Two-body gravitational scattering in effective field theory

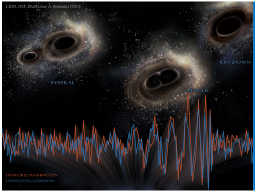
Pierre Vanhove

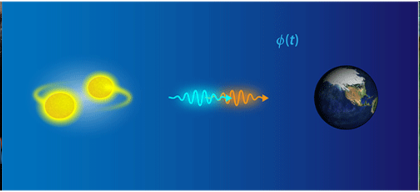
IPhT Saclay

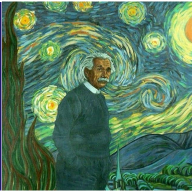
Strong and Electro-Weak Matter 2022, Saclay, France

based on work done in collaboration with
N.E.J. Bjerrum-Bohr, Poul Damgaard, Ludovic Planté, Stavros Mougiakakos

Einstein theory of gravity is the main paradigm for understanding the structure and dynamic of our observable Universe







Black holes, compact stars and gravitational waves are amongst the most spectacular predictions of general relativity and natural probes of the fundamental principles of Einstein's theory and its extension

"Gravitational waves enable tests of general relativity in the highly dynamical and strong-field regime. Using events detected by LIGO-Virgo up to 1 October 2019, we evaluate the consistency of the data with predictions from the theory. [...] *We find no evidence for new physics beyond general relativity, for black hole mimickers, or for any unaccounted systematics.*"[1]

[1] [LIGO and Virgo collaboration 2010.14529]

Gravity effective field theories

Still it is important to have an better understanding of how Einstein gravity fits within our current EFT approach of particle physics

By embedding the *classical* Einstein gravity effects in a quantum EFT framework we will shed a new light on subtle gravitational effects

We will be working in the context of an effective field theory assuming :

- Standard QFT (local, unitary, lorentz invariant, ...)
- The low-energy DOF: graviton, usual matter fields
- Standard symmetries: General relativity as we know it

$$\mathcal{S}_{\rm eff} = \mathcal{S}_{\rm eff}^{\rm gravity} + \mathcal{S}_{\rm eff}^{\rm matter}$$

Classical gravity from quantum amplitudes

THE GENERATION OF GRAVITATIONAL WAVES.
IV. BREMSSTRAHLUNG*†‡

SÁNDOR J. KOVÁCS, JR.
W. K. Kellogg Radiation Laboratory, California Institute of Technology

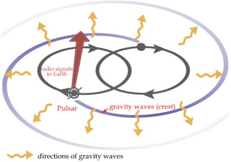
AND

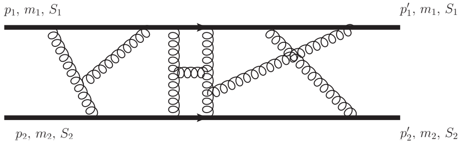
KIP S. THORNE
Center for Radiophysics and Space Research, Cornell University; and
W. K. Kellogg Radiation Laboratory, California Institute of Technology

Received 1977 October 21; accepted 1978 February 28

g) *The Feynman-Diagram Approach*

Any classical problem can be solved quantum-mechanically; and sometimes the quantum solution is easier than the classical. There is an extensive literature on the Feynman-diagram, quantum-mechanical treatment of gravitational bremsstrahlung radiation (e.g., Feynman 1961, 1963; Barker, Gupta, and Kaskas 1969; Barker and Gupta

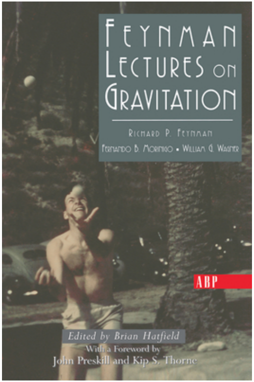


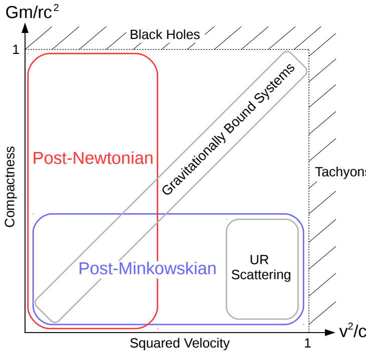


We will then present a scheme using the modern quantum scattering amplitudes to provide an optimal framework for gravitational observable that will be the starting point of the wave-form analysis

Precison graviton from amplitudes

Nature isn't classical, dammit, and if you want to make a simulation of nature, you'd better make it quantum mechanical, and by golly it's a wonderful problem, because it doesn't look so easy. (Richard Feynman, 1981)





EFT are designed for high precision (analytic!) computations applied to the inspiral phase, which is the largest part of the detected signal!

We look to understand the inspiraling regime in almost all relevant theories of gravity (Einstein gravity, Supergravity, higher dimensions, higher derivative operators, . . .)

Classical gravitational observables

There are various approaches for computing the classical gravitational observables developped at IPhT in particular

in-in formalism – [arXiv:1811.10950]

Amplitudes, Observables, and Classical Scattering

David A. Kosower,[a] Ben Maybee,[b] Donal O'Connell[b]

Effective field theory worldline – [arXiv:2204.06556] & [arXiv:2205.15295]

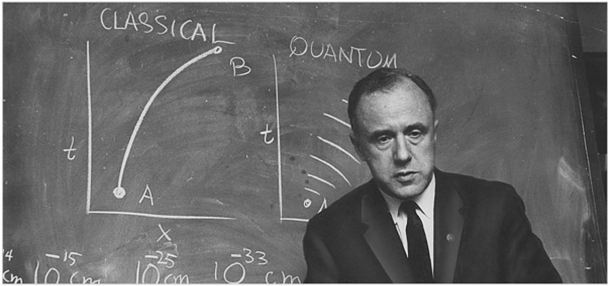
Gravitational Bremsstrahlung with tidal effects in the post-Minkowskian expansion

Stavros Mougiakakos,[1,2] Massimiliano Maria Riva,[1] and Filippo Vernizzi[1]

Gravitational Bremsstrahlung from Spinning Binaries in the Post-Minkowskian Expansion

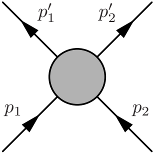
Massimiliano Maria Riva, Filippo Vernizzi, and Leong Khim Wong

Classical Gravity from Quantum Scattering



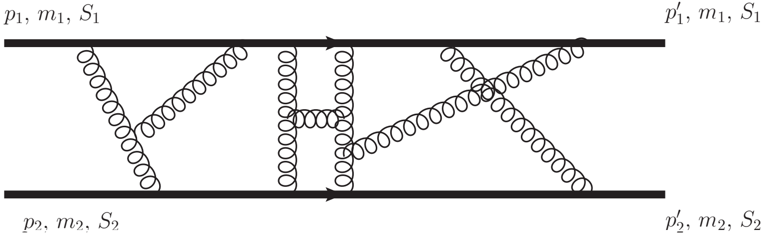
One important **new** insight is that the **classical** gravitational two-body interactions (conservative and radiation) can be extracted from **quantum scattering amplitudes**

$$\mathcal{M}(p_1 \cdot p_2, q^2) =$$



$$= \sum_{L=0}^{+\infty} G_N^{L+1} \mathcal{M}^{\mathrm{L-loop}}$$

Classical physics from quantum loops



In the limit $\hbar, q^2 \to 0$ with $\underline{q} = \frac{q}{\hbar}$ fixed at each loop order of the quantum amplitude has the Laurent expansion[2] $\gamma = \frac{p_1 \cdot p_2}{m_1 m_2}$ and $q^2 = (p_1 - p'_1)^2$

$$\mathfrak{M}_L(\gamma, \underline{q}^2, \hbar) = \frac{\mathcal{M}_L^{(-L-1)}(\gamma, \underline{q}^2)}{\hbar^{L+1} |\underline{q}|^{\frac{L(4-D)}{2}+2}} + \cdots + \frac{\mathcal{M}_L^{(-1)}(\gamma, \underline{q}^2)}{\hbar |\underline{q}|^{\frac{L(4-D)}{2}+2-L}} + O(\hbar^0)$$

- At all loop orders there is a classical contribution of order $1/\hbar$
- **classical gravity contributions** are determined by the **unitarity**
- The quantum amplitude has IR and UV divergences. But the classical amplitude is finite

2 [Iwasaki; Holstein, Donoghue; Bjerrum-Bohr, Damgaard, Planté, Vanhove; Kosower, Maybee, O'Connell]

Exponentiation of the S-matrix

Using an exponential representation of the $\widehat{S}$ matrix[3]

$$\widehat{S} = \mathbb{I} + \frac{i}{\hbar}\widehat{T} = \exp\left(\frac{i\widehat{N}}{\hbar}\right)$$

doing the Dyson expansion with the conservative and radiation part

$$\hat{T} = G_N \sum_{L\geqslant 0} G_N^L \hat{T}_L + G_N^{\frac{1}{2}} \sum_{l\geqslant 0} G_N^L \hat{T}_L^{\rm rad}, \qquad \hat{N} = G_N \sum_{L\geqslant 0} G_N^L \hat{N}_L + G_N^{\frac{1}{2}} \sum_{l\geqslant 0} G_N^L \hat{N}_L^{\rm rad}$$

The classical radial action $\hat{N}^{\rm classical}$ does not have any $\hbar$. The higher power of $1/\hbar$ more singular than the classical **are needed for the consistency of the full quantum amplitude and the correct exponentiation of the amplitude**

$$\mathfrak{M}_L(\gamma,\underline{q}^2,\hbar) = \frac{\mathcal{M}_L^{(-L-1)}(\gamma,\underline{q}^2)}{\hbar^{L+1}|\underline{q}|^{\frac{L(4-D)}{2}+2}} + \cdots + \frac{\mathcal{M}_L^{(-1)}(\gamma,\underline{q}^2)}{\hbar|\underline{q}|^{\frac{L(4-D)}{2}+2-L}} + O(\hbar^0)$$

[3] [Damgaard, Planté, Vanhove]

Classical physics from loops : ħ counting

In QFT the propagator has inverse ħ that the traditional counting disregards (cf . [Quantum Field Theory, Itzykson Zuber, §6-2-1 page 288])[4]

> To find the connection between L and the power of $\hbar$, we collect all factors $\hbar$. We leave aside the factor $\hbar$ that gives the mass term a correct dimension. In other words, the Klein-Gordon equation should read $[\partial_x^2 + (mc/\hbar)^2]\varphi = 0$, indicating that the mass term is of quantum origin. This phenomenon is disregarded in the sequel. There are thus two origins of such factors. First the

At the $L+1$PM order, the two-body scattering amplitude scales with the masses as

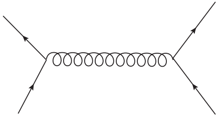
$$\mathcal{M}_L(\gamma, q^2) = \frac{G_N^{L+1} m_1^2 m_2^2}{\underline{q}^{2+\frac{(2-D)L}{2}}} \sum_{i=0}^{L} c_{L-i+2,i+2}(\gamma) m_1^{L-i} m_2^i$$

This piece will emerge from a L quantum amplitude as follows

$$\mathfrak{M}^L\Big|_{\text{classical}} \propto \frac{m_1^2 m_2^2}{\underline{q}^{2+\frac{(2-D)L}{2}}} \hbar^{L-1} G_N^{L+1} \sum_i \left(\frac{m_1 c}{\hbar}\right)^{L-i} \left(\frac{m_2 c}{\hbar}\right)^i \propto \frac{\mathcal{M}_L(\gamma, \underline{q}^2)}{\hbar}$$

4 [B. R. Holstein and J. F. Donoghue, [arXiv:hep-th/0405239 [hep-th]].]

One graviton exchange : tree-level amplitude



$$\mathfrak{M}_1 = -16\pi G_N \hbar \frac{2(p_1\cdot p_2)^2 - m_1^2 m_2^2 - |\hbar \underline{\vec q}\,|^2 (p_1\cdot p_2)}{|\hbar \underline{\vec q}\,|^2}$$

The $\hbar$ expansion of the tree-level amplitude

$$\mathfrak{M}_1 = \frac{\mathcal{M}_1^{(-1)}(p_1\cdot p_2)}{\hbar|\underline{q}|^2} + \hbar 4\pi G_N p_1\cdot p_2$$

The higher order in q^2 are quantum with powers of $\hbar$

The *classical* potential is obtained by taking the 3d Fourier transform $E_i = \sqrt{p_i^2 + m_i^2}$

$$\mathcal{V}_1(p_1\cdot p_2, r) = \int \frac{d^3\underline{\vec q}}{(2\pi)^3} \frac{\mathcal{M}_1^{(-1)}(\vec q)\, e^{i\underline{\vec q}\cdot\vec r}}{4E_1E_2} = \frac{G_N}{E_1E_2}\frac{m_1^2m_2^2 - 2(p_1\cdot p_2)^2}{r}$$

Velocity cuts

We need to expand the amplitude to get a prescribed polynomial mass dependence. When expanding the amplitude we get combination of linear propagators[5]

$$\left(\frac{1}{(p_A\cdot\ell_A+i\varepsilon)(p_A\cdot\ell_B-i\varepsilon)}-\frac{1}{(p_A\cdot\ell_B+i\varepsilon)(p_A\cdot\ell_A-i\varepsilon)}\right)\times$$
$$\left(\frac{1}{(p_B\cdot\ell_A-i\varepsilon)(p_B\cdot\ell_C+i\varepsilon)}-\frac{1}{(p_B\cdot\ell_C-i\varepsilon)(p_B\cdot\ell_A+i\varepsilon)}\right)$$

can be expressed in terms of delta functions

$$\left(\frac{\delta(p_A\cdot\ell_A)}{p_A\cdot\ell_B+i\varepsilon}-\frac{\delta(p_A\cdot\ell_B)}{p_B\cdot\ell_A+i\varepsilon}\right)\times\left(\frac{\delta(p_B\cdot\ell_C)}{p_B\cdot\ell_A+i\varepsilon}-\frac{\delta(p_B\cdot\ell_A)}{p_B\cdot\ell_C+i\varepsilon}\right)$$

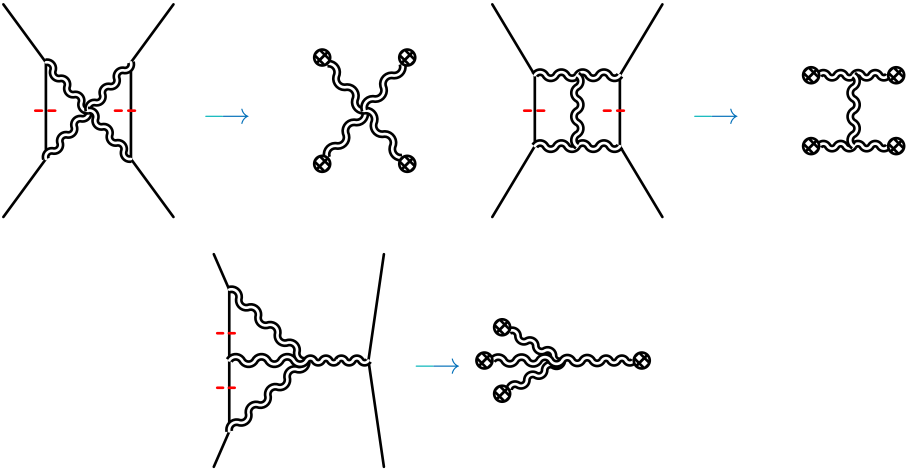
thanks to the identity

$$\frac{1}{x+i\varepsilon}-\frac{1}{x-i\varepsilon}=-2i\pi\delta(x)$$

[5] [Bjerrum-Bohr, Damgaard, Planté, Vanhove]

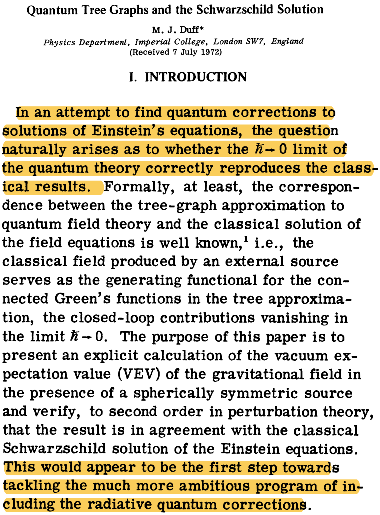
Velocity cuts and Classical amplitude

The classical amplitude arises from the contribution with L-delta functions on the massive propagators projecting the integral on world-line like graph[6]



6 [Bjerrum-Bohr, Damgaard, Planté, Vanhove]

Classical black hole metric from quantum amplitudes



Quantum Tree Graphs and the Schwarzschild Solution

M. J. Duff*
Physics Department, Imperial College, London SW7, England
(Received 7 July 1972)

I. INTRODUCTION

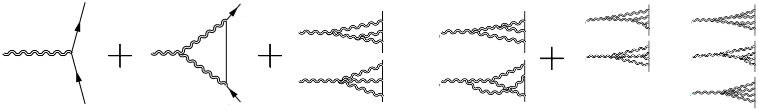
In an attempt to find quantum corrections to solutions of Einstein's equations, the question naturally arises as to whether the $\hbar \to 0$ limit of the quantum theory correctly reproduces the classical results. Formally, at least, the correspondence between the tree-graph approximation to quantum field theory and the classical solution of the field equations is well known,[1] i.e., the classical field produced by an external source serves as the generating functional for the connected Green's functions in the tree approximation, the closed-loop contributions vanishing in the limit $\hbar \to 0$. The purpose of this paper is to present an explicit calculation of the vacuum expectation value (VEV) of the gravitational field in the presence of a spherically symmetric source and verify, to second order in perturbation theory, that the result is in agreement with the classical Schwarzschild solution of the Einstein equations. This would appear to be the first step towards tackling the much more ambitious program of including the radiative quantum corrections.

In 1973 Duff asked the question about the classical limit of quantum gravity.[a] He showed how to reproduce the Schwarzschild back hole metric from quantum tree graphs to G_N^3 order

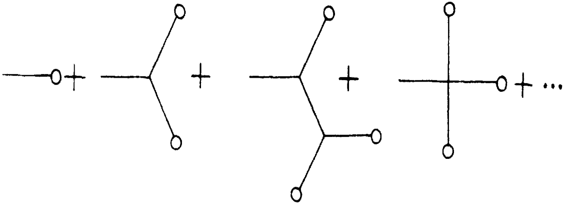
The double expansion in G_N and $\hbar$ give a new perspective on the classical limit of gravitational scattering amplitudes

[a] M. J. Duff, "Quantum Tree Graphs and the Schwarzschild Solution," Phys. Rev. D **7** (1973), 2317-2326

Black hole metric from amplitudes



$$\frac{GM}{r} + \left(\frac{GM}{r}\right)^2 + \left(\frac{GM}{r}\right)^3 + \left(\frac{GM}{r}\right)^4$$

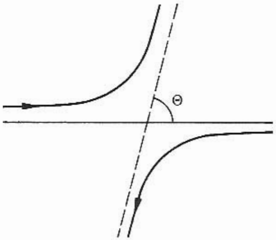


- The tree skeleton graphs are the one computed by Duff
- Reproduces the Schwarzschild-Tangherlini metric in $d \geqslant 4$ dimensions[7]

[7] [Mougiakakos, Vanhove]

The scattering angle

One important observable that allows to analytically continue from the scattering regime to the bound state regime is the scattering angle



Scattering angle from the (classical) radial action

$$N_{4PM}(\gamma, |b|) = \int_{\mathbb{R}^2} e^{i\underline{q}\cdot b} \frac{N_{4PM}(\gamma, \underline{q}^2)}{4m_1 m_2 \sqrt{\gamma^2-1}} \frac{d^2\underline{q}}{(2\pi)^2}$$

as $\chi_{4PM}(\gamma) = -\partial N_{4PM}(\gamma, J)/\partial J$ with the angular momentum $J = m_1 m_2 \sqrt{\gamma^2-1} b/E_{\rm C.M.}$

$$\frac{\chi}{2}\Big|_{1PM+2PM} = \frac{(2\gamma^2-1)}{\gamma^2-1}\left(\frac{G_N m_1 m_2}{J}\right) + \frac{3\pi(m_1+m_2)(5\gamma^2-1)}{8(m_1^2+m_2^2+2m_1 m_2\gamma)}\left(\frac{G_N m_1 m_2}{J}\right)^2$$

Angle for a test mass in the Schwarzschild black hole of mass $M = m_1 + m_2$.

The 3PM scattering angle

$$\frac{\chi}{2}\Big|_{3PM} = \left(\frac{G_N m_1 m_2}{J}\right)^3 \sqrt{\gamma^2-1}\left(\frac{(64\gamma^6-120\gamma^4+60\gamma^2-5)}{3\left(\gamma^2-1\right)^2}\right.$$
$$-\frac{4m_1m_2}{3\mathcal{E}_{C.M.}^2}\gamma\left(14\gamma^2+25\right)+\frac{m_1m_2}{\mathcal{E}_{C.M.}^2}\frac{4(3+12\gamma^2-4\gamma^4)\,\text{arccosh}(\gamma)}{\sqrt{\gamma^2-1}}$$
$$\left.+\frac{2m_1m_2(2\gamma^2-1)^2}{\mathcal{E}_{C.M.}^2\sqrt{\gamma^2-1}}\left(-\frac{11}{3}+\frac{d}{d\gamma}\left(\frac{(2\gamma^2-1)\,\text{arccosh}(\gamma)}{\sqrt{\gamma^2-1}}\right)\right)\right)$$

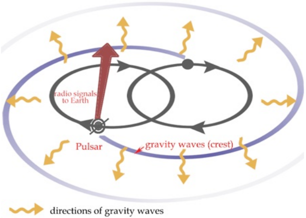
At 3PM (two-loop) new phenomena arise

- The conservative part deviates from Schwarzschild as we have contributions which depends (linearly) on the relative mass[8] $\nu = \frac{m_1 m_2}{(m_1+m_2)^2}$
- And the important Radiation-reaction terms[9]

8 [Damour; Bern et al.; di Vecchia et al.;Bjerrum-Bohr et al.]

9 [Damour; di Vecchia et al.;Bjerrum-Bohr et al.]

Radiation reaction



The problem of **radiation reaction** has been one of the fundamental theoretical issues in general relativity. This is a needed contribution to match the binary-pulsar observations.

At 3PM a consistent derivation of radiation-reaction was missing. The amplitude approach clarified that

- The radiation-reaction from the soft region of the amplitude (not in the potential region of[10])
- The radiation-reaction is needed for restoring a smooth continuity between the non-relativitic, relativistic and ultra-relativistic regimes[11]

The complete classical scattering amplitude gives a clear-cut unified and unambiguous resolution of these issues at 3PM[12]

[10] [Bern et al.]

[11] [Damour, Veneziano et al.]

[12] [Bjerrum-Bohr, Damgaard, Planté, Vanhove]

Post-Minkowskian expansion

The advantages of the scattering amplitude approach is that

- No resummation in velocity is needed as this is automatically given by the relativistic expression in the relative velocity u

$$\mathfrak{M}^{\rm GR}(p_1\cdot p_2, q) = \sum_{m,n} G_N^n u^m c^{(n,m)}(q) \qquad u^2 \sim \frac{G_N(m_1+m_2)}{r} \ll 1$$

  Compare to post-Newtonian expansion known up to 4PN

- One only gets velocities no higher derivative : acceleration terms &c in the Lagrangian are removed by appropriate coordinate transformation. No need to use equation of motions
- The amplitude gives a way to analyse the transition between the small velocity regime and the ultra-relativistic regime and the effect of gravitational radiation[13]

[13] [Damour; di Vecchia, Heisenber, Russo, Veneziano]

Beyond Eintein gravity

The same formalism can be used to include effects beyond Einstein gravity

From the exponentiation of the S-matrix $S \simeq \exp(N(J,E))$ one can include corrections beyond the classical Einstein gravity and study bulk causality and how classical observables are modified

- Star light bending angle[14]

$$\theta_S \simeq \frac{\partial N(J,E)}{\partial J} \simeq \frac{4GM}{b} + \frac{15\pi}{4}\left(\frac{GM}{b}\right)^2 + \frac{8bu^S - 48\log(b/b_0)}{\pi}\frac{G\hbar M}{b^3}$$

- Bulk causality and UV completion can be studied using the Shapiro time delay/advance[15]

$$\delta T = 2\frac{\partial N(J,E)}{\partial E} \geqslant 0$$

Causality is preserved thanks to various positivity constraints on Wilson coefficients

14 [Bjerrum-Bohr, Donoghue, Holstein, Planté, Vanhove]

15 [Donoghue et al.; Camanho et al. ; Arkani-Hamed et al.; Bellazini et al.; Huber et al.]

Outlook

This approach gives a new understanding on the relation between classical (Einstein) general relativity and the quantum theory of gravity

1. The 2-body gravitational scattering amplitude leads to the classical observables : potential, scattering angle, and radiation
2. The amplitude approach is much simpler that the traditional approach from solving Einstein's equation, and analytic relativistic expressions. The velocity cut method is very efficient method for extracting the classical part
3. This is a very useful framework for studying subtle effects like radiation-reactions and memory effects where subtle non-linear effects arise from 5PN order [Blanchet; Damour; ...]
4. The approach applies to any EFT of gravity where one can compute amplitudes. Therefore this is a power approach to derive new constraints for modified gravity scenarios.