Diluted spin glass models

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Overview

Applications

- random 2-SAT
- the Potts antiferromagnet
- random k-SAT

Random 2-SAT

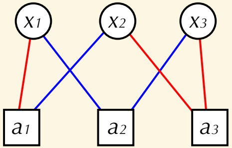
The 2-SAT problem

- variables $x_1, \ldots, x_n$ with $\Omega = \{\pm 1\}$ representing truth values
- four types of clauses:

$$x_i \vee x_j \qquad x_i \vee \neg x_j \qquad \neg x_i \vee x_j \qquad \neg x_i \vee \neg x_j$$

- a 2-SAT formula is a conjunction $\Phi = \bigwedge_{i=1}^{m} a_i$ of clauses
- $S(\Phi) =$ set of satisfying assignments and $Z(\Phi) = |S(\Phi)|$

Random 2-SAT



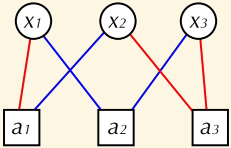
Example

- $\Phi = (\neg x_1 \vee x_2) \wedge (x_1 \vee x_3) \wedge (\neg x_2 \vee \neg x_3)$
- $Z(\Phi) = 2$ and $S(\Phi)$ consists of the two assignments

$$\sigma_{x_1} = +1 \qquad \sigma_{x_2} = +1 \qquad \sigma_{x_3} = -1$$
$$\sigma_{x_1} = -1 \qquad \sigma_{x_2} = -1 \qquad \sigma_{x_3} = +1$$

- “glassy” because variables may appear with opposing signs

Random 2-SAT



Random formulas

- for a fixed $0 < d < \infty$ let $\boldsymbol{m} = \mathrm{Po}(dn/2)$
- $\Phi =$ conjunction of $\boldsymbol{m}$ independent random clauses
- variable degrees have distribution $\mathrm{Po}(d)$
- *Goal:* to compute $Z(\Phi)$
- *the threshold for $Z(\Phi) = 0$ occurs at $d = 2$* *[CR92,G96]*

Random 2-SAT

Theorem [ACOHKLMPZ20]

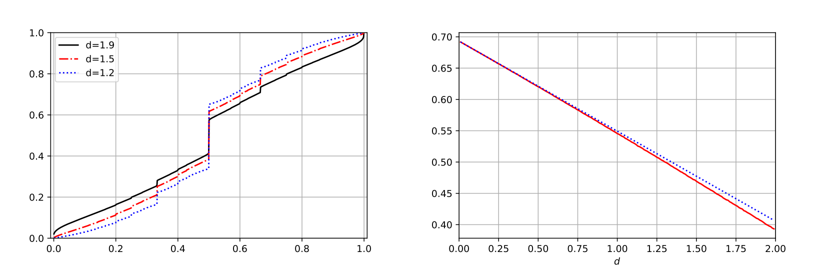
For $d < 2$ there is a unique distribution π_d on $(0,1)$ s.t.

$$\boldsymbol{\mu}_0 \stackrel{\mathrm{d}}{=} \frac{\prod_{i=1}^{\boldsymbol{d}_+} \boldsymbol{\mu}_i}{\prod_{i=1}^{\boldsymbol{d}_+} \boldsymbol{\mu}_i + \prod_{i=1}^{\boldsymbol{d}_-} \boldsymbol{\mu}_{i+\boldsymbol{d}^+}}$$

and

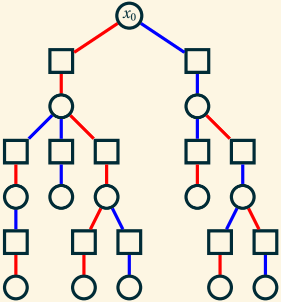
$$\lim_{n\to\infty} \frac{1}{n} \log Z(\Phi) = \mathrm{E}\left[\log\left(\prod_{i=1}^{\boldsymbol{d}_+} \boldsymbol{\mu}_i + \prod_{i=1}^{\boldsymbol{d}_-} \boldsymbol{\mu}_{i+\boldsymbol{d}_+}\right) - \frac{d}{2}\log\left(1 - \boldsymbol{\mu}_1 \boldsymbol{\mu}_2\right)\right]$$

Random 2-SAT



d=1.9
d=1.5
d=1.2
0.0
0.2
0.4
0.6
0.8
1.0
0.40
0.45
0.50
0.55
0.60
0.65
0.70
0.00
0.25
0.50
0.75
1.00
1.25
1.50
1.75
2.00
d

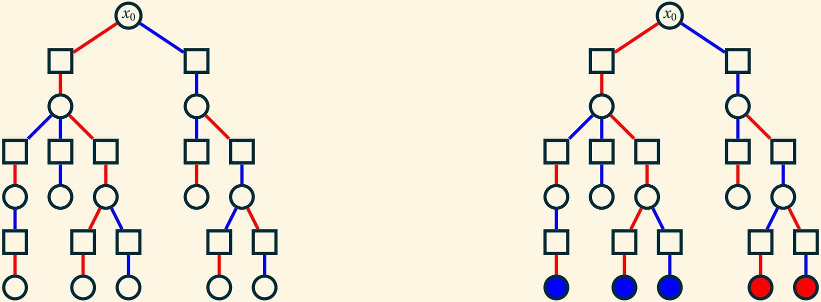
Random 2-SAT



The Galton-Watson tree

- a random tree $\boldsymbol{T}$ comprising variable and clause nodes
- the root x_0 is a variable
- each variable node spawns $\mathrm{Po}(d)$ clause nodes
- each clause node has one variable node child

Random 2-SAT



Gibbs uniqueness

- the influence of the boundary condition disappears
- therefore, there is *a unique* BP fixed point
- consequently, there is only one Bethe state
- hence, the model is replica symmetric for all $d < 2$

The Potts antiferromagnet

The model

- $\mathbb{G}(n, p)$ with $p = d/n$
- q-spin Potts antiferromagnet
- β =inverse temperature

Random k-SAT

The k-SAT threshold

- consider a random k-SAT formula $\mathbf{\Phi}$ with $k \geq k_0$
- let $\boldsymbol{m} = \mathrm{Po}(dn/k)$ be the number of clauses
- the *satisfiability threshold* occurs at [COP16,DSS15]

$$d/k = d^*(k)/k = 2^k \log 2 - \frac{1+\log 2}{2} + o_k(1)$$

- *... as predicted by Survey Propagation* *[MPZ02]*

Random k-SAT

Replica symmetry breaking

- consider an inverse temperature parameter $\beta > 0$
- we expect *replica symmetry breaking* for large β for d/k prior to the satisfiability threshold [KMRTSZ07]
- that is, we expect that

$$\left|\mu_{\mathbf{\Phi},\beta}(\{\boldsymbol{\sigma}_{x_1} = \boldsymbol{\sigma}_{x_2} = 1\}) - \mu_{\mathbf{\Phi},\beta}(\{\boldsymbol{\sigma}_{x_1} = 1\})\mu_{\mathbf{\Phi},\beta}(\{\boldsymbol{\sigma}_{x_2} = 1\})\right| > 0$$

Random k-SAT

Theorem [CORM22]

Assume $k \geq k_0$ and β is large enough. Then replica symmetry breaking occurs in random k-SAT for

$$2^k \log 2 - \frac{3}{2} \log 2 + o_k(1) \leq d/k \leq d^*(k)/k$$

Random k-SAT

Belief Propagation on the Galton-Watson tree

- consider a Galton-Watson tree $\boldsymbol{T}_{d,k}$ that mimics the local structure of the random formula $\boldsymbol{\Phi}$
- truncate the tree at the 2ℓ-th layer
- on the boundary feed in independent messages drawn from a distrubtion π on $[0,1]$
- does Belief Propagation converge to a "sound" limit?
- say that a the distribution π has slim tails if

$$\pi\left(\left[0,1/2-2^{-k/10}\right]\cup\left[1/2+2^{-k/10},1\right]\right)\leq 2^{-k/10}$$

Lemma [CORM22]

For any distribution π with slim tails BP converges weakly to a distribution π^* with slim tails as $\ell \to \infty$ *Proof via the contraction method*

Random k-SAT

Lemma [CORM22]

Assume $d < d^*(k)$ and β is large enough. If $\mu_{\Phi,\beta}$ is replica symmetric, then the empirical distribution

$$\pi_{\Phi,\beta} = \frac{1}{n} \sum_{i=1}^{n} \delta_{\mu_{\Phi,\beta,x_i}(1)}$$

has slim tails with high probability. *Proof via a combinatorial analysis of the solution space*

Random k-SAT

Lemma [CORM22]

Assume $d < d^*(k)$ and β is large enough. If $\mu_{\Phi,\beta}$ is replica symmetric, then

$$\lim_{n\to\infty}\frac{1}{n}\log Z_\beta(\Phi) = \mathscr{B}(\pi^*) \quad \text{where}$$

$$\mathscr{B}(\pi) = \mathrm{E}\Bigg[\log\Bigg(\prod_{i=1}^{\boldsymbol{\gamma}^+} 1-(1-\mathrm{e}^{-\beta})\prod_{j=1}^{k-1}\boldsymbol{\mu}_{\pi,i,j} + \prod_{i=1}^{\boldsymbol{\gamma}^-} 1-(1-\mathrm{e}^{-\beta})\prod_{j=1}^{k-1}\boldsymbol{\mu}_{\pi,i+\boldsymbol{\gamma}^-,j}\Bigg) - \frac{d(k-1)}{k}\log\Bigg(1-(1-\mathrm{e}^{-\beta})\prod_{j=1}^{k}\boldsymbol{\mu}_{\pi,1,j}\Bigg)\Bigg].$$

This follows from the general theorem about RS factor graphs

Random k-SAT

Lemma [CORM22]

Assume $k \geq k_0$ and β is large enough. If

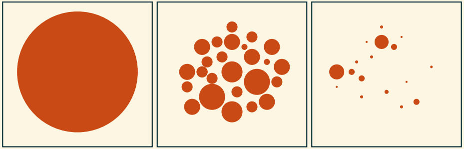
$$2^k \log 2 - \frac{3}{2} \log 2 + o_k(1) \leq d/k \leq d^*(k)/k$$

then

$$\lim_{n\to\infty} \frac{1}{n} \log Z_\beta(\Phi) < \mathscr{B}(\pi^*)$$

Proof via the 1RSB interpolation method

The Potts antiferromagnet



Replica symmetry breaking?

- suppose we fix β and vary d
- for what d is the model replica symmetric, viz. for $1 \le s, t \le q$,

$$\lim_{n\to\infty} \mathrm{E}\left|\mu_{\mathbb{G},\beta}(\{\boldsymbol{\sigma}_1 = s, \boldsymbol{\sigma}_2 = t\}) - q^{-2}\right| = 0$$

- *if the model fails to be replica symmetric, we know that a non-trivial Bethe state decomposition occurs*

The Potts antiferromagnet

Theorem [COKPZ18,COEJKK18]

For $\beta > 0$, $d > 0$ let

$$\mathscr{B}^*_{q,\beta}(d) = \sup\left\{\mathscr{B}_{q,\beta,d}(\pi) : \int \mu(i)\, d\pi(\mu) = 1/q\right\} \qquad \text{where}$$

$$\mathscr{B}_{q,\beta,d}(\pi) = \mathrm{E}\left[\frac{\Lambda(\sum_{\sigma=1}^{q}\prod_{i=1}^{\boldsymbol{\gamma}} 1-(1-e^{-\beta})\boldsymbol{\mu}_i^{(\pi)}(\sigma))}{q(1-(1-e^{-\beta})/q)^{\boldsymbol{\gamma}}} - \frac{d}{2}\frac{\Lambda(1-(1-e^{-\beta})\sum_{\sigma=1}^{q}\boldsymbol{\mu}_1^{(\pi)}(\sigma)\boldsymbol{\mu}_2^{(\pi)}(\sigma))}{1-(1-e^{-\beta})/q}\right]$$

Then

$$d_{\mathrm{RSB}}(q,\beta) = \inf\left\{d > 0 : \mathscr{B}^*_{q,\beta}(d) > \log q + \frac{d}{2}\log(1-(1-e^{-\beta})/q)\right\}$$

The Potts antiferromagnet

	$\sigma^{*\,-1}(1)$	$\sigma^{*\,-1}(2)$	$\sigma^{*\,-1}(3)$
$\sigma^{*\,-1}(1)$	$e^{-\beta}$	1	1
$\sigma^{*\,-1}(2)$	1	$e^{-\beta}$	1
$\sigma^{*\,-1}(3)$	1	1	$e^{-\beta}$

The stochastic block model

- random colouring $\boldsymbol{\sigma}^* : V \to \{1, \ldots, q\}$
- for each pair $e = \{v, w\}$ independently,

$$\mathrm{P}\left[e \in \mathbb{G}^* | \boldsymbol{\sigma}^*\right] = \frac{d}{n} \cdot \frac{q}{q - 1 + e^{-\beta}} \cdot \begin{cases} e^{-\beta} & \text{if } \boldsymbol{\sigma}^*(v) = \boldsymbol{\sigma}^*(w), \\ 1 & \text{if } \boldsymbol{\sigma}^*(v) \neq \boldsymbol{\sigma}^*(w) \end{cases}$$

- $d =$ signal strength; $e^{-\beta} =$ noise

The Potts antiferromagnet

	$\sigma^{*\,-1}(1)$	$\sigma^{*\,-1}(2)$	$\sigma^{*\,-1}(3)$
$\sigma^{*\,-1}(1)$	$e^{-\beta}$	1	1
$\sigma^{*\,-1}(2)$	1	$e^{-\beta}$	1
$\sigma^{*\,-1}(3)$	1	1	$e^{-\beta}$

The stochastic block model

- the overlap of $\sigma, \tau : V \to \{1, \ldots, q\}$ is

$$\alpha(\sigma, \tau) = \frac{1}{q-1} \max_{\kappa \in \mathbb{S}_q} \left\{ \frac{q}{n} \sum_{v \in V} \mathbf{1}\{\sigma(v) = \kappa \circ \tau(v)\} - 1 \right\}.$$

- for what d, β is it possible to infer $\tau_{\mathbb{G}^*}$ such that

$$\mathrm{E}[\alpha(\boldsymbol{\sigma}^*, \tau_{\mathbb{G}^*})] \geq \Omega(1) \ \ ?$$

The Potts antiferromagnet

	$\sigma^{*-1}(1)$	$\sigma^{*-1}(2)$	$\sigma^{*-1}(3)$
$\sigma^{*-1}(1)$	$e^{-\beta}$	1	1
$\sigma^{*-1}(2)$	1	$e^{-\beta}$	1
$\sigma^{*-1}(3)$	1	1	$e^{-\beta}$

Lemma [COKPZ18]

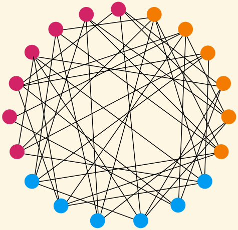
For given d, β the following are equivalent:

- the Potts antiferromagnet on $\mathbb{G}(n,p)$ is replica symmetric
- non-trivial inference of $\boldsymbol{\sigma}^*$ is impossible
- we have

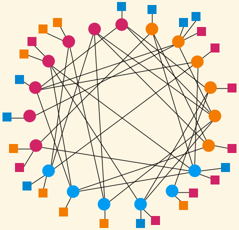
$$\lim_{n\to\infty} \frac{1}{n} \mathrm{E} \log Z_{q,\beta}(\mathbb{G}^*) = \log q + \frac{d}{2} \log\left(1 - \frac{1 - \mathrm{e}^{-\beta}}{q}\right)$$

Proof: truncated moment computation

The Potts antiferromagnet



The interpolation bound



[COKPZ18]

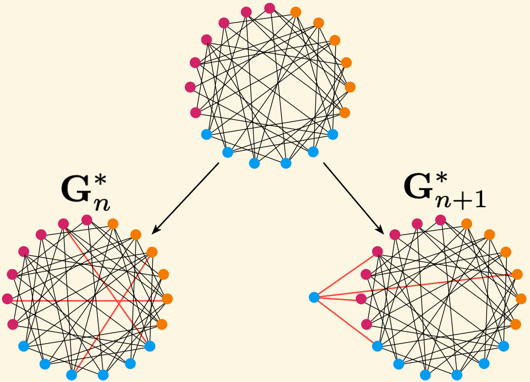
$$\frac{1}{n}\mathrm{E}\log Z_{q,\beta}(\mathbb{G}^*) \geq \mathscr{B}^*_{q,\beta}(d)$$

The Potts antiferromagnet

Pinning to the ground truth [COKPZ18]

- the pinning operation precipitates replica symmetry…
- …without significantly affecting $Z_{q,\beta}(\mathbb{G}^*)$
- we use the ground truth $\boldsymbol{\sigma}^*$ as the reference configuration
- this is possible thanks to the *Nishimori identity*

The Potts antiferromagnet



Aizenman-Sims-Starr [COKPZ18]

$$\frac{1}{n}\mathrm{E}\log Z_{q,\beta}(\mathbb{G}^*) \leq \mathscr{B}_{q,\beta,d}(\pi^*) \leq \mathscr{B}^*_{q,\beta}(d)$$

The Potts antiferromagnet

Theorem [COKPZ18]

For all d, β we have

$$\lim_{n\to\infty} \frac{1}{n} \mathrm{E} \log Z_{q,\beta}(\mathbb{G}^*) = \mathscr{B}^*_{q,\beta}(d)$$

this implies the theorem about the Potts antiferromagnet on $\mathbb{G}(n,p)$

The Potts antiferromagnet

Corollary [COEJKK18]

For all $d < d_{\mathrm{RSB}}(q, \beta)$ the stochastic block model and the Potts antiferromagnet on $\mathbb{G}(n, p)$ are mutually contiguous.

Summary

- Random 2-SAT: Gibbs uniqueness
- Random k-SAT: replica symmetry breaking “by contradiction”
- Potts model: replica symmetry breaking via statistical inference

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