Gravitational Wave Analysis : Matched Filtering

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This document briefly presents the way to compute the signal-to-noise ratio (SNR) through the Optimal Matched Filtering and to find gravitational waves inside a signal.

The signal s can be split in two parts. The noise n in one hand and in the other hand the gravitational wave h :

$$s(t) = n(t) + h(t)$$

In practice, the signal is translated in the frequency domain : $\tilde{s}(f) = \tilde{n}(f) + \tilde{h}(f)$
In order to analyse the signal and find the wave inside, a matched filter is applied on the signal with some templates. This matched filter consist in an inter-correlation between the signal and a template :

$$S = \int_{-\infty}^{+\infty} \tilde{s}(f)\tilde{Q}^*(f)\mathrm{d}f$$

with $\tilde{Q}(f)$, the frequency-domain template. The significance of the filtered signal is given by the signal overnoise ratio (SNR) defined as :

$$SNR = \frac{\langle S \rangle}{\sigma_N}$$

where σ_N is the standard deviation of the filtered noise N : $\sigma_N^2 = \langle N^2 \rangle - \langle N \rangle^2 = \langle N^2 \rangle$ since $\langle n \rangle = 0$ in the case of a gaussian noise, and $\langle \cdot \rangle$ denote the expectation value. The SNR can then be expressed as follows :

$$SNR = \sqrt{\frac{\langle S \rangle^2}{\langle N^2 \rangle}}$$

From the definition of S, N can be expressed as :

$$N = \int_{-\infty}^{+\infty} \tilde{n}(f)\tilde{Q}^*(f)\mathrm{d}f = S - \langle S \rangle$$

hence :

$$\langle N^2 \rangle = \langle S^2 \rangle - \langle S \rangle^2 = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \langle \tilde{n}(f)\tilde{n}^*(f') \rangle \,\mathrm{d}f' |\tilde{Q}(f)|^2 \mathrm{d}f = \int_0^{+\infty} S_n(f)|\tilde{Q}(f)|^2\mathrm{d}f$$

Here S_n is the unilateral Power Spectral Density (PSD) and is defined from the bilateral PSD S_n^{bi} which is the fourier transform (noted $\mathcal{F}$) of the noise auto-correlation :

$$S_n^{bi}(f) = \int_{-\infty}^{+\infty} \left[\lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{+T} n^*(t)n(t-\tau)\mathrm{d}t \right] e^{-2i\pi f\tau}\mathrm{d}\tau$$

$$= \mathcal{F}\left(\lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{+T} n^*(t)n(t-\tau)\mathrm{d}t\right) = \mathcal{F}(n * n^*) = \tilde{n}\cdot\tilde{n}^*$$

hence,

$$S_n(f) = \begin{cases} 0 & \text{if} \quad f \leq 0 \\ S_n^{bi}(-f) + S_n^{bi}(f) & \text{if} \quad f > 0 \end{cases}$$

The filtering is optimal if the template Q maximizes the SNR. It is then necessary for Q to be proportional to $\frac{\tilde{h}}{S_n}$.

In general the case of a template proportional to $\frac{\tilde{h}}{S_n}$ is :

$$\tilde{Q}(f) = \alpha\frac{\tilde{T}(f)}{S_n(f)}e^{2i\pi f t_0}$$

with $\tilde{h}(f) = 2\alpha\tilde{T}(f)e^{2i\pi f t_0}$ where $\tilde{T}(f)$ is the expected normalised waveform, α and t_0 are respectively a normalisation factor and a time shift. Let replace this expression of the template in the matched filtering :

$$S(t_0) = 4\alpha\mathcal{R}e\left\{\int_0^{+\infty}\frac{\tilde{s}(f)\tilde{T}^*(f)}{S_n(f)}e^{2i\pi f t_0}\right\}$$

In this way the maximal SNR is obtained when the time of the wave t_GW fits with the template and it has avalue of α.

$$SNR^2 = 2\alpha^2\frac{\mathcal{R}e\left\{\int_0^{+\infty}\frac{|\tilde{T}(f)|^2}{S_n(f)}e^{2i\pi f(t_{GW}-t_0)}\mathrm{d}f\right\}^2}{\int_0^{+\infty}\frac{|\tilde{T}(f)|^2}{S_n(f)}\mathrm{d}f} = \langle S\rangle^2 = \alpha^2 \qquad \text{if } \mathrm{t_{GW}} = \mathrm{t_0}$$

In that case we can see that the noise follows a normal distribution with $\langle N^2\rangle = 1$ ($\tilde{T}(f)$ can be fixed such as $\int_0^{+\infty}\frac{|\tilde{T}(f)|^2}{S_n(f)}\mathrm{d}f = \frac{1}{2}$).