Basic Concepts of Statistics

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School of Statistics

ATLAS EXPERIMENT

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General introduction

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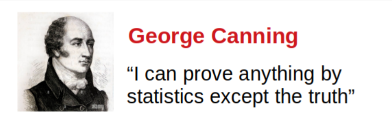

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Goals of the lecture

- recap the basics needed for the SOS
- learn how to be critical with statistics (in science, but not only)
- *focus* on meaning and (mis)intuition rather than mathematical rigour

Statistics versus probability (according to Persi Diaconis)

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Few personal tips for this lecture

- keywords/concepts will be listed at the end of each section
  $\rightarrow$ make sure you know the ideas behind them!

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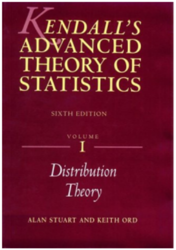
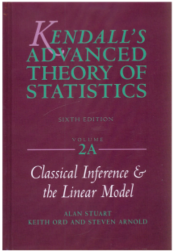
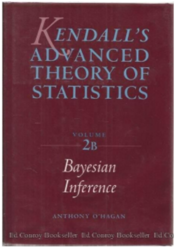
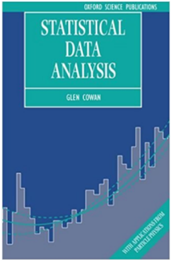
- keywords/concepts will be listed at the end of each section
  → make sure you know the ideas behind them!
- statistics is almost like a language: you need practice to learn it!
  → compute/code as much as simple examples as you can **by yourself!**

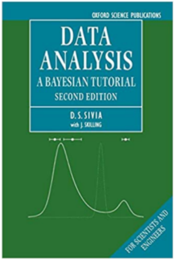
Some references

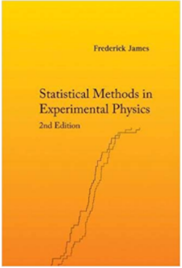


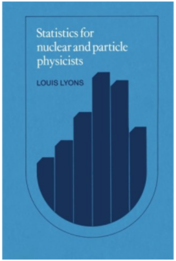


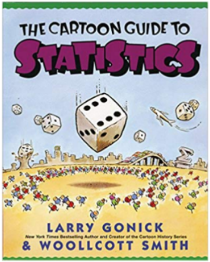












Content

1. Statistics
2. Probability
3. Statistical model
4. The two big schools
5. Parameter estimation and hypothesis testing

Statistics

Descriptive statistics

Definitions:

- Descriptive statistics $\sim$ "summarize" a sample
- sample = set of observations $\mathcal{S} \equiv \{x_1, x_2, ..., x_n\}$

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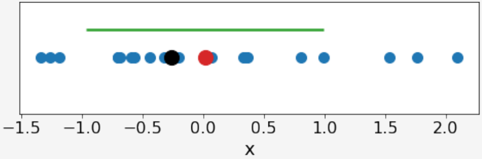
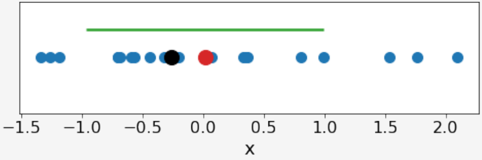
  - Skewness: $\gamma_x = \overline{\left(\frac{x-\overline{x}}{\sigma_x}\right)^3}$ - asymmetry
  - Kurtosis: $\beta_x = \overline{\left(\frac{x-\overline{x}}{\sigma_x}\right)^4}$ - importance of tails

Sample caracterisation - illustrations



blue: x_i, red: mean. black: median, green: σ_x



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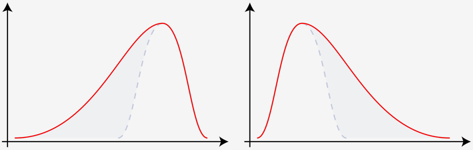
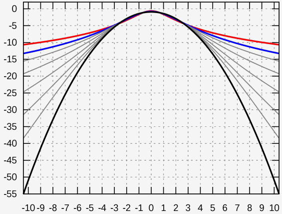
Skewness and Kurtosis (using probability functions)





Right plot: Kurtosis $\gamma = \infty$ (red), 2 (blue), $1, 1/2, 1/4, 1/8$, and $1/16$ (gray), 0 (black)

Sample caracterisation - comments

Notion of estimator (more on this later)

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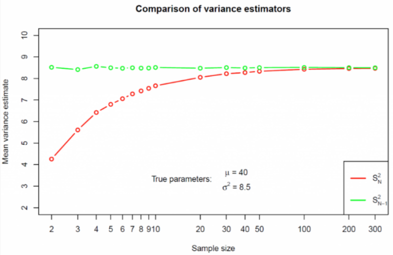

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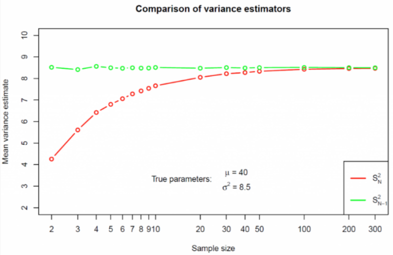

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Statistical moments (more on this later)

- Order-r moment: $m_r = \overline{\left(\frac{x-\overline{x}}{\sigma_x}\right)^r}$ (relates directly to the mean of x^r)
- probability theory: all truth moments $\equiv$ exact underlying probability
- first moments $\equiv$ "main" features of the sample

Correlations

Multidimensional sample

- single observation i = several numbers: $x_i \rightarrow (x_i^{(1)}, x_i^{(2)}, ...x_i^{(p)})$
- *e.g.* biological dataset: person size, weight, age and genre

Previous description applies to each variable $x_i^{(j)}$ but one can now explore how variables behave wrt each other.

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Covariance and correlations between two variables a and b:

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- probes if fluctuations around the mean are coherent for a and b
- covariance (and correlation) are symetric - fortunate
- covariance of x with itself is the variance
- $\rho_{a,b} \in [-1, 1]$; 0 = uncorrelated ($\neq$ indep!), (-)1 = (anti-)correlated

More on correlations

Covariance matrix or error matrix

- $C_{ij} = \rho_{ij} \times \sigma_i \sigma_j$ - real and symmetric.
- ρ_{ij} is the correlation matrix - symmetric with 1's on diagonal.

Why is this object so important?

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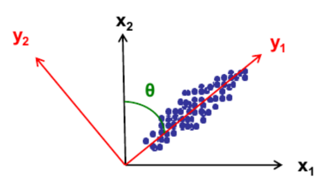

- x_1 and x_2 both have a large σ
- but, they are highly correlated
- most of the information is in y_1 (largest σ)
  $\rightarrow$ idea of dimension reduction
  $\rightarrow$ idea of pre-processing in ML

Correlation and dependence

Correlation $\equiv$ *linear* dependence $\Rightarrow$ dependence

BUT

Non-correlation *dosen't* imply independence (matter of vocabulary)

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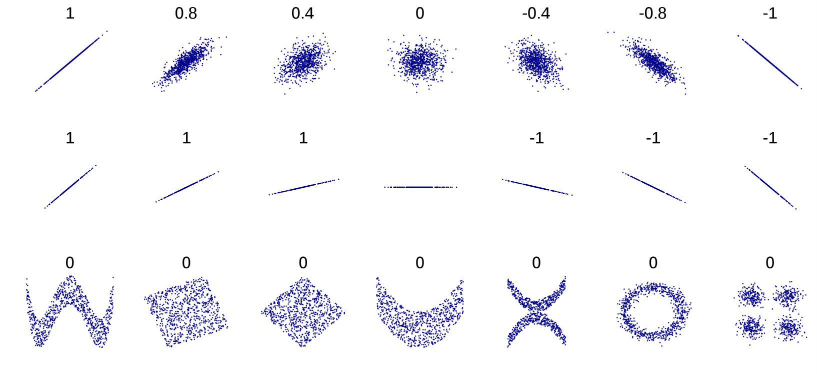

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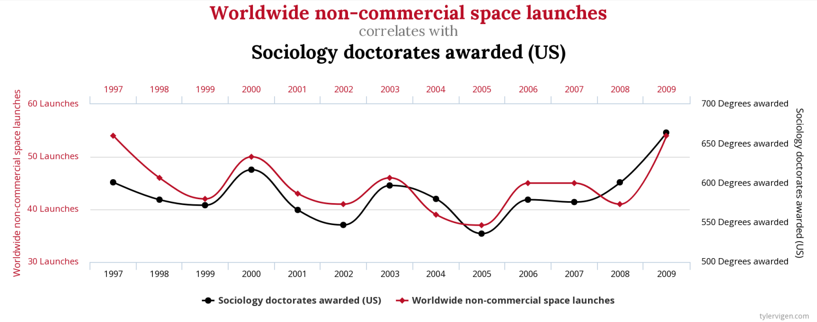

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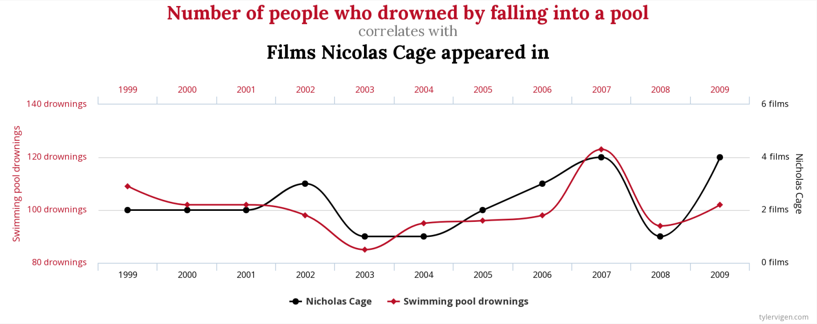

Keywords and concepts

Part I

descriptive statistics – sample – mean – (co)variance – (de)correlation

Probability

Some definitions

Caution: what follows is *not* mathematically rigorous

Random variable and associated probability

- a random variable X describes an observable which is not certain
- all possible outcomes - realisations - of X form a set Ω
- a probability P_i is associated to each realisation i of Ω
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Simple concrete example: a flippin coin

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$\rightarrow$ these notions can be defined and manipulated without any sample

Coming back to estimators - I

Previously: sample mean $\neq$ "true mean". What is the true mean?

$$\mu = \sum_{\Omega} P_i x_i \quad ; \quad \sigma^2 = \sum_{\Omega} P_i \times (x_i - \mu)^2 \quad ; \quad m_r = \sum_{\Omega} P_i \times \left(\frac{x_i - \mu}{\sigma}\right)^r$$

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E.g. of the flipping coin

- $\mu = 1/2$, $\sigma = 1/2$, $m_r = 1$ if r is even and 0 if r is odd

Conditional probabilities and bias theorem

Bias theorem - math version

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Comments

- many ways to *understand* this fundamental equation
- in some case, each of these term has a clear meaning
- these two posts are quit interesting post 1 and post 2

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- $P(evidence|hypothesis)$: proba that there is smoke if there is fire
  → **easy to assess** (fire produces smoke)
  → That is **the** interst of bias theorem

Understanding Bias theorem

Example: $hypothesis = fire$ and $evidence = smoke$

$$P(fire|smoke) = P(fire) \times \frac{P(smoke|fire)}{P(smoke)}$$

- $P(hypothesis|evidence)$: proba that there is a fire if there is smoke
  $\rightarrow$ difficult to assess (many sources of smoke), that's **the posterior**
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N.B.: $P(evidence)$ is independent from the hypothesis, and is sometime impossible to compute. It is often seen as a "normalization factor" and dropped while comparing different hypothesis.

Everyday life questions are often bayesian

Few examples:

- I'm not feeling so well $\rightarrow$ Am I sick ?
- There are clouds $\rightarrow$ will it rain?
- I go out in a bar $\rightarrow$ will I end up drunk?
- I attend to a school statistics $\rightarrow$ will I learn something?

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Always the same thinking:

1. you observe a fact
2. you wonder the probability of something, given you this fact happened
3. you have (somtimes rough/wrong) prior, based on past knowledge
4. your brain applies Bias theorem, even you don't know it!

Continous random variables

Generalization to the continuous case

- There is a whole continuum of outcome (realization) for X
- Probability described by a density probability function (PDF), $f(x)$:

$$P(x \in [x_1, x_2]) = \int_{x_1}^{x_2} f(x)\mathrm{d}x \quad ; \quad \int_{\Omega} f(x)\mathrm{d}x = 1$$

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Moments definitions

$$\mu = \int_{\Omega} x\, f(x)\mathrm{d}x \ ; \ \sigma^2 = \int_{\Omega} (x-\mu)^2 f(x)\mathrm{d}x \ ; \ m_r = \int_{\Omega} \left(\frac{x-\mu}{\sigma}\right)^r f(x)\mathrm{d}x$$

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Characteristic function of a PDF

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- moments are the Taylor expension coefficients: $m_r = (-i)^r \left.\frac{\mathrm{d}^r \varphi_X}{\mathrm{d}t^r}\right|_{t=0}$

Important PDF examples

Binomial law: efficiency, trigger rates, …

$$B(k;n,p) = C_k^n p^k (1-p)^{n-k}, \mu = np, \sigma = \sqrt{np(1-p)}$$

Poisson distribution: counting experiments, hypothesis testing

$$P(n;\lambda) = \frac{\lambda^n e^{-\lambda}}{n!}, \mu = \lambda, \sigma = \sqrt{\lambda}$$

Gauss distribution (aka Normal): many use-case (asymptotic convergence)

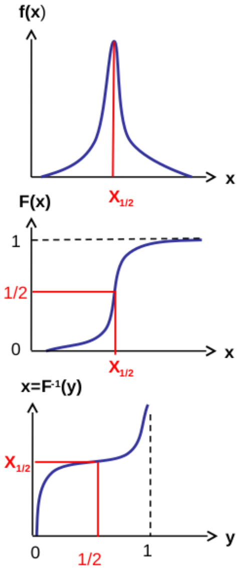
$$f(x;\mu,\sigma) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

Cauchy distribution (aka Breit-Wigner): particle decay width, ...

$$f(x;x_0,\gamma) = \frac{1}{\pi\gamma\left[1+\left(\frac{x-x_0}{\gamma}\right)^2\right]}$$

μ and σ not defined (divergent integral)

Cumulative distribution and quantiles



Probability density function: $f(x)$

Cumulative distribution: $F(x)=y$

Inverse cumulative distribution: $x=F^{-1}(y)$

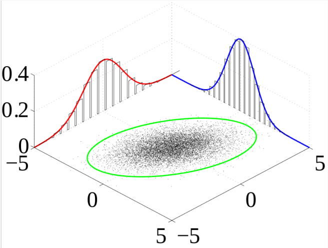
Median: x such that $F(x)=1/2 \rightarrow x_{1/2} = F^{-1}(1/2)$

Quantile of order α: $x_\alpha = F^{-1}(\alpha)$

Multidimensional PDF

How to describe several random variables simulataneously?

- X and Y are two random variables $\rightarrow$ PDF is f_{XY},
- several questions can be asked about X, Y or both.



- Probability that $X \in [x, x + \mathrm{d}x]$ and $Y \in [y + \mathrm{d}y]$:
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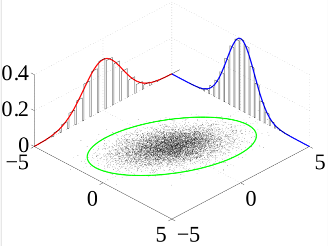

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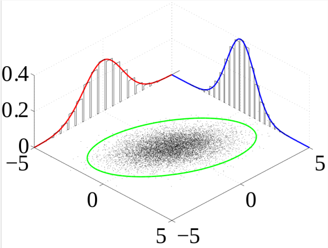

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Multidimensional normal distribution

$$f(\vec{x};\vec{\mu},\Sigma) \;=\; \frac{1}{\sqrt{(2\pi)^n \det\Sigma}}\,\exp\left(-\frac{1}{2}\,(\vec{x}-\vec{\mu})^T\,\Sigma^{-1}\,(\vec{x}-\vec{\mu})\right)$$

- $\vec{\mu}$ mean position of $\vec{x}$, Σ covariance matrix

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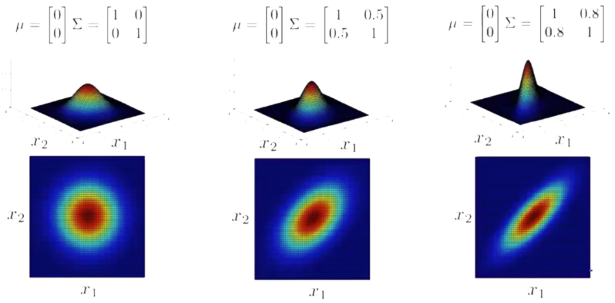

Central limit theorem

Caution: what follows is *not* mathematically rigorous

If n random variables $\{X_i\}$ are distributed according to the same PDF f_X with a defined mean μ_x and a std σ_x, **then** the random variable $Y = \frac{1}{n}(X_1 + ... + X_n)$ is following a normal distribution of mean μ_x and std $\sigma_x/\sqrt{n}$.

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For 2 variables $Y = X_1 + X_2$

- The PDF of Y is $f_Y(y) = \int f_{X_1}(x_1) \times f_{X_2}(y - x_1)\mathrm{d}x_1 \to$ convolution!
- Caracteristic function: $\varphi_Y(t) = \varphi_{X_1}(t) \times \varphi_{X_2}(t) = \varphi_X(t)^2$ - same PDF!
- 1st and 2nd moments known : $\varphi_x(t) \sim$ 2nd order Taylor expansion

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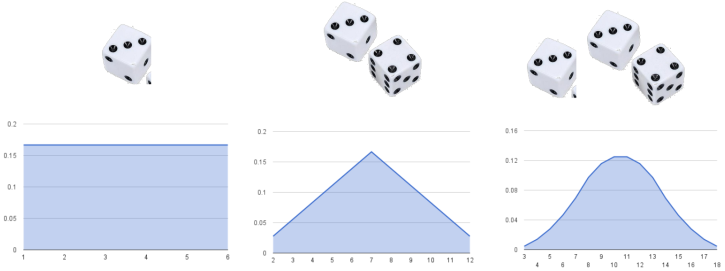
Generalizing for sum of n variables:

- $\varphi_Y(t) = \varphi_x(t)^n \sim \left(1 - \frac{t^2}{n}\right)^n \rightarrow e^{-t/2}$ for $n \rightarrow \infty$
- going back to real space, a normal distribution is obtained

N.B. this reasonning doesn't explain why $\sigma_Y = \sigma_x/\sqrt{n}$, this needs to properly re-scale Y.

Central limit theorem – continued

One way to understand why it works



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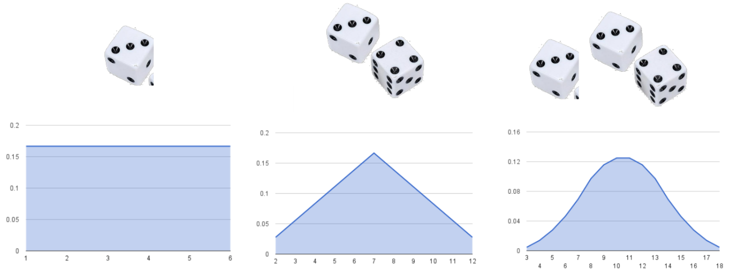

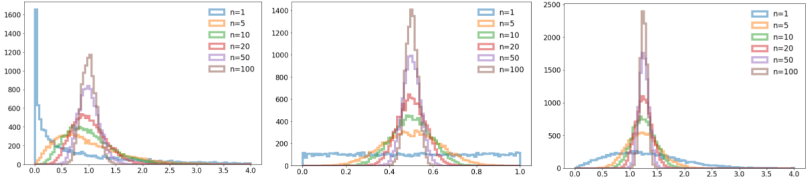

Central limit theorem - homework

Proof

Proove that $\sigma_Y = \sigma_X/\sqrt{n}$ with the proper scalings to define Y.

Application

Proove, using the CLT, that a Poisson distribution $P(n;\lambda)$ tends to a normal distribution for large numbers.

Hint: $N = 1 + 1 + 1.... + 1$ N-times

Function of random variables

Final observable is very often a combination of (random) variable.

- $\mathcal{O} = g(X_1, X_2, ..., X_n) \equiv g(\vec{X})$. $\mathcal{O}$ is also a random variable
- what is the PDF of $\mathcal{O}$, knowing $f_{\vec{X}}$? *Not trival* (think about a sum)!

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Comments:

- these equations are known as error propagation
- **this procedure is not exact** and relies on Taylor expansion
- only 1st and 2nd moments of $\vec{X}$ are needed (or their estimators)

Error propagation formula is not exact

(Counter) example with one variable

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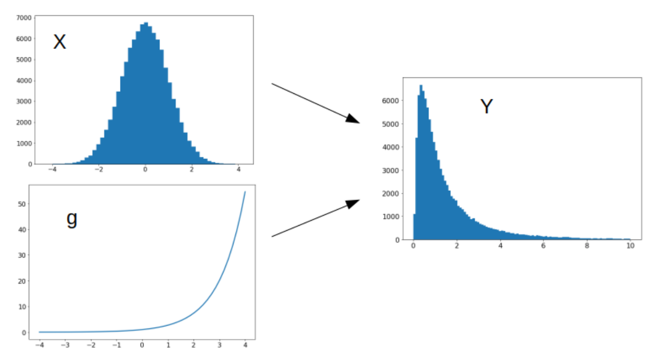


Keywords and concepts

Part I: statistics

descriptive statistics – sample – mean – (co)variance – (de)correlation

Part II: probability

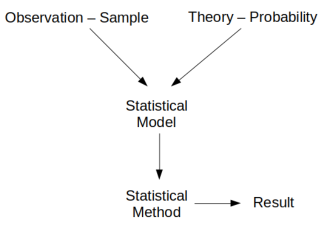
Bias theorem – prior – posterior – random variable – (marginal) PDF –
moments – caracteristic function – (in)dependent variables –
CLT – error propagation

Content

1. Statistics
2. Probability
3. Statistical model
4. The two big schools
5. Parameter estimation and hypothesis testing

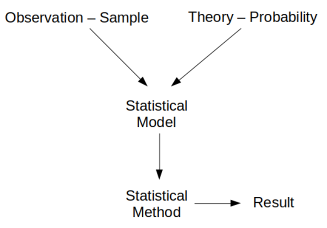
Statistical model

Statistical model: what, why, how



What? missing piece between the "sample" and "probablity"

Statistical model: what, why, how



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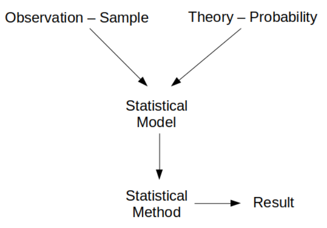

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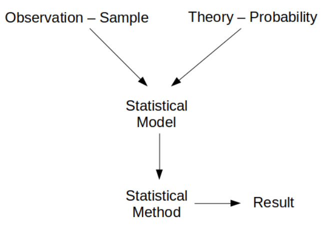

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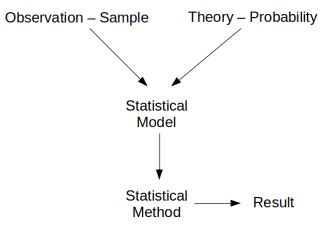
How? physical model + fluctuation model = statistical model

Statistical model: what, why, how



What? missing piece between the "sample" and "probablity"

Why? because a measurement is always one realization of a random variable.

N.B. Statistical methods will be introduced in the next sections

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Statistical model ingredients:

- (pseudo-)observations, written $\vec{x}$ (or x)
- parameters we want: parameter(s) of interest, written $\vec{\mu}$ or μ (POI)
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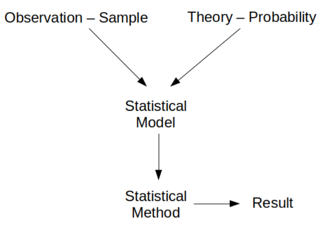

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A statistical model is also called likelihood function $\mathcal{L}(\vec{\mu}, \vec{\theta}; \vec{x})$. It can be seen as the probability that the physical model predicts the observable $\vec{x}$, given the parameters $(\vec{\mu}, \vec{\theta})$.

Statistical model: particle physics experiment - I

Model ingredients:

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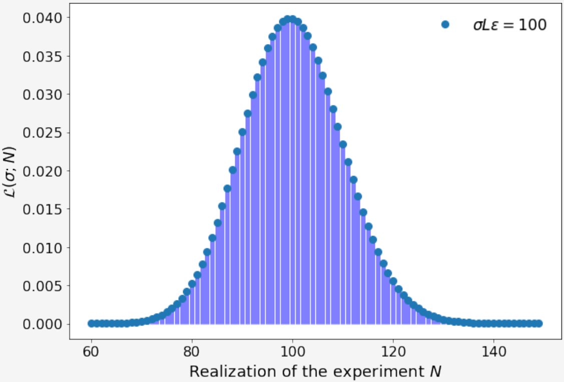
Statistical model

$$\mathcal{L}(\sigma; N) = e^{-\sigma L \epsilon} \frac{(\sigma L \epsilon)^N}{N!}$$

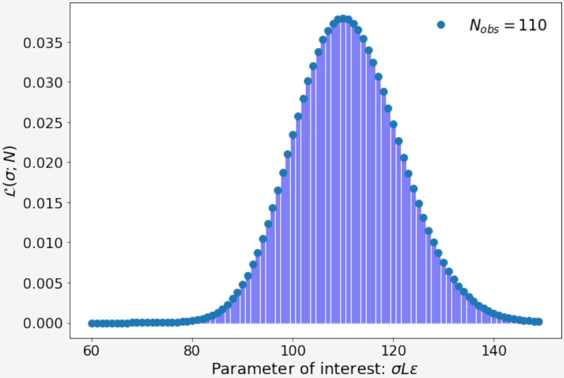
Illustration of the Likelihood

Given a value of σ, what's the "probability" to observe N ?



Anticipation: frequentist "usage" of the likelihood

If we observed a value for N, what's the "probability" that $\sigma = X$?



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Systematic uncertainties turn numbers into new random variables.
They PDFs depends on parameters, we don't really care about: nuisances parameters. *Example* of systematic parametrization:

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More realistic statistical model

In realistic experiment:

- histograms are used - not only event counts
- several samples can be considered simultaneously
- Many processes are usually needed to describe data
- Some are known (backgrounds), others are to be measured (signals)

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- Why do we multiply terms?

A first discussion on uncertainties

Caution

Systematic uncertainty estimation *and* treatment is not an exact science.

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Implications:

- arbitrariness (and a loooot of discussion that go with it)
- always check the robustness of the conclusion wrt to those
- that's the way it is, no choice! $\rightarrow$ be *smartly* practical!

Keywords and concepts

Part I: statistics

descriptive statistics – sample – mean – (co)variance – (de)correlation

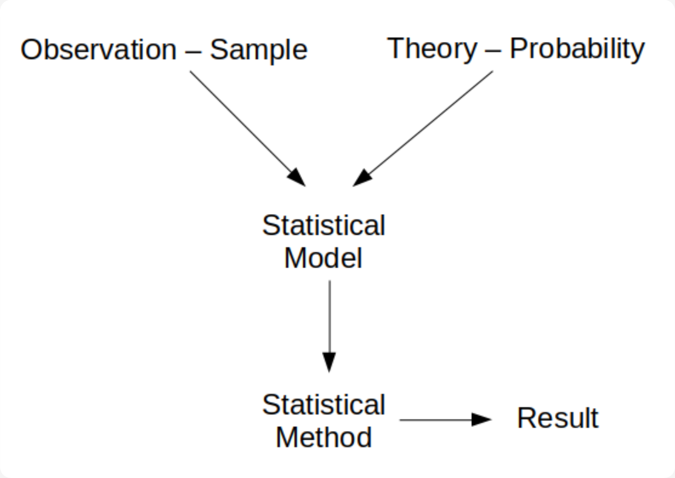
Part II: probability

Bias theorem – prior – posterior – random variable – (marginal) PDF – moments – caracteristic function – (in)dependent variables – CLT – error propagation

Part III: statistical model

Likelihood – nuisance parameter – parameter of interest – systematic uncertainties

Overview



Observation – Sample
Theory – Probability
Statistical Model
Statistical Method
Result

Content

1. Statistics
2. Probability
3. Statistical model
4. The two big schools
5. Parameter estimation and hypothesis testing

The two big schools

Fequentist versus bayesian

	Frequentist	**Bayesian**
probability	frequency of occurence	degree of belief

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- frequenstist $\rightarrow$ probability of observation, given a model
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Methodologies

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- *bayesian:* exploits the Bayes theorem to compute the posterior $P(para|obs)$, using the prior $P(para)$ and $P(obs|para)$ - the **likelihood**

Is a flipping coin tricked?

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We toss a coin 113 times and we got 'tail' 68 times. Is the coin tricked?

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- N (known parameter): number of tosses
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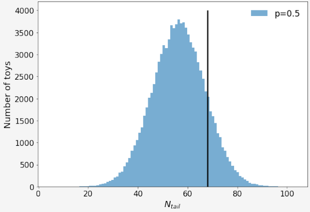
Let's try to analyze the same experiment with both frequentist and bayesian approaches

Is a flipping coin tricked? Frequentist approach



Toys with a normal coin

14.1% of pseudo-experiments using an normal coin would lead to $N_{tail} \geq 68$

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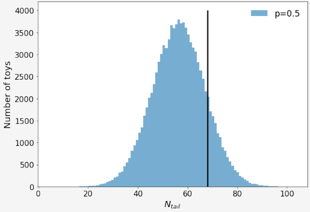

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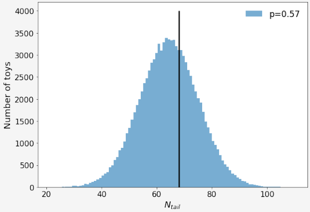

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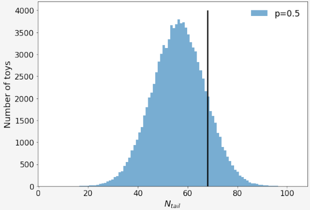

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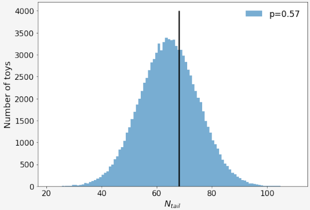

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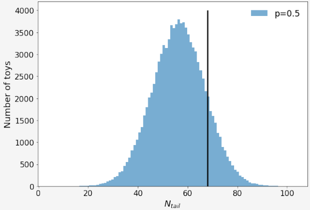

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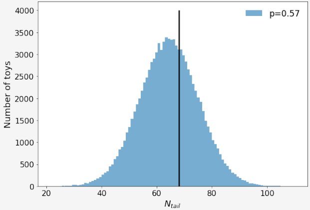

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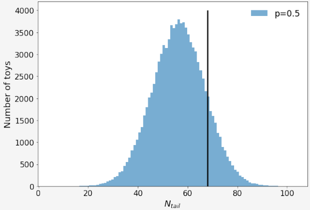

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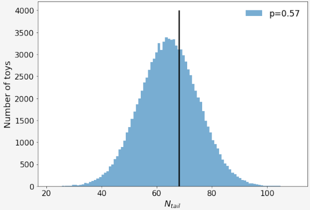

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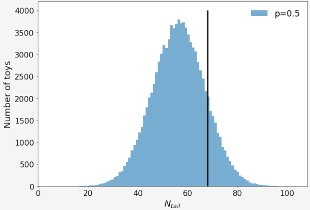

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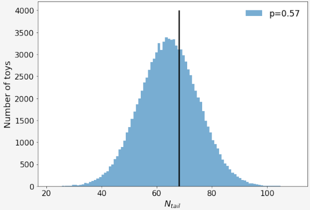

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  $\rightarrow$ this question has no sense in frequentist

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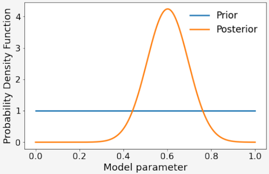

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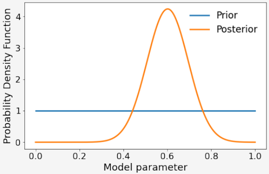

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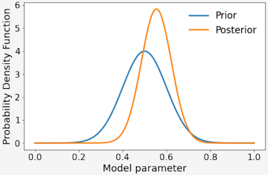

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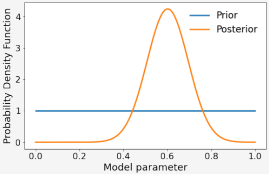

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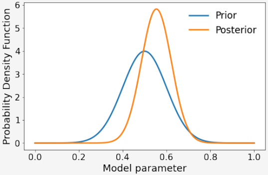

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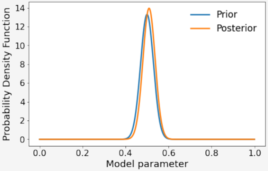

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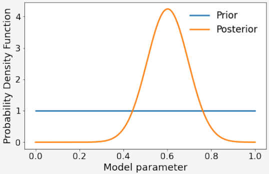

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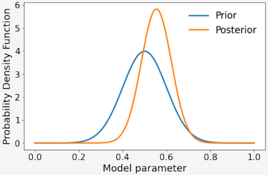

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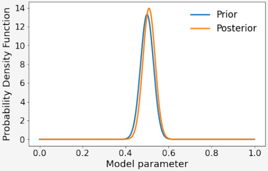

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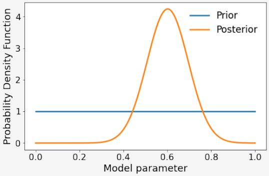

Most probable value is $p = 0.60$

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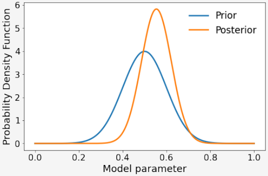

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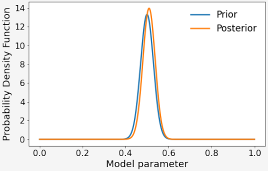

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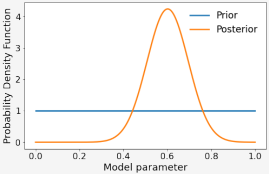

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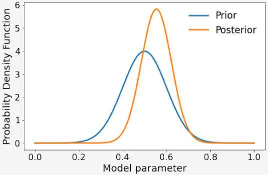

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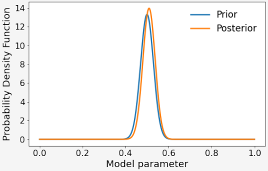

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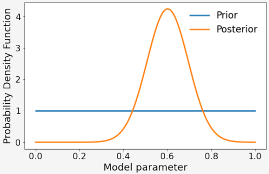

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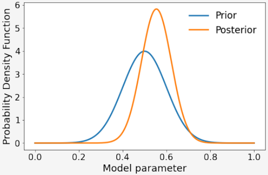

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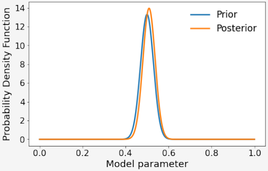

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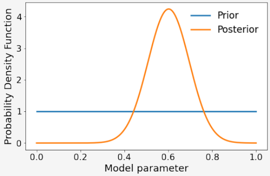

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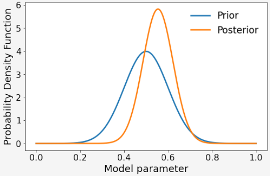

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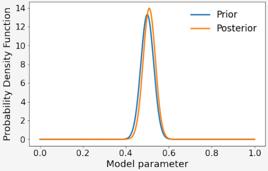

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  $\rightarrow$ expect it depends on the choice of the prior ...

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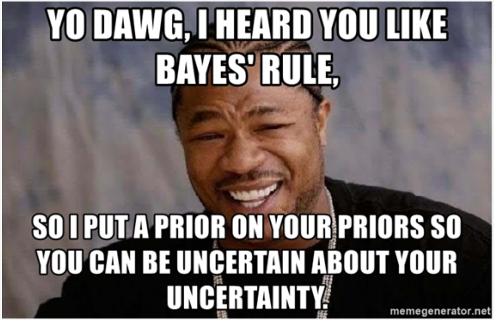

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Well … statistics can't say for sure (science of handling the "not fully certain"). The unambiguous answer exists only in the limit of infinite number of measurements. **What both methods say in that case?**

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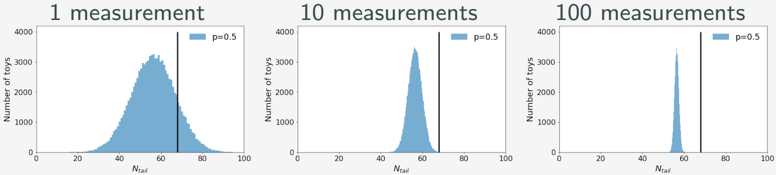

Frequentists say "Yes, the coin is tricked!"

Certainty comes from the extremely low fraction of pseudo-experiments of a normal coin, that would lead the observed result.

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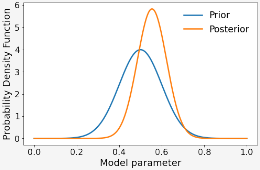
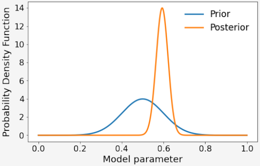
Bayesian, wide prior

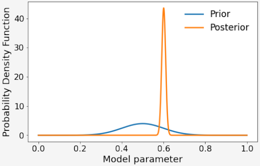
$N_{meas} = 1$



$N_{meas} = 10$



$N_{meas} = 100$



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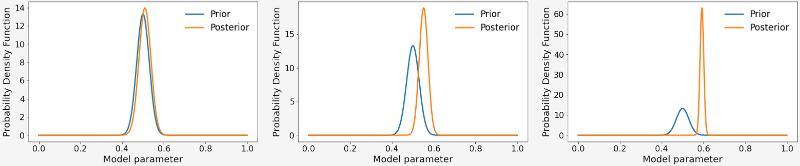
Bayesian, narrow prior

$N_{meas} = 1$ $N_{meas} = 10$ $N_{meas} = 100$



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Bayesian, posterior for various priors

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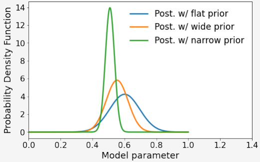

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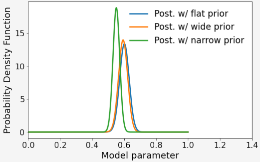

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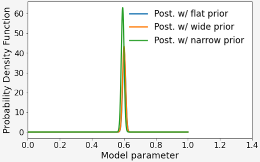

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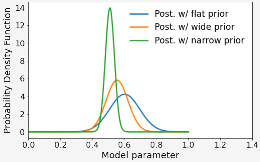

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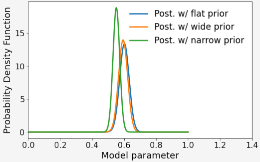

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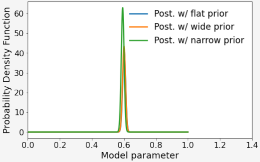

Bayesians also say "Yes, the coin is tricked!"

Frequentist v.s. Bayesian: what to take away

1. Both approaches handle differently the "non fully certain"

2. Final conlusions should be compatible, even if the question they adress are not exaclty the same.

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One thing I like from the two approaches

- probability intepretation from the frequentist
- ranking two theories using their probability, called Bias factors

Keywords and concepts

Part I: statistics

descriptive statistics – sample – mean – (co)variance – (de)correlation

Part II: probability

Bias theorem – prior – posterior – random variable – (marginal) PDF – moments – caracteristic function – (in)dependent variables – CLT – error propagation

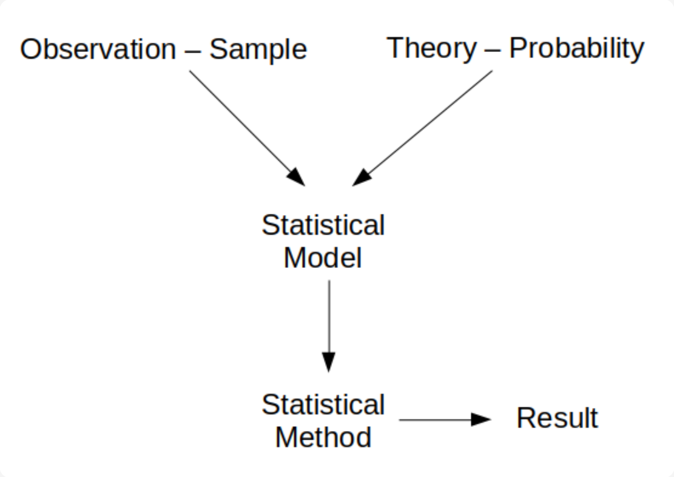
Part III: statistical model

Likelihood – nuisance parameter – parameter of interest – systematic uncertainties

Part IV: The two big school

Frequentist – occurence frequency – pseudo-data (toys) – bayesian – degree of belief

Overview


Observation – Sample
Theory – Probability
Statistical
Model
Statistical
Method
Result

Content

1. Statistics
2. Probability
3. Statistical model
4. The two big schools
5. Parameter estimation and hypothesis testing

Parameter estimation and hypothesis testing

Program of this section

Baics of parameter estimation in both frequentist and bayesian,
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3. **Coming back on nuisance parameters** (*i.e.* uncertainties on the model)

Frequentist approach: estimators

Definition: random variable which gives a 'good' estimate of your parameter of interest ($\hat{\mu} = \frac{1}{N} \sum_i x_i$ as estimator of $\mathbb{E}[X]$). Estimator depends on observation $\hat{\mu}(x_1, ..., x_n)$ and is *not* constant. N_{meas} needed to assess its quality.

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Two important examples of estimators

1. Maximum likelihood estimator (MLE): $\hat{\mu}$ which maximizes $\mathcal{L}(\mu; x)$
   $\to$ numerically easier to minimze $-2\ln \mathcal{L}(\mu; x)$ - negative log likelihood (NLL)
2. χ^2 estimator: $\hat{\mu}$ which minimizes $\chi^2(\mu) \equiv \sum_i w_i \left(X_i^{pred}(\mu) - x_i\right)^2$

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Question 2 for the audience:
Why consistency and bias of an estimator are different?

Example: linear fit

Model $N^{pred}(p_0, p_1; t) = p_0 + p_1 t$

4 estimators (or "cost function") are used:

$$-2 \log \mathcal{L}_{poisson}$$

$$\chi^2(p_0, p_1) = \sum_i (N_i^{pred}(p_0, p_1) - N_i)^2$$

$$\chi^2_{Pearson}(p_0, p_1) = \sum_i \left(\frac{N_i^{pred}(p_0, p_1) - N_i}{\sqrt{N_i^{pred}(p_0, p_1)}} \right)^2$$

$$\chi^2_{Neyman}(p_0, p_1) = \sum_i \left(\frac{(N_i^{pred}(p_0, p_1) - N_i)^2}{\sqrt{N_i}} \right)^2$$

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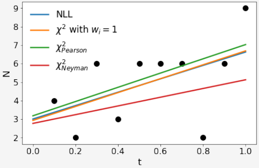

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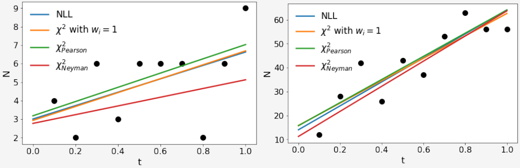

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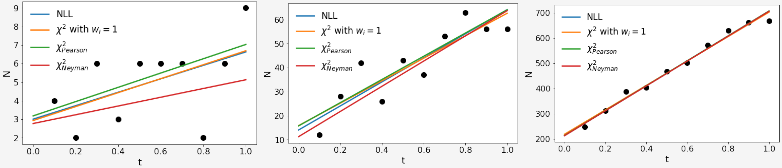

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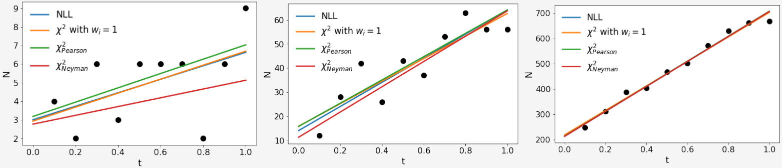

Comments:

- $\chi^2_{pearson} \equiv -2\log\mathcal{L}_{Gauss} \approx -2\log\mathcal{L}_{Poiss}$ for large numbers

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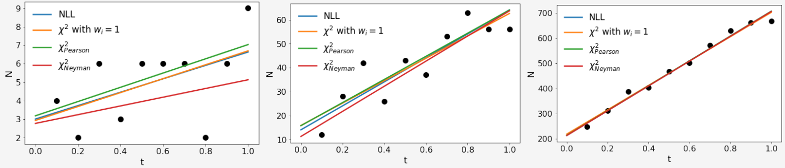

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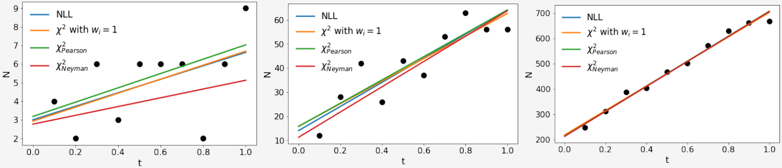

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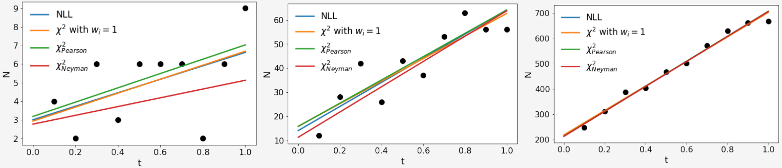

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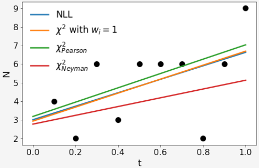

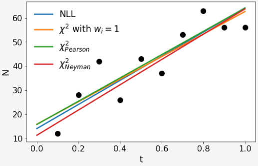

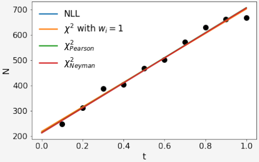

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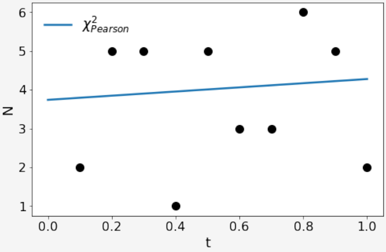
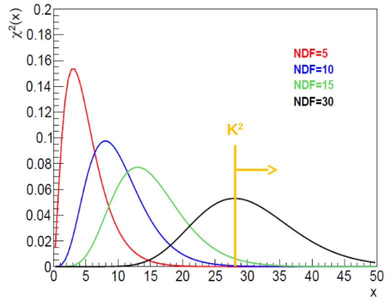
  $\rightarrow$ goodness-of-fit is possible to evaluate since χ^2 PDF is known

The basics of goodness-of-fit





$\chi^2_{min} = 6.7$ with 10 data points ($nDoF = 10$) $\rightarrow$ blue PDF tells us this is a good fit, *even if* not a point is on the line.

We can actually compute the fraction of pseudo-data that would lead to a higher χ^2 (p-value), to quantify this statement.

Food for thought

1. Perform a fit of an histogram in ROOT, with quite wide binning. Do you recover the true value? Does the result depends on the number of bins? How to solve it?

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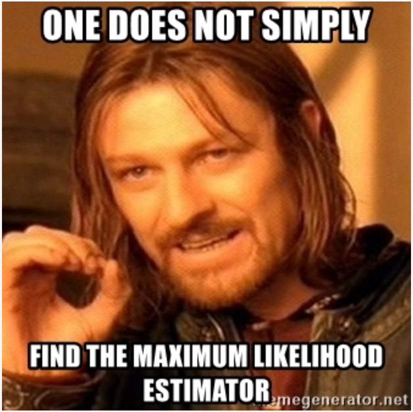
2. Imagine you have one dataset, but you want to fit simultaneously two distributions of these events. How to write the χ^2?

Frequentist parameter uncertainty



Frequentist parameter uncertainty

Confidence interval and level $\mu \in [\mu_{min}, \mu_{max}]$ @ α CL

- $\equiv$ the true value is in $[\mu_{min}, \mu_{max}]$ in α% of all possible realisations
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n is called "number of σ" and $\alpha(n)$ is known for a normal PDF:

- $\alpha(1) = 68\%$
- $\alpha(1.64) = 90\%$
- $\alpha(1.95) = 95\%$
- $\alpha(2) = 95.4\%$
- $\alpha(3) = 99.7\%$
- $\alpha(5) = 99.99994\%$

Frequentist parameter uncertainty

Quality of a given confidence interval

- CI $\equiv$ random variable: consider the limit of ∞ number of meas.
- Coverage $\equiv$ probability P that the true parameter *actually is* in C
- "Confidence level = what we target" while "coverage = what we get"

The 3 cases

1. $P = \alpha$: perfect coverage $\rightarrow$ ideal
2. $P > \alpha$: over-coverage $\rightarrow$ acceptable (conservative conclusions)
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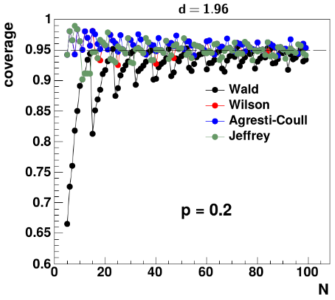
In practice: estimating coverage can be done using toys experiment (CPU-intensive for realistic models).

Example: binomial distribution, with parameter of interest p

$$P(k; N, p) = \binom{N}{k} p^k (1-p)^{N-k}$$

$$\hat{p} = \frac{k}{N}$$

$$p \in \left[\hat{p} - d\sqrt{\frac{\hat{p}(1-\hat{p})}{N}}; \hat{p} + d\sqrt{\frac{\hat{p}(1-\hat{p})}{N}}\right] \quad \text{(Wald interval)}$$



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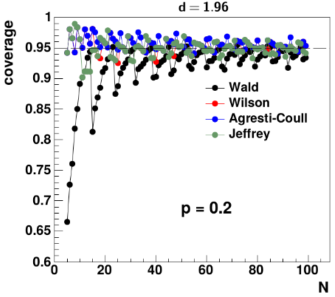

Take away messages:

- notation $\mu = X^{+Y}_{-Z}$ (assuming 68% C.L.) is sometimes only indicative
- only object which contains the full information is likelihood
- OK to manipulate these approximate quanties - just know what they are(n't)

Bayesian parameter estimation

From the posterieur to the final value: given $f(\mu) \equiv P(\mu|data)$

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  - Replace $\mathbb{E}[\mu]$ by the mode, or the median ...

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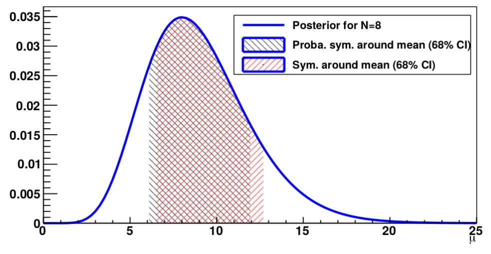

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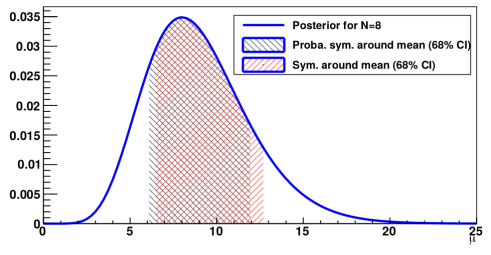

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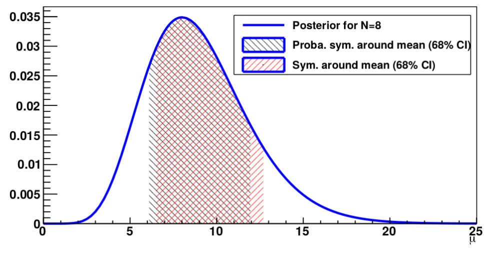

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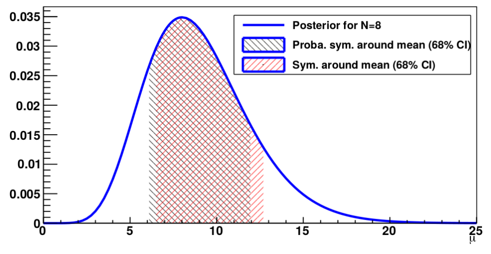

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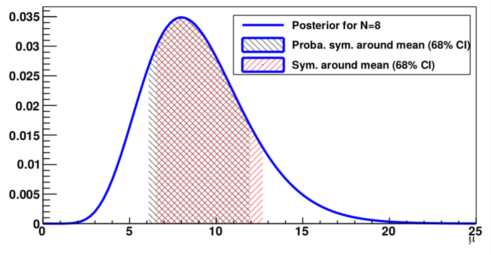

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  (2) Why the 3 properties of frequentist estimator are defined in baysien?

Coming back to model uncertainties - I

Frequentist approach imagine you measure energy response r_E of a detector using a dedicated data d_E

- this measure is described by a likelihood $\mathcal{L}_{energy}(r_E, d_E)$
- the parameter of interest will be better known with more data
- this unknown can be added to the stat model using the full likelihood

$$\mathcal{L}(\mu, r_E; data, d_E) = \mathcal{L}(\mu, ; data)\mathcal{L}_{energy}(r_E, d_E)$$

- this is notion of auxiliary measurement.
- $\mathcal{L}_{energy}(r_E, d_E)$ is usally too complex to be implemented.
- One uses its approximation (Taylor Expension of order 2 of NLL around the min, leading to a gaussian likelihood)

Coming back to model uncertainties - II

Bayesian approach imagine you have a calculation with some approximations, to which an uncertainty is associated.

- this uncertainty is closer to a degree of beleif
- a prior $\pi(\theta)$ is required to quantify, were the true value of θ is more likely to be
- this unknown can be added to the stat model using the full likelihood

$$\mathcal{L}(\mu, \theta; \mathit{data}) = \mathcal{L}(\mu, ; \mathit{data})\, \pi(\theta)$$

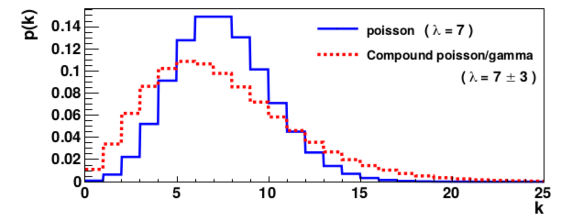
- this final likelihood is marginalized over θ:

$$\mathcal{L}_m(\mu; \mathit{data}) = \int \mathcal{L}(\mu, \theta; \mathit{data})\, \pi(\theta) \mathrm{d}\theta$$

- Interpretation: average all possible situations (defined by a θ value), accounting for the probability to actually have this value

Coming back to model uncertainties - III

Example of marginalization



Coming back to model uncertainties - III

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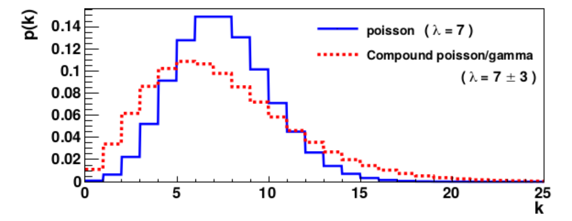

What's the proper way to implement uncertainties?

- no absolute answer to this question → arbitrariness
- make your choice depending on the context (ease interpretation or calculation, or ...?)
- always check the robustness of your conclusion wrt these choices

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Most naive approch: event count as test statistics $t = N$

- *e.g.* H_1 predicts $N_1 = 110$, while H_0 predicts $N_1 = 100$
- observation $N_{obs} = 112$: do I reject the signal hypothesis?
- Steps of test hypothesis
  - find distribution of t in both hypothesis $f(t|H_0)$ and $f(t|H_1)$
  - check where t_{obs} fall wrt to $f(t|H_0)$ and $f(t|H_1)$
  - conclude with a confidence level ($p-$value)

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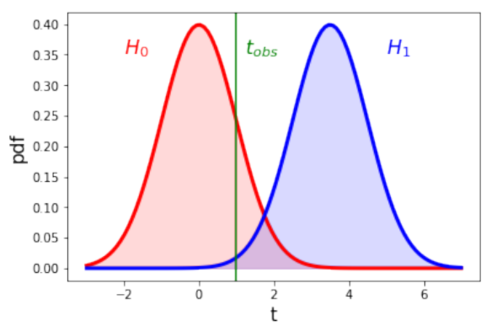

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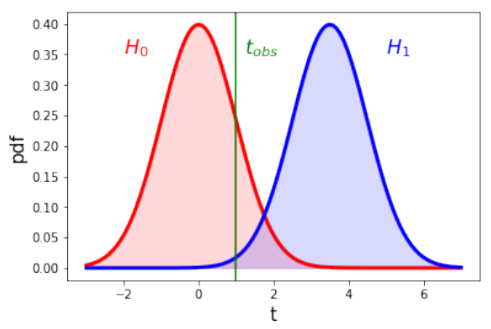
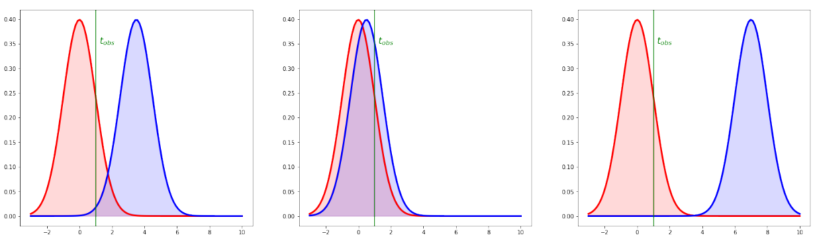


Quantitative agreement with an hypothsis: *p*-value

p-value = probability to observe what you observed in measurement or "more extreme" values

How to find exclusion limit



→ Increase the signal until the signal hypothesis get rejected (at a given confidence level).

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Egon Pearson


Jerzy Neyman

Pearson-Neyman Lemma (1933)

- the most powerful statistical test is **N**egative **L**og **L**ikelihood ratio

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In practice: hunders or thousands of event counts!

Keywords and concepts

Part I: statistics

descriptive statistics – sample – mean – (co)variance – (de)correlation

Part II: probability

Bias theorem – prior – posterior – random variable – (marginal) PDF – moments – caracteristic function – (in)dependent variables – CLT – error propagation

Part III: statistical model

Likelihood – nuisance parameter – parameter of interest – systematic uncertainties

Part IV: The two big school

Frequentist – occurence frequency – pseudo-data (toys) – bayesian – degree of belief

Part VI: Parameter estimation & hypothesis testing

estimator and its properties – χ^2 – confidence/credibility level/interval – coverage – p-value – LLR

Concluding remarks

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$\rightarrow$ not a single way $\rightarrow$ some arbitrariness

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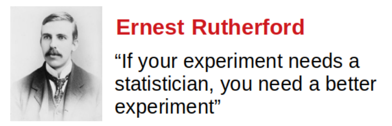

Thanks for you attention !