Diluted spin glass models

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The Potts model

- a model of magnetism
- a model of assortative (ferromagnetic) or disassortative (antiferromagnetic) interaction
- complexity of counting/sampling

The Potts model

- G = a (sparse) graph
- $[q] = \{1, \ldots, q\}$ = set of spins/colours
- for $\sigma : V(G) \to [q]$ define

$$H_G(\sigma) = \sum_{vw \in E(G)} \mathbf{1}\{\sigma_v = \sigma_w\} = \#\text{monochromatic edges}$$

- the special case $q = 2$ is called the Ising model
- in the Ising model, spins are labeled ± 1

The Potts model

- the Boltzmann distribution

$$\mu_{G,\beta}(\sigma) = \frac{\exp(\beta H_G(\sigma))}{Z_{G,\beta}}$$

$$Z_G(\beta) = \sum_{\sigma} \exp(\beta H_G(\sigma))$$

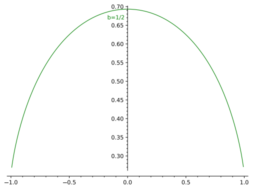
- ferromagnetic: $\beta > 0$
- anti-ferromagnetic: $\beta < 0$

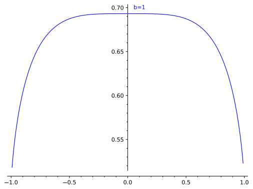
The Curie-Weiss model

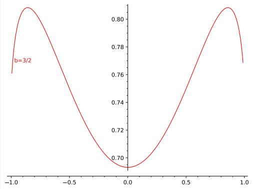
The Ising ferromagnet on the complete graph

- let $G = K_n$ be the complete graph
- let $q = 2$, $\beta = b/n$ with $b > 0$
- *even in this simple case, a phase transition occurs!*

The Curie-Weiss model







The partition function

We have

$$\lim_{n\to\infty} \frac{1}{n} \log Z_{K_n,\beta} = \max_{-1\le m\le 1} \phi(b,m)$$

$$\phi(b,m) = \frac{bm^2}{2} - \frac{1+m}{2}\log\frac{1+m}{2} - \frac{1-m}{2}\log\frac{1-m}{2}$$

The Curie-Weiss model

Proof

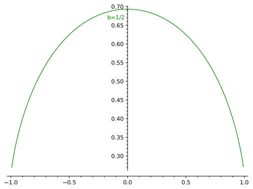
- for $\sigma \in \{\pm 1\}^n$ define $\lambda(\sigma) = \frac{1}{n}\sum_{i=1}^n \sigma(x_i)$
- then $\dfrac{1}{n}\sum_{i<j} \sigma(x_i)\sigma(x_j) = \dfrac{n}{2} - \dfrac{n}{2}\lambda(\sigma)^2$
- therefore,

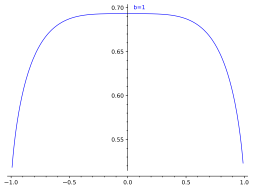
$$Z = \exp(-\beta n/2)\sum_{\lambda}\binom{n}{n(1+\lambda)/2}\exp\left[\frac{\beta n}{2}\cdot\lambda^2\right].$$

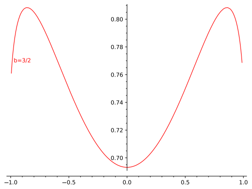
- by Stirling's formula $\frac{1}{n}\ln\binom{n}{n(1+\lambda)/2} \sim h((1+\lambda)/2)$.
- hence,

$$Z = \exp(o(n))\sum_{\lambda}\exp(n\phi_\beta(\lambda)) = \max_{\lambda}\exp(n\phi_\beta(\lambda) + o(n))$$

The Curie-Weiss model







The Boltzmann distribution

- for any spin i the Boltzmann distribution satisfies

$$\left\langle \boldsymbol{\sigma}_i, \mu_{K_n,\beta} \right\rangle = 0$$

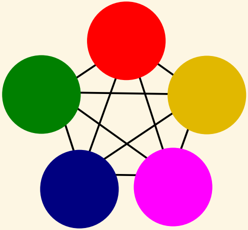
- if $b \leq 1$ then pairwise correlations disappear:

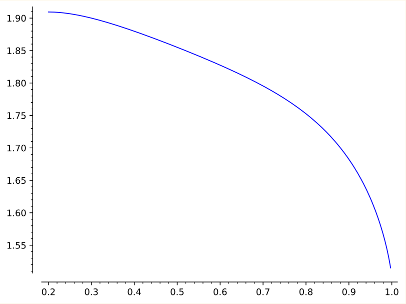
$$\left\langle \boldsymbol{\sigma}_i \boldsymbol{\sigma}_j, \mu_{K_n,\beta} \right\rangle = 0$$

- but for $b > 1$ pairwise correlations persist:

$$\left\langle \boldsymbol{\sigma}_i \boldsymbol{\sigma}_j, \mu_{K_n,\beta} \right\rangle > 0$$

The Curie-Weiss Potts model

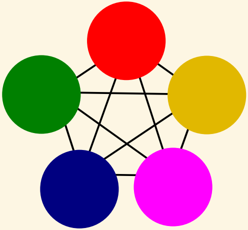


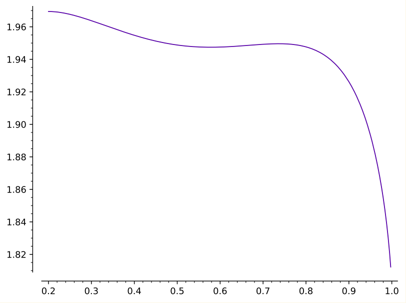


The Potts ferromagnet with $q = 5$ colours [L21]

- unique paramagnetic
- non-unique paramagnetic
- non-unique ferromagnetic
- unique ferromagnetic

The Curie-Weiss Potts model





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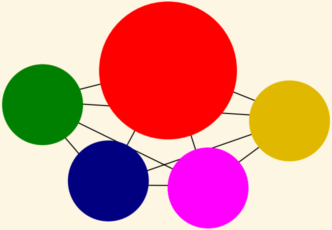

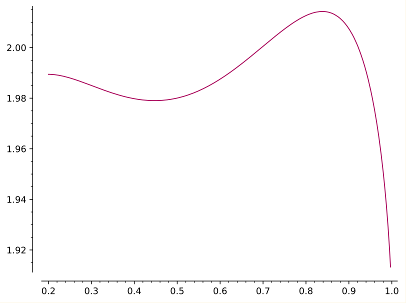

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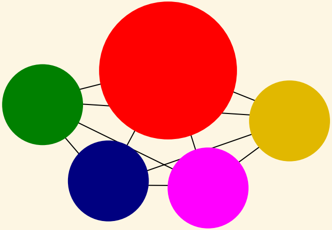

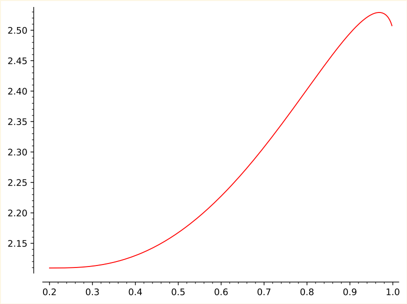

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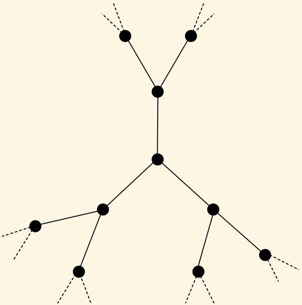
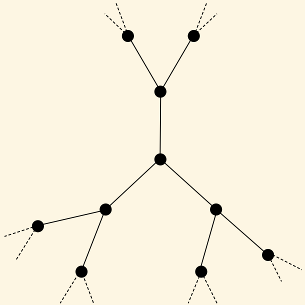
The Potts model on sparse random graphs



- $\mathbb{G} = \mathbb{G}(n, d)$ =random d-regular graph
- $Z_{\mathbb{G},\beta}$, $\mu_{\mathbb{G},\beta}$ are random
- *what does $\mu_{\mathbb{G},\beta}$ look like with high probability?*

The Potts model on sparse random graphs



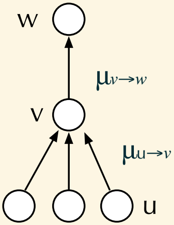
Global intuition

- the edges of $\mathbb{G}$ are spread uniformly
- perhaps $\mathbb{G}$ is not so different from the complete graph

Local geometry

- locally, $\mathbb{G}$ converges to a d-regular tree
- short-range interactions matter!

The Potts model on sparse random graphs



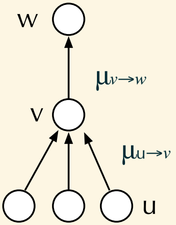
The replica symmetric ansatz [MP00]

- assume the absence of long-range correlations
- we can write the Belief Propagation equations for messages

$$\mu_{v\to w}(c) \propto \prod_{u\in\partial\setminus\{w\}} \left(1+(e^{\beta}-1)\mu_{u\to v}(c)\right)$$

- these are exact on *acyclic* graphs

The Potts model on sparse random graphs



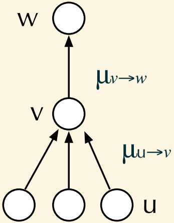
The replica symmetric ansatz [MP00]

- the “correct” messages maximise the Bethe free energy

$$\mathscr{B}_{\mathbb{G},\beta}(\boldsymbol{\mu}) = \frac{1}{n}\sum_{v\in V(\mathbb{G})}\log\sum_{c\in[q]}\prod_{u\in\partial v}\left(1+(\mathrm{e}^{\beta}-1)\mu_{u\to v}(c)\right) - \frac{1}{n}\sum_{uv\in E(\mathbb{G})}\log\left(1+(\mathrm{e}^{\beta}-1)\sum_{c\in[q]}\mu_{u\to v}(c)\mu_{v\to u}(c)\right)$$

- the max on $\boldsymbol{\mu}$ ought to approximate $\frac{1}{n}\log Z_{\mathbb{G},\beta}$ w.h.p.

The Potts model on sparse random graphs



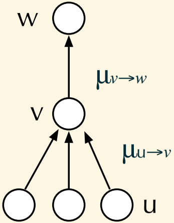
The replica symmetric ansatz

- due to regularity, we expect that all messages are *identical*

$$\mu(c) \propto \left(1+(e^{\beta}-1)\mu(c)\right)^{d-1}$$

$$\mathscr{B}_{d,\beta}(\mu) = \log \sum_{c\in[q]} \left(1+(e^{\beta}-1)\mu(c)\right)^{d} - \frac{d}{2}\log\left(1+(e^{\beta}-1)\sum_{c\in[q]}\mu(c)^2\right)$$

The Potts model on sparse random graphs



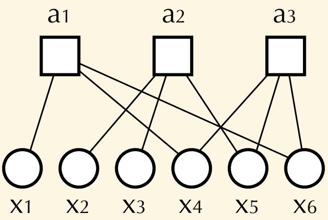
Theorem [GSVY13]

For all $d \geq 3, q \geq 2, \beta > 0$ we have

$$\lim_{n \to \infty} \frac{1}{n} \log Z_{\mathbb{G},\beta} = \max_{\mu} \mathscr{B}_{d,\beta}(\mu)$$

in probability.

Belief Propagation on factor graphs

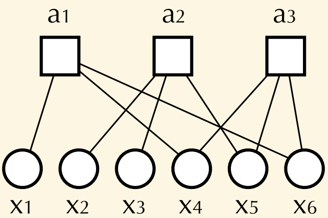


Factor graphs

A factor graph G comprises

- variable nodes $V(G)$ ranging over a finite $\Omega \neq \emptyset$
- factor nodes $F(G)$
- a weight function $\psi_a : \Omega^{\partial a} \to [0,\infty)$ for each $a \in F(G)$

Belief Propagation on factor graphs

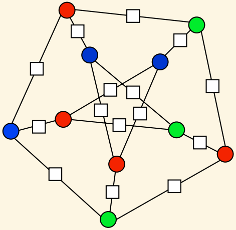


The Boltzmann distribution

$$\mu_G(\sigma) = \frac{1}{Z_G} \prod_{a \in F(G)} \psi_a(\sigma_{\partial a}) \qquad (\sigma \in \Omega^{V(G)})$$

$$Z_G = \sum_{\sigma \in \Omega^{V(G)}} \prod_{a \in F(G)} \psi_a(\sigma_{\partial a}) \qquad (\sigma \in \Omega^{V(G)})$$

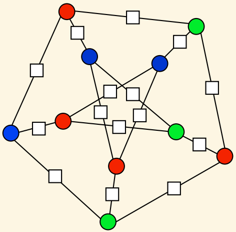
Belief Propagation on factor graphs



Example: Potts model

- $\Omega = [q] = \{1, \ldots, q\}$
- $\psi_a(\sigma) = \exp(\beta \mathbf{1}\{\exists c \in [q] \forall x \in \partial a : \sigma_x = c\})$

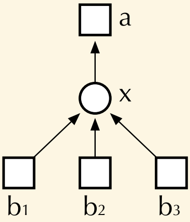
Belief Propagation on factor graphs

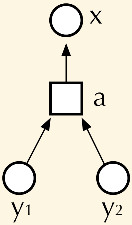


Random factor graphs

- a lot of the time we study sparse *random* factor graphs
- examples of this include binomial random graphs or random regular graphs
- “planted” models such as the stochastic block model

Belief Propagation on factor graphs

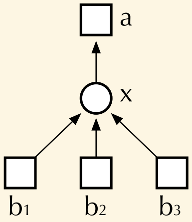


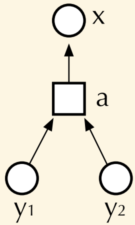


Belief Propagation

- associate messages $\mu_{G,x\to a}(\cdot)$, $\mu_{G,a\to x}(\cdot)$ with the edges
- each message is a probability distribution on Ω

Belief Propagation on factor graphs



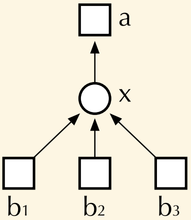


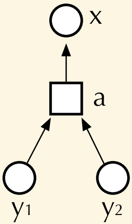
Belief Propagation

$$\mu_{G,x\to a}(s) \propto \prod_{b\in\partial x\setminus a} \mu_{G,b\to x}(s)$$

$$\mu_{G,a\to x}(s) \propto \sum_{\sigma\in\Omega^{\partial a}} \mathbf{1}\{\sigma_x = s\}\psi_a(\sigma) \prod_{y\in\partial a\setminus x} \mu_{G,y\to a}(\sigma_y)$$

Belief Propagation on factor graphs

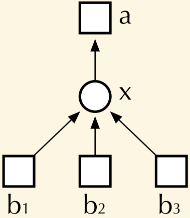


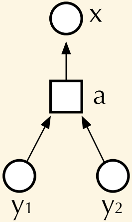


The Bethe free entropy

$$\mathscr{B}_G = \sum_{x \in V(G)} \log \sum_{s \in \Omega} \prod_{a \in \partial x} \mu_{G,a \to x}(s) + \sum_{a \in F(G)} \log \sum_{\sigma \in \Omega^{\partial a}} \psi_a(\sigma) \prod_{y \in \partial a} \mu_{G,y \to a}(\sigma_y) - \sum_{a \sim x} \log \sum_{s \in \Omega} \mu_{G,a \to x}(s) \mu_{G,x \to a}(s)$$

Belief Propagation on factor graphs





The replica symmetric “solution”

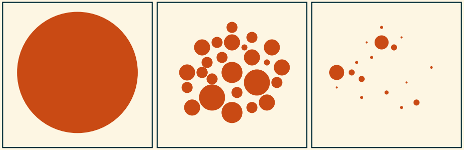
$$\log Z_G = \mathscr{B}_G + o(n)$$

Bethe states and replica symmetry breaking

Replica symmetry breaking [MP00,KMRTSZ07]

- the replica symmetric "solution" sometimes turns out to be correct
- Example: the Potts ferromagnet on $\mathbb{G}(n, d)$
- but not *always*
- Example: the Potts anti-ferromagnet on $\mathbb{G}(n, d)$ at low temperature

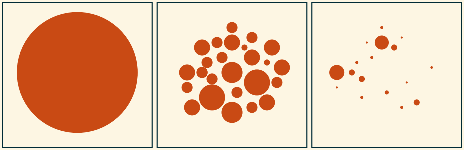
Bethe states and replica symmetry breaking



Replica symmetry breaking [MP00,KMRTSZ07]

- replica symmetry
- dynamic replica symmetry breaking
- (static) replica symmetry breaking

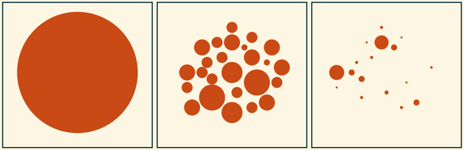
Bethe states and replica symmetry breaking



Replica symmetry [MP00,KMRTSZ07]

$$\mu_{\mathbb{G},\beta}(\{\boldsymbol{\sigma}_{x_1} = s, \boldsymbol{\sigma}_{x_2} = t\}) \sim \mu_{\mathbb{G},\beta}(\{\boldsymbol{\sigma}_{x_1} = s\})\mu_{\mathbb{G},\beta}(\{\boldsymbol{\sigma}_{x_2} = t\})$$

Bethe states and replica symmetry breaking



Bethe states [MPRTRLZ99,MP00,KMRTSZ07]

- the phase space decomposes into pure states
- in disordered system their locations are “unpredictable”
- each of them induces a BP fixed point (but not vice versa)
- in the presence of hard constraints, freezing may occur

Summary

- ferromagnetic models possess the “obvious” pure states
- these can be captured cleanly be BP
- replica symmetry breaking in disordered systems
- prediction: non-trivial decomposition into Bethe states