time-dependent scaling form $f(p_\perp, p_z, \tau) = \tau^{\alpha(\tau)} f_S\left(\tau^{\beta(\tau)} p_\perp, \tau^{\gamma(\tau)} p_z\right)$.



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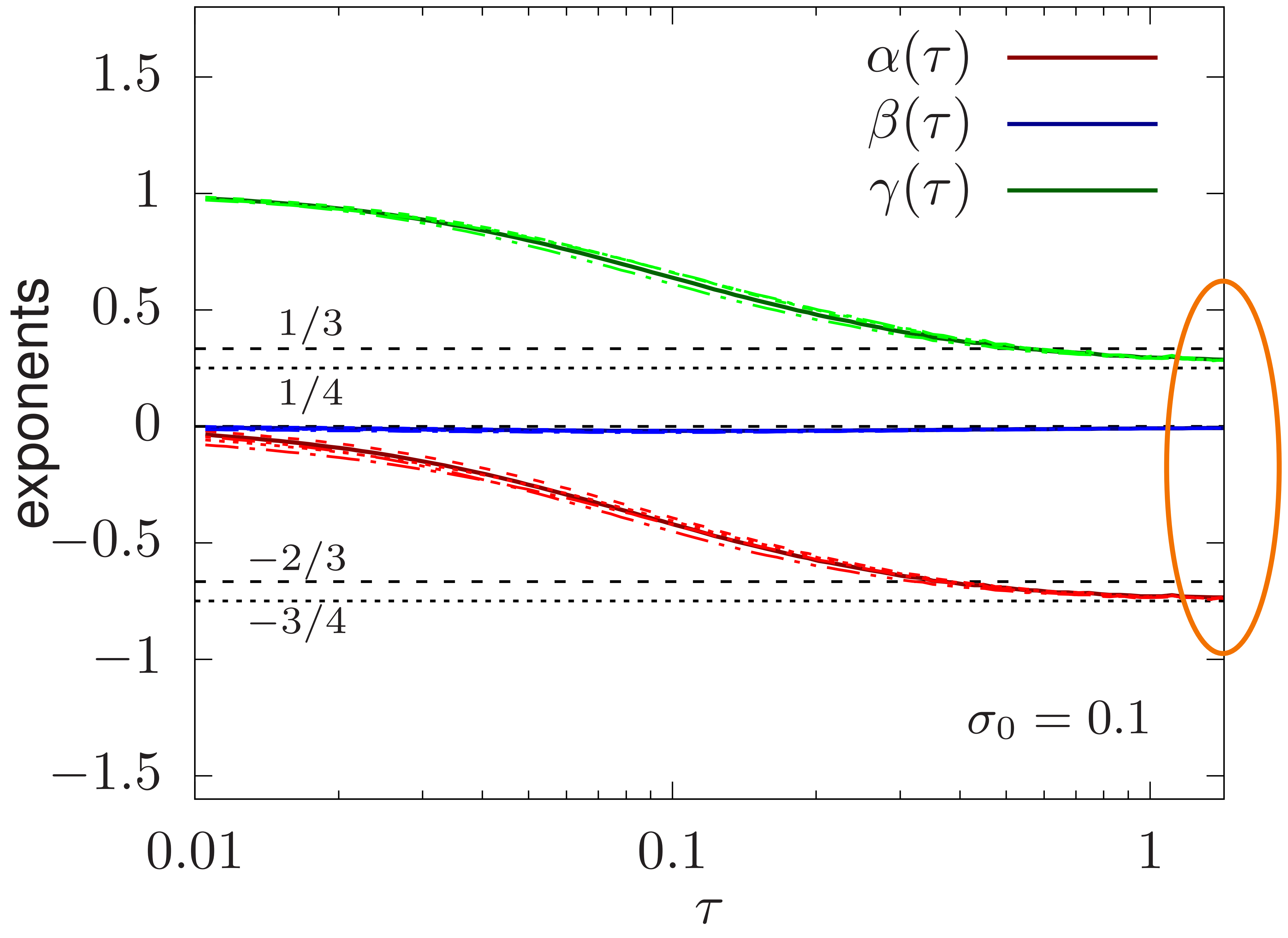

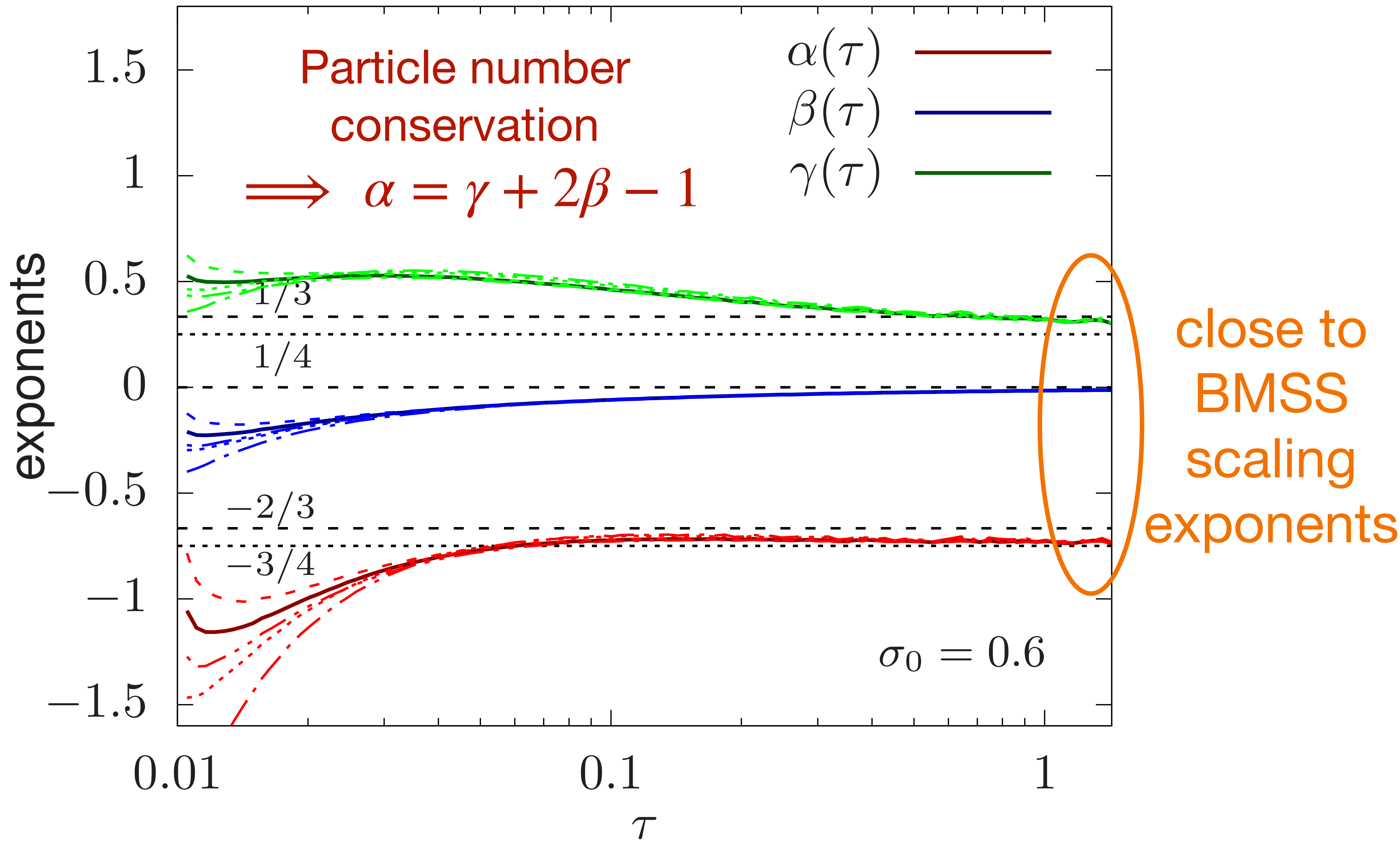

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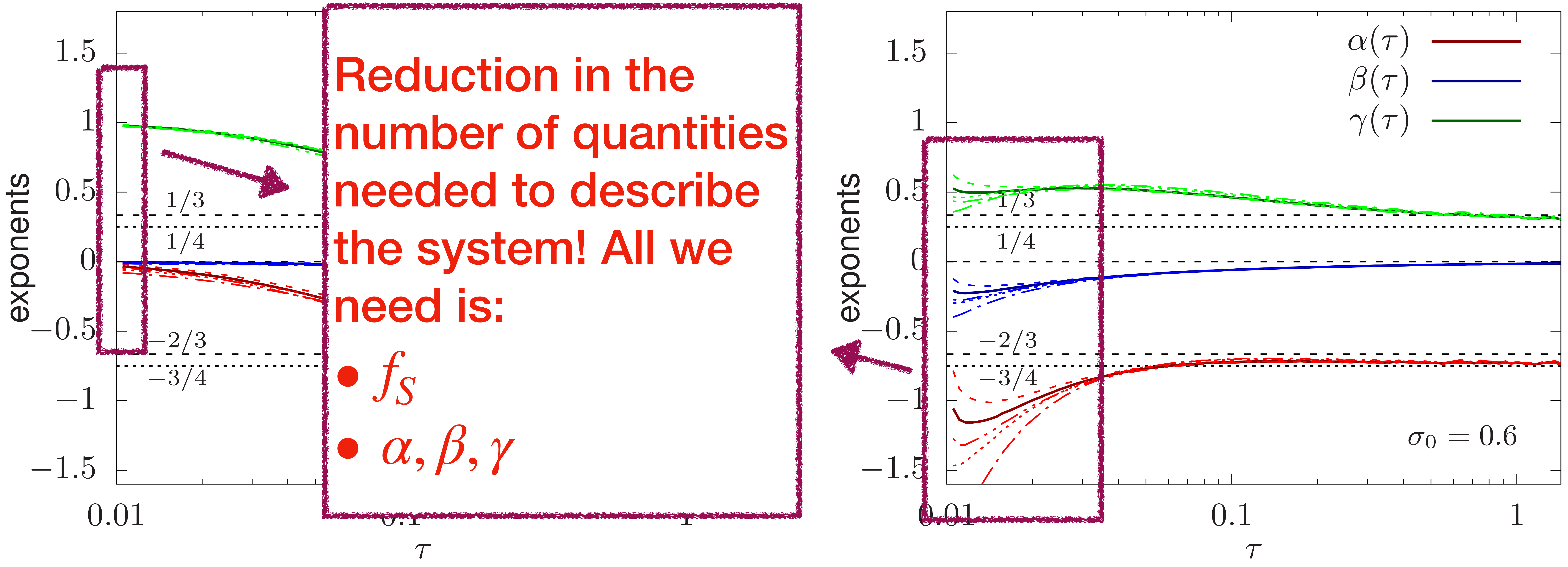

Reduction in the number of quantities needed to describe the system! All we need is:

- f_S
- α, β, γ

[4] A. Mazeliauskas, J. Berges, “Prescaling and far-from-equilibrium hydrodynamics in the quark-gluon plasma” Phys. Rev. Lett. 122, 122301 (2019)

The gluon collision kernel

in the small-angle scattering approximation [5]

- To get some analytic control, we [6] work in the small-angle scattering approximation [5]

$$\partial_\tau f - \frac{p_z}{\tau}\partial_{p_z} f = 4\pi\,\alpha_s^2 N_c^2 l_{\mathrm{Cb}}[f]\Big[I_a[f]\,\nabla_{\mathbf{p}}^2 f + I_b[f]\,\nabla_{\mathbf{p}}\cdot\big(\hat{p}(1+f)f\big)\Big],$$

where

$$I_a[f] = \int_{\mathbf{p}} (1+f)f\,,\quad I_b[f] = \int_{\mathbf{p}} \frac{2}{p} f = \frac{m_D^2}{2N_c g_s^2}\,,\quad l_{\mathrm{Cb}}[f] = \ln\left(\frac{p_{\mathrm{UV}}}{p_{\mathrm{IR}}}\right) \approx \frac{1}{2}\ln\left(\frac{\langle p_\perp^2\rangle}{m_D^2}\right)$$

[5] A.H. Mueller, "The Boltzmann equation for gluons at early times after a heavy ion collision," Phys. Lett. B 475, 220 (2000)

[6] J. Brewer, B. Scheihing-Hitschfeld, Y. Yin "Scaling and adiabaticity in a rapidly expanding gluon plasma" JHEP 05 (2022) 145

- Furthermore, for the first stage of the bottom-up scenario we can consider the approximations [6]: [6] J. Brewer, B. Scheihing-Hitschfeld, Y. Yin "Scaling and adiabaticity in a rapidly expanding gluon plasma" JHEP 05 (2022) 145

$$\frac{\langle p_z^2 \rangle}{\langle p_\perp^2 \rangle} \ll 1 \,, \quad f \gg 1 \,,$$

with which the kinetic equation simplifies to

$$\partial_\tau f - \frac{p_z}{\tau} \partial_{p_z} f = 4\pi \, \alpha_s^2 N_c^2 l_{\mathrm{Cb}}[f] I_a[f] \, \nabla_{\mathbf{p}}^2 f \,.$$

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We find:

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$$\frac{\langle p_z^2 \rangle}{\langle p_\perp^2 \rangle} \ll 1 \,, \quad f \gg 1 \,,$$

Form of initial condition ∀plots:

$$f(\tau_I) = \frac{\sigma_0}{g_s^2} \exp\left(-\frac{p_\perp^2 + \xi^2 p_z^2}{Q_s^2} \right)$$

$\xi = 2$

with which the kinetic equation simplifies to

$$\partial_\tau f - \frac{p_z}{\tau} \partial_{p_z} f = 4\pi \alpha_s^2 N_c^2 l_{\mathrm{Cb}}[f] I_a[f] \nabla_{\mathbf{p}}^2 f \,.$$

We find:

Scaling and adiabaticity

‘Optimizing’ adiabaticity

rescaling the degrees of freedom

- From the previous discussion, we see that scaling plays a crucial role in this problem.
- This gives us a very useful tool to ‘optimize’ adiabaticity. For instance, if we have a distribution function evolving as

$$f(p_\perp, p_z, \tau) = A(\tau)\, w\Big(p_\perp/B(\tau), p_z/C(\tau); \tau\Big),$$

then we can look for the choice of A, B, C that maximize the degree to which the dynamics of w is adiabatic.

$q = 4\pi\alpha_s^2 N_c^2 l_{\mathrm{Cb}}[f] I_a[f]\tau$

'Optimizing' adiabaticity

in practice

- The original kinetic equation has the form

$$\tau\partial_\tau f - p_z \partial_{p_z} f = q(\tau)\,\nabla^2_{\mathbf{p}} f\,.$$

- Then, by introducing $\zeta = p_\perp/B$, $\xi = p_z/C$, $y = \log(\tau/\tau_I)$, and the scaling exponents $\alpha = \partial_y \ln A$, $\beta = -\partial_y \ln B$, $\gamma = -\partial_y \ln C$, one obtains that

$$\partial_y w = -\mathscr{H} w\,,$$

with $\mathscr{H} = \alpha - (1-\gamma)\left[\tilde{q}\,\partial_\xi^2 + \xi\,\partial_\xi\right] + \beta\left[\tilde{q}_B\,(\partial_\zeta^2 + \frac{1}{\zeta}\partial_\zeta) + \zeta\,\partial_\zeta\right].$

$\tilde{q} = \dfrac{q}{C^2(1-\gamma)}$, $\tilde{q}_B \equiv -\dfrac{q}{B^2\beta}$

What is the advantage of this?

- Because A, B, C are a choice of coordinates (a “gauge” choice to describe the system), we can choose them such that $\tilde{q} = \tilde{q}_B = 1$.
- Then, we get

$$\mathscr{H} = \alpha - (1-\gamma)\Big[\partial_\xi^2 + \xi\,\partial_\xi\Big] + \beta\left[\partial_\zeta^2 + \frac{1}{\zeta}\partial_\zeta + \zeta\,\partial_\zeta\right],$$

which is a separable Hamiltonian of the form

$$\mathscr{H} = f_0(y)\,H_0 + f_1(y)\,H_\xi + f_2(y)\,H_\zeta,$$

where the Hamiltonians H_0, H_ξ, H_ζ are constant and can be “diagonalized” simultaneously. In this situation, the adiabatic approximation is exact.

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Results
low-lying energy states

- We can choose A such that $\alpha = \gamma + 2\beta - 1$ to set the ground state energy $\mathscr{E}_{0,0} = 0$.
- The eigenvalues of $\mathscr{H}$ are $\mathscr{E}_{n,m} = 2n(1-\gamma) - 2m\beta \ , \quad n, m = 0, 1, 2, \ \dots$
- The left and right eigenstates are:

$$\phi^L_{n,m} = \mathrm{He}_{2n}\left(\frac{\xi}{\sqrt{\tilde{q}}}\right) {}_1F_1\left(-2m, 1, \frac{\zeta^2}{2\tilde{q}_B}\right),$$

$$\phi^R_{n,m} = \frac{1}{\sqrt{2\pi\tilde{q}}(2n)!}\frac{1}{\tilde{q}_B}\mathrm{He}_{2n}\left(\frac{\xi}{\sqrt{\tilde{q}}}\right) {}_1F_1\left(-2m, 1, \frac{\zeta^2}{2\tilde{q}_B}\right) e^{-\frac{\xi^2}{2\tilde{q}} - \frac{\zeta^2}{2\tilde{q}_B}}$$

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Gapped energy levels!

$\implies$ Ground state will dominate after a transient time

$$\phi^L_{n,m} = \mathrm{He}_{2n}\left(\frac{\xi}{\sqrt{\tilde{q}}}\right) {}_1F_1\left(-2m, 1, \frac{\zeta^2}{2\tilde{q}_B}\right),$$

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Left and right eigenstates differ because $\mathscr{H}$ is not hermitian

Results

evolution equations for the scaling exponents

- This “diagonalization” was achieved by taking $\tilde{q} = \tilde{q}_B = 1$. This implies evolution equations for the scaling exponents:

$$\partial_y \beta = (\partial_y \ln q + 2\beta)\,\beta\;, \;\; \partial_y \gamma = (\partial_y \ln q + 2\gamma)(\gamma - 1).$$

- To close the system, one needs to specify how q evolves.
- However, since we showed that the system is gapped, we can get a good description of the evolution by solving for $q[f;\tau]$ assuming w is in its ground state.
    - Corrections from excited states can also be included systematically.

Flow of γ, β under time evolution

Open circles: fixed points with $\dot{l}_{\rm Cb} = 0$, Filled circles: fixed points with $\dot{l}_{\rm Cb} = 0.4$

$\text{over-occupied } (A \gg 1 \iff ''f \gg 1''):$

$$\partial_y \beta = \left(\gamma + 4\beta - 1 + \dot{l}_{\rm Cb}\right)\beta\,,$$

$$\partial_y \gamma = \left(3\gamma + 2\beta_S - 1 + \dot{l}_{\rm Cb}\right)(\gamma - 1)\,.$$

$\text{dilute } (A \ll 1 \iff ''f \ll 1''):$

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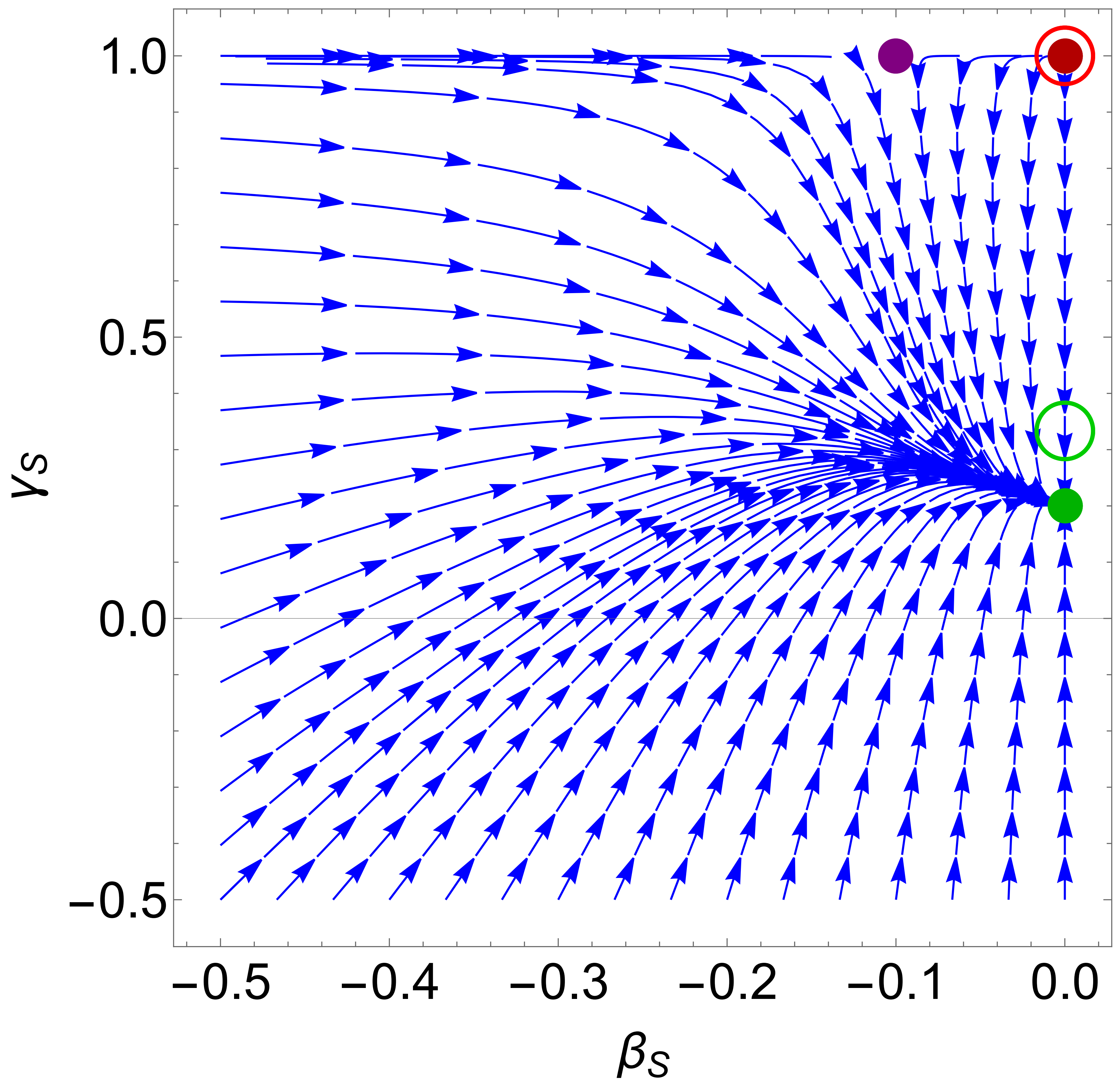

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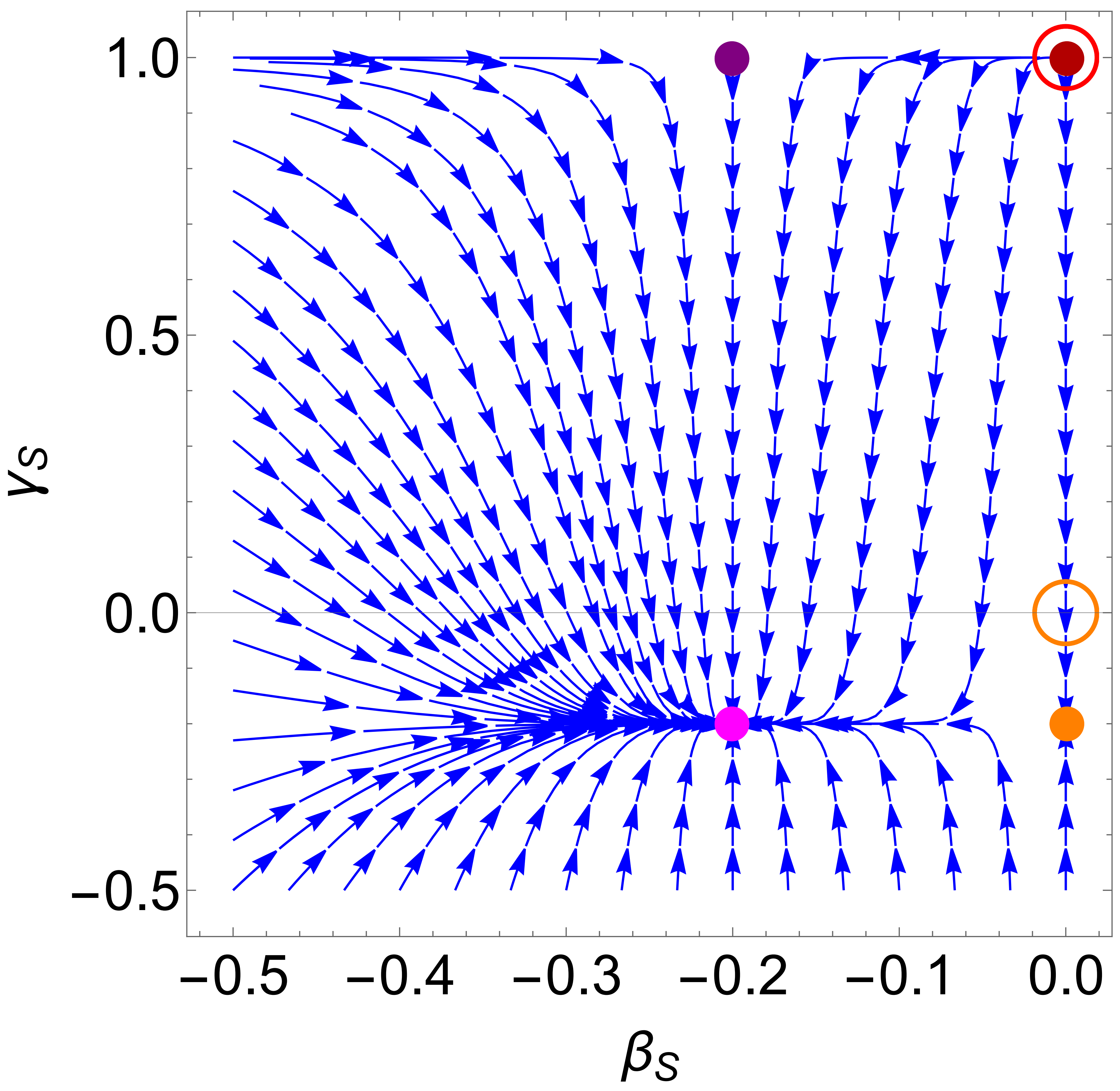


Scaling exponents

comparison with QCD EKT

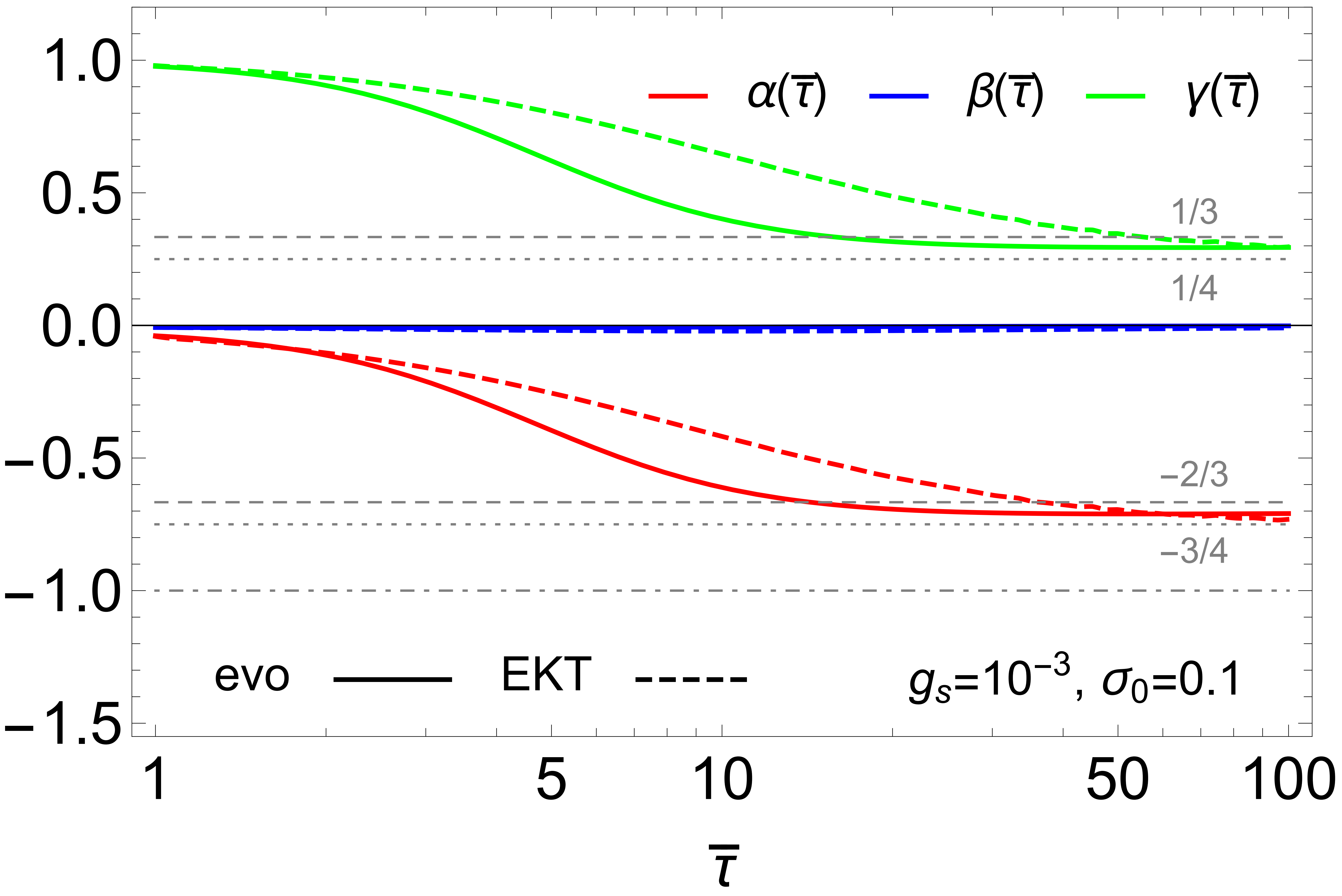
- We compare our results with those of [4], using the same initial condition:

$$f(\tau_I) = \frac{\sigma_0}{g_s^2} \exp\left(-\frac{p_\perp^2 + \xi^2 p_z^2}{Q_s^2}\right).$$

- In our description, for this initial condition we predict a deviation from the BMSS scaling exponents given by:

$$\delta\gamma \equiv \gamma - \frac{1}{3} = -\frac{1}{3\ln\left(\frac{4\pi\tau}{N_c\tau_I\sigma_0}\right)}$$

Flow equations [6] (solid) versus QCD EKT [4] (dashed)



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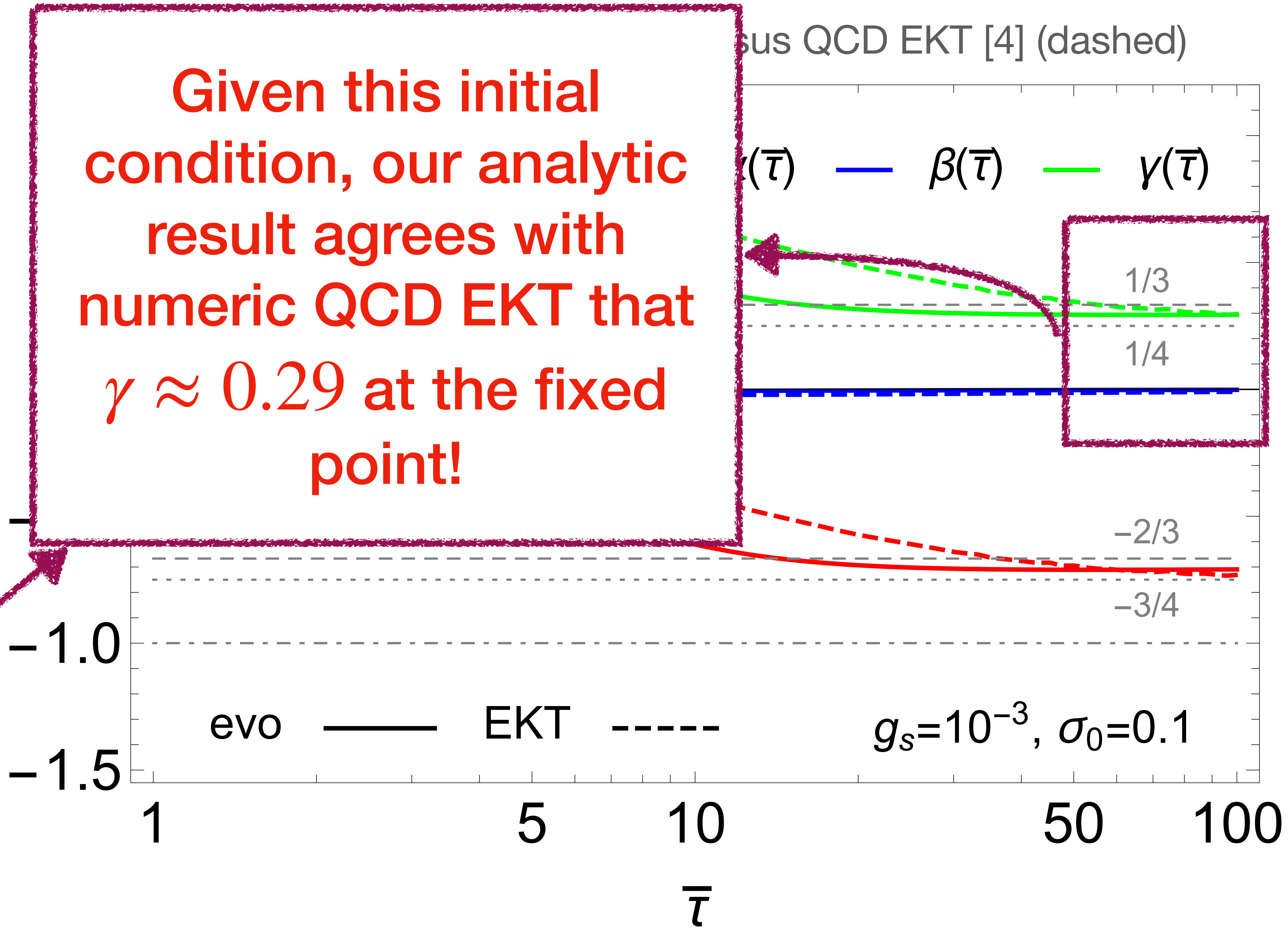

Given this initial condition, our analytic result agrees with numeric QCD EKT that $\gamma \approx 0.29$ at the fixed point!

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Summary

We conclude that the first stage of the ‘bottom-up’ thermalization scenario is an example of adiabatic hydrodynamization. Furthermore, our results explain:

1. How an out-of-equilibrium weakly-coupled gluon plasma rapidly approaches a pre-hydrodynamic stage whose subsequent evolution has little memory of its initial conditions, all long before hydrodynamization.

2. The emergence of time-dependent scaling as a feature of QCD kinetic theory.

3. The fixed points of the (non-linear) dynamical evolution as instantaneous ground states of an effective Hamiltonian.

Outlook

Possible generalizations we have in mind:

- Include radial expansion in the kinetic equation (relevant for HIC)
- Generalize the analysis to a broader class of collision kernels
- Identify the adiabatic aspects of hydrodynamization in strongly coupled theories (e.g., using AdS/CFT)

Thank you!