GDR QCD HIC School

Heavy Quarks as hard probes of the QGP

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Menu:

- Day 1: introduction, motivation, and some selected topics of HQ energy loss & transport (beginning)
- Day 2: some aspects of HQ energy loss & transport (end) + overall tour of HQ production in HIC

Do not hesitate to interrupt, ask questions, …

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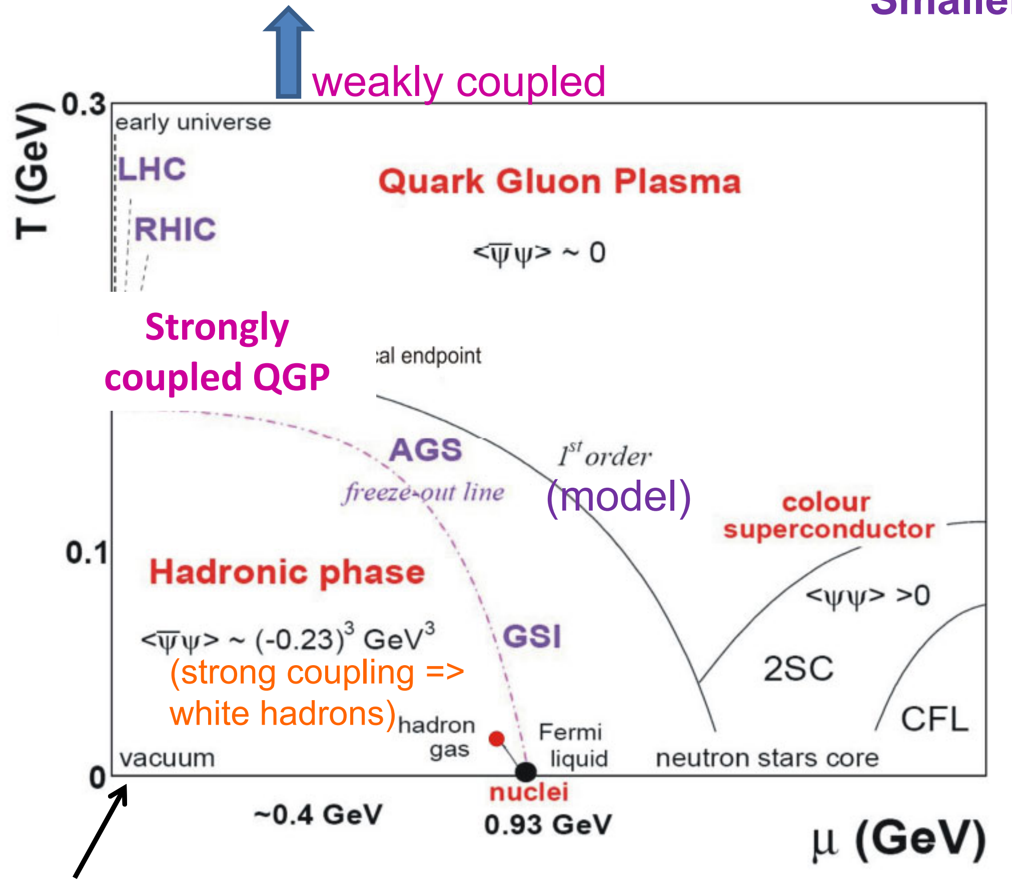
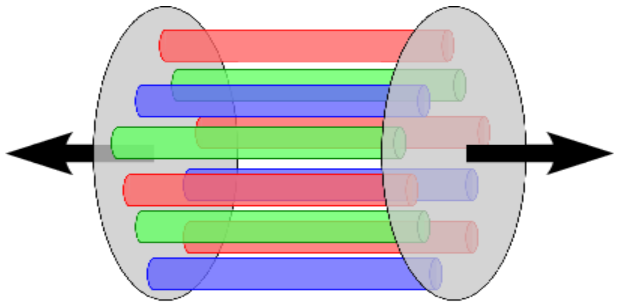
Matter under Extreme Conditions

Smaller μ at larger collider energies ?

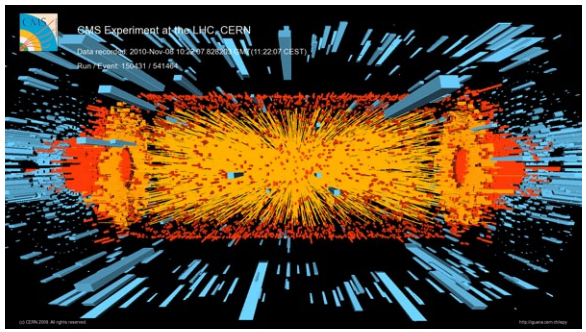


From J. Wambach (The Phase Diagram of Strongly Interacting Matter); 2006

Initial state



Final state

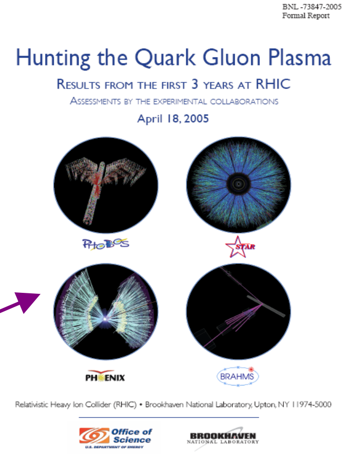


Net-Baryon rich | Net-Baryon poor | Net-Baryon rich

Investigating the QGP, Why ?

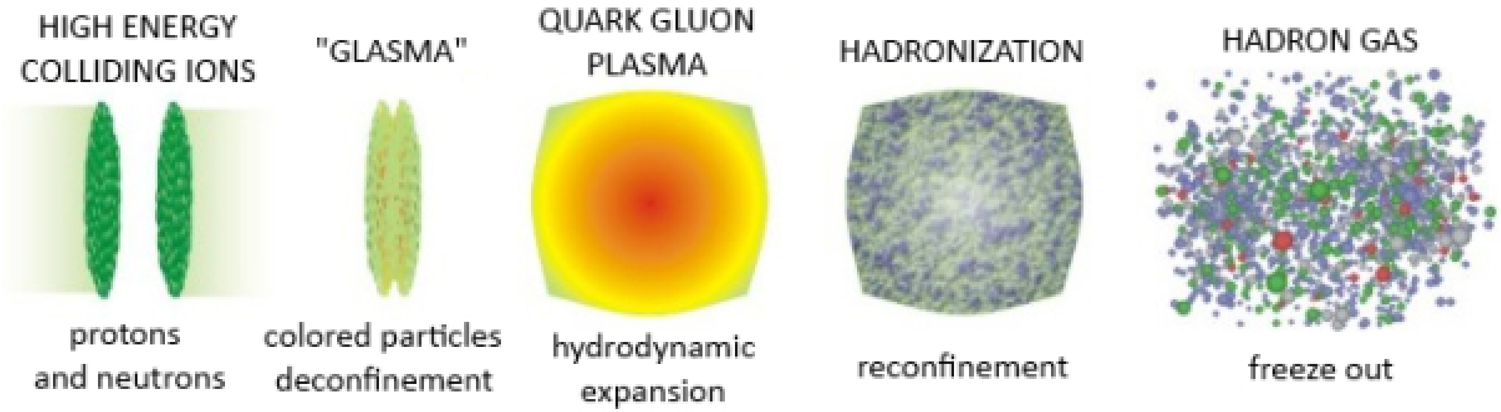
Possible interests (intrinsic & extrinsic) of QGP study:

- One of the strongest coupled many-body system (new techniques, new concepts) $\Rightarrow$ Challenging per se
- Could help in understanding *some aspects* of confinement
- Ingredient of the astrophysical "standard model"
- It has probably been (re)created in earth during the last decades thanks to URHIC: it *EXISTS* and should be characterized!



Ultra-Relativistic Heavy Ion Collisions

Schematic view I (artistic):



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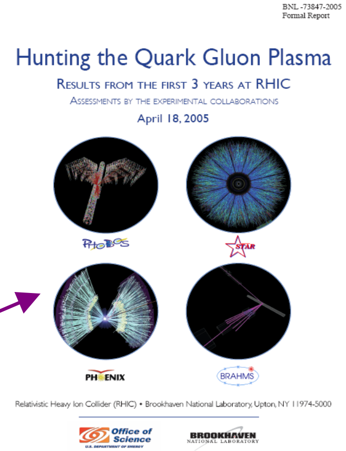
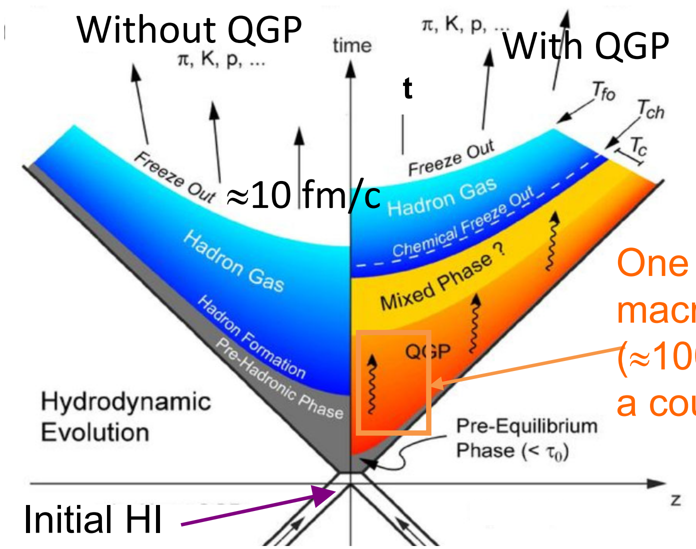


Ultra-Relativistic Heavy Ion Collisions

Schematic view II

(time – long. direction)

Since mid-80’s $\rightarrow$ now (AGS, SPS, RHIC, LHC): more and more energy deposit in the central overlapping region.

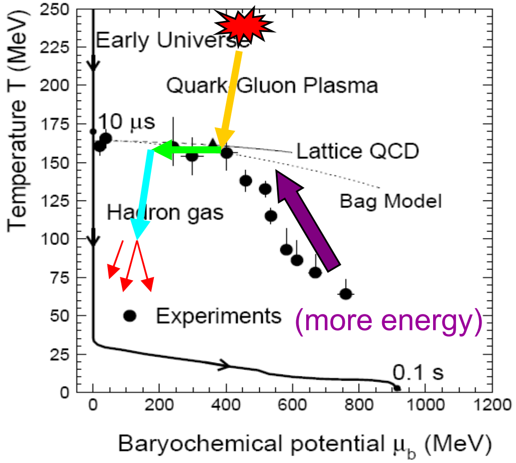


One of the smallest macroscopic system ($\approx$100 fm^3) surviving for a couple of fm/c only.

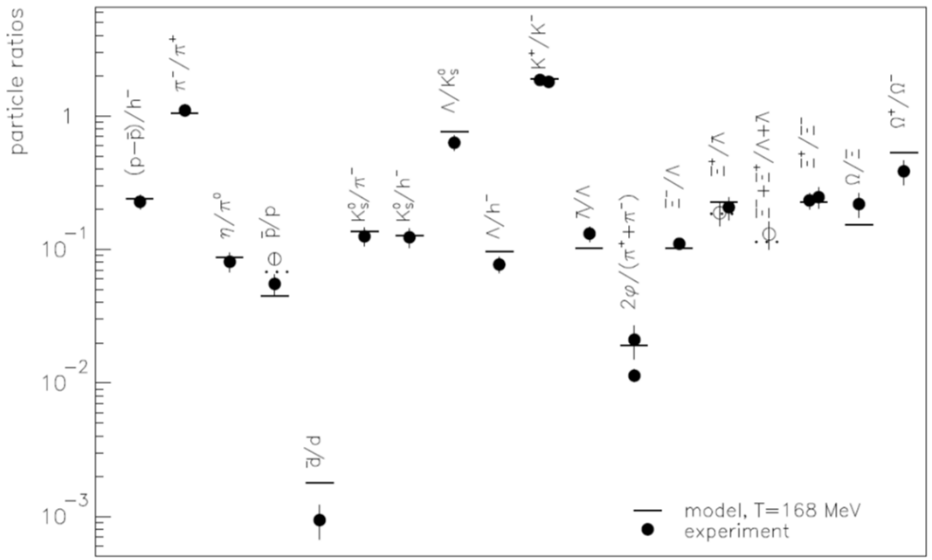
Investigating the QGP, How ?

Q: Does the system created in central region reach and maintain equilibrium long enough (10 fm/c) to be understood in terms of a quasi-stationnary state ?

=> Hadro-chemistry of the final state as a *thermometer* (# and spectra):



P. Braun-Munzinger & J. Wambach (arXiv:0801.4256)



Experiments seem to reveal the freeze-out "horizon", i.e. the frontier between a hadron gas and a state "beyond"

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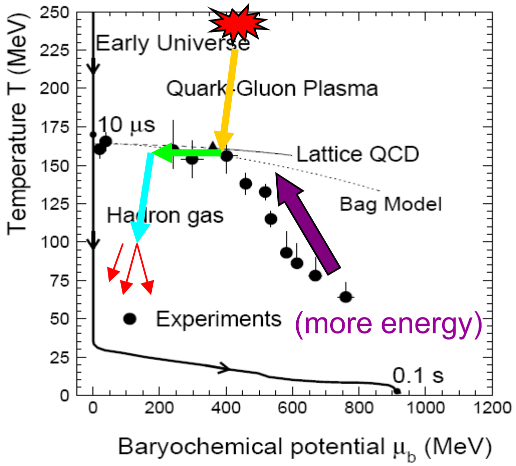

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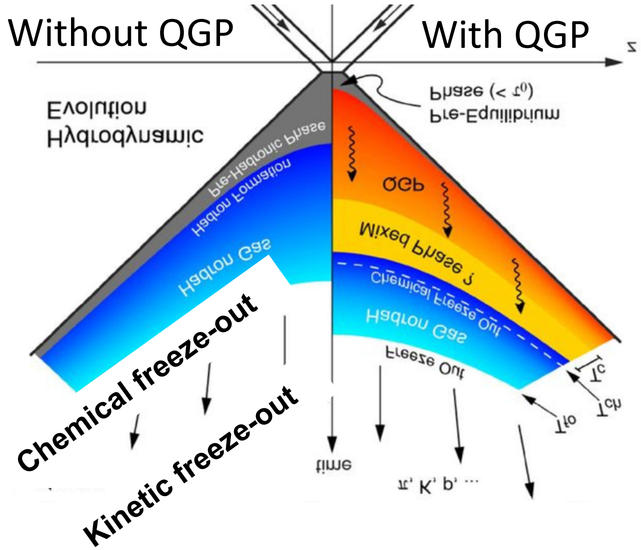
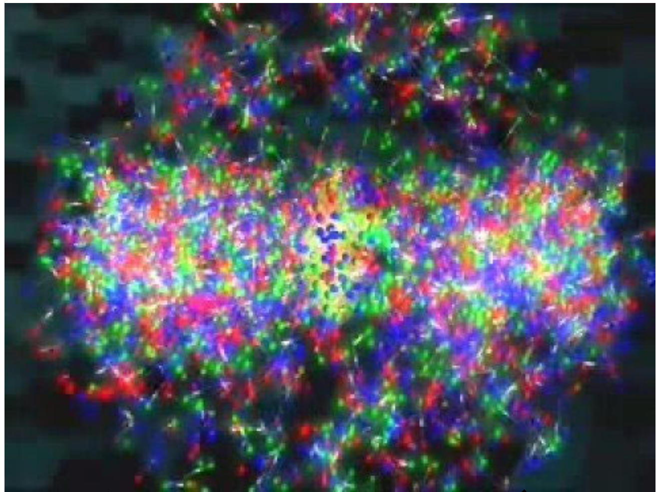
T(t)



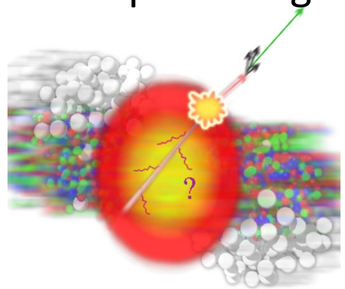
Memory Loss !

But How can we proceed to probe the QGP “beyond the horizon”, that is “before the freeze out” ?

Investigating the QGP, How ?



High p_T partons quenching



Method #1 : "soft probes"

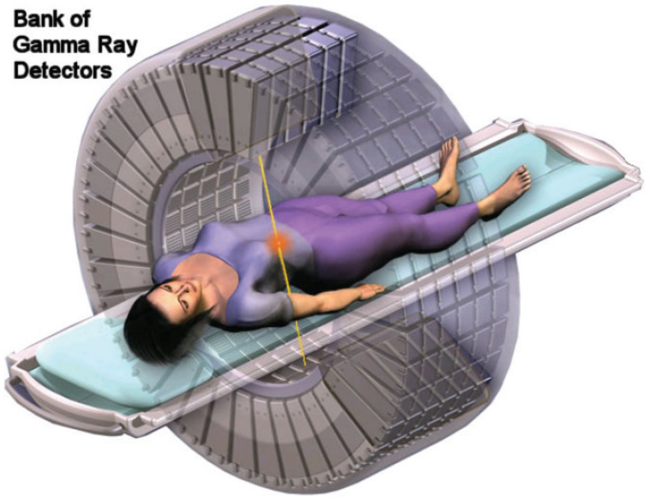
Although memory lost locally, the final stage results from the convolution of 1) initial stage & 2) **(fluid) dynamical evolution of QGP**, sensitive to various key aspects : EOS & transport coeffcients (viscosity η,...) => allows to constrain QGP properties

Method #2 : "penetrating probes" aka "hard probes"

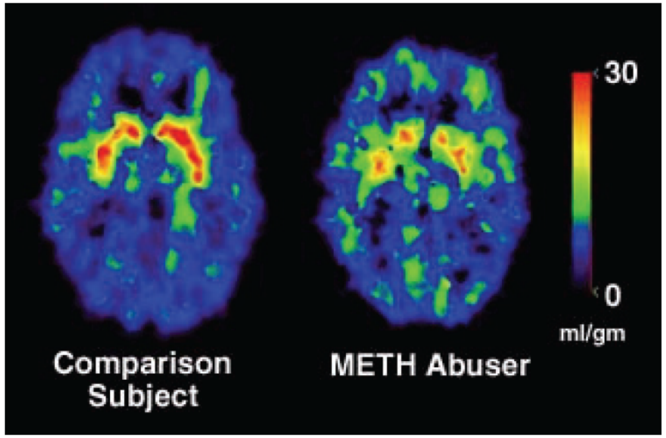
- Identify some particle / object / mode that does not (completely) loses its memory through QGP evolution.
- Energy scale >> T > Λ_{QCD}

A13

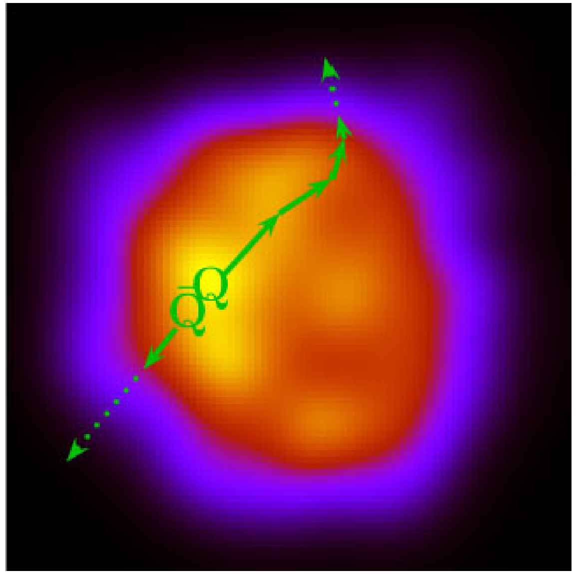
Extracting density profiles with HF Tomography ?



Schematic diagram of a PET scanner



PET scan showing abnormal brain function of a METH user



Diffusion of heavy-quarks

QGP tomography with Q-Qbar pairs

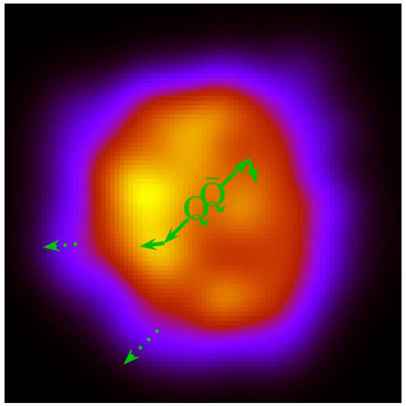
Seems pretty attractive concept…

← Well formulated inverse problem.

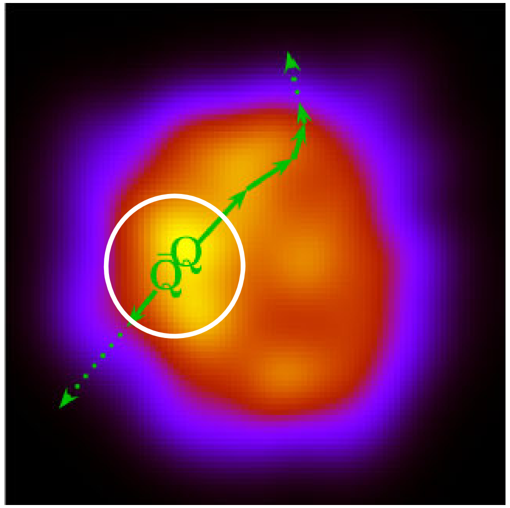
Hard probing QGP in URHIC with the help of HF ?

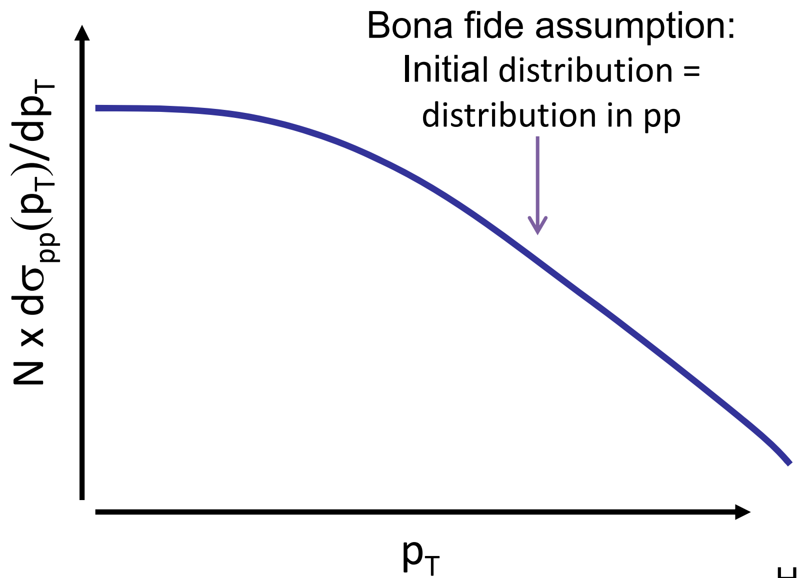
1. Φ: Heavy quarks are produced early, number conserved through time evolution (even at LHC) $\Rightarrow$ signature of early (hot) phase
2. Φ: *Strongly (even too strongly ?) affected by the QGP phase… without complete thermalisation*
3. Do not flow hydrodynamically but propagate/interact inside the medium via other processes $\Leftrightarrow$ sensitive to its properties *(*Deconfined ? Density and T ? Transport properties ? … *)*
4. Φ: Weakly affected by late time evolution in hadronic matter (heavy, colour transparency)
5. Theory: $m_Q \gg T > \Lambda_{QCD}$:Allows *some* pQCD calculations for the initial production, propagation and annihilation (even at low p_T)
6. Exp: Quarkonia suppression: clear decay channel $\rightarrow$ leptons

Usually advocated as an ideal *probe* of dense matter



Extracting density profiles with Tomography





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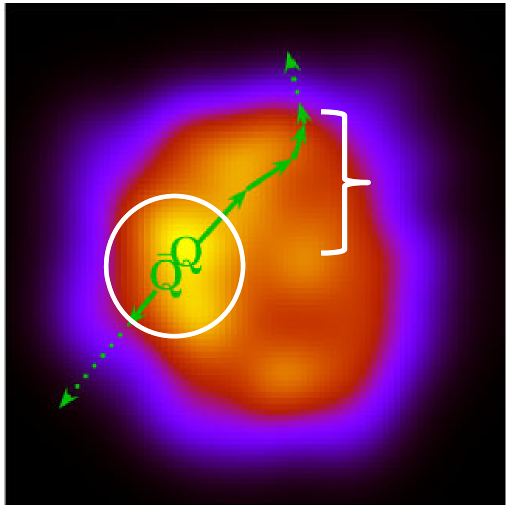
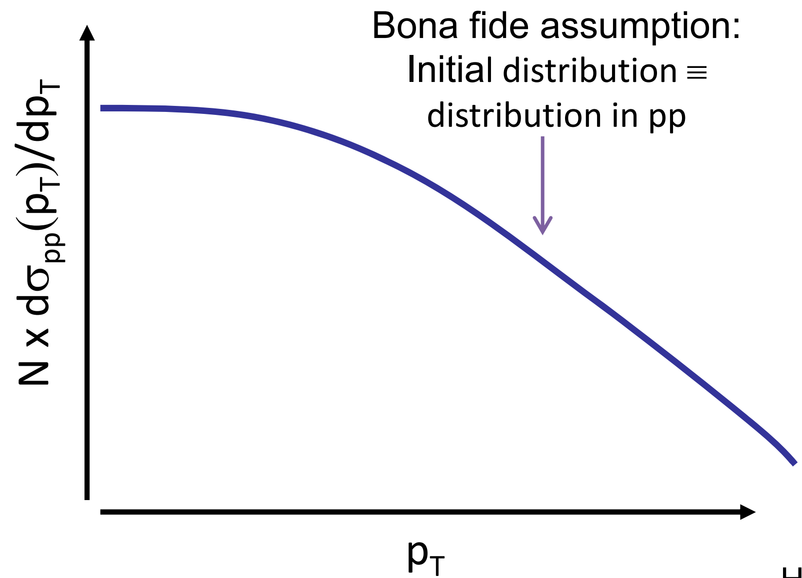


Naive assumption: $$\frac{dp}{dl} = \rho \times f(p, \ldots)$$

$$\Delta p(p, \ldots) = \int_0^L dl \, \frac{dp}{dl} = f(p, \ldots) \times \int_0^L dl \, \rho$$

accumulate history of QGP

(≠ particle emission from the freeze out boundary => complimentary information)



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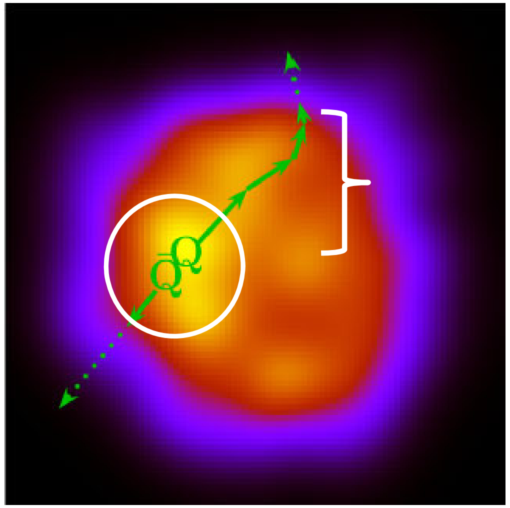

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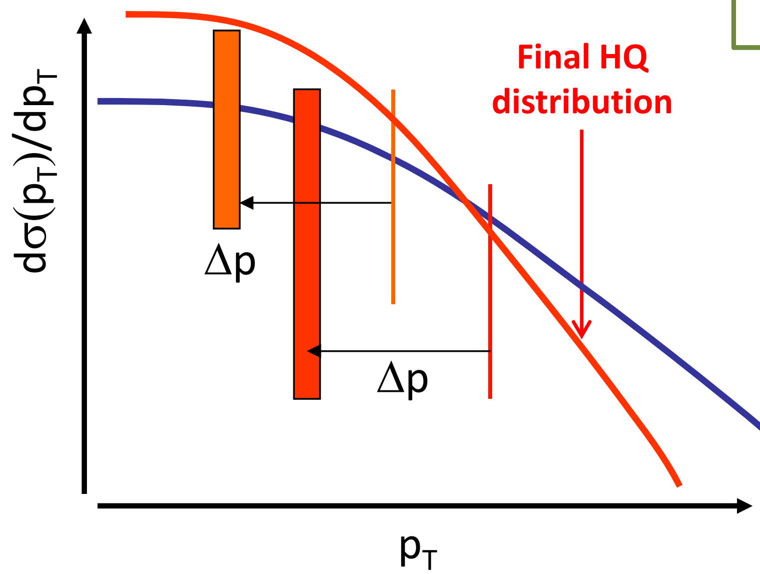
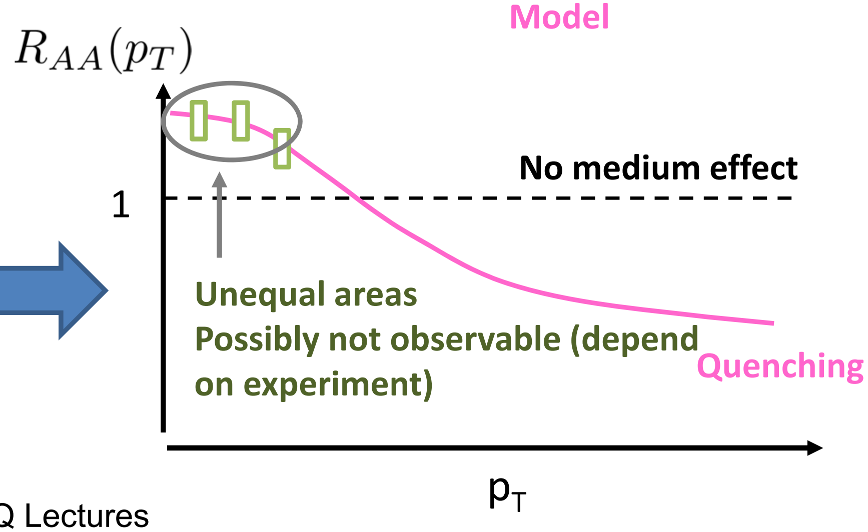
Exp obs.

$$R_{AA}(p_T) = \frac{\frac{dN_{AA}}{dp_T}}{\langle N_{\rm col} \rangle \times \frac{dN_{pp}}{dp_T}} \approx \frac{\frac{d\sigma_{pp}}{dp_T}(p_T + \Delta p(p_T))}{\frac{d\sigma_{pp}}{dp_T}(p_T)}$$

Model





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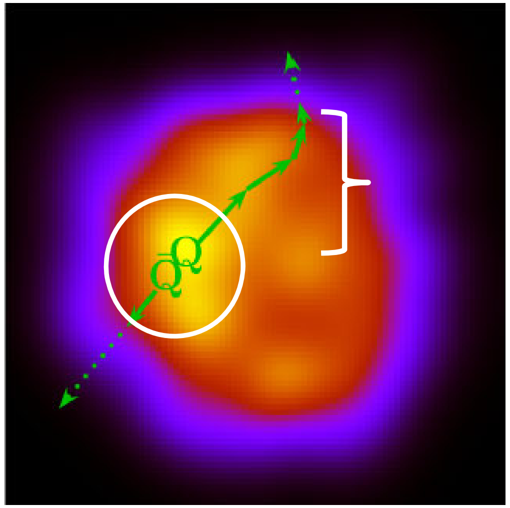
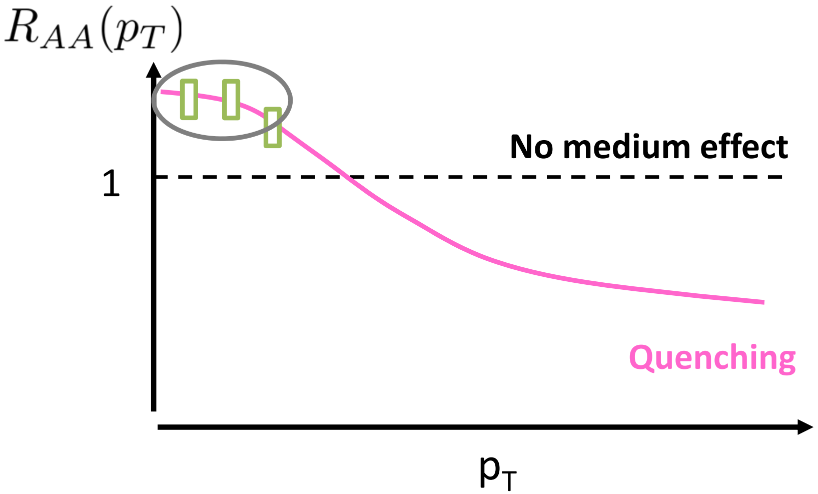




Model

$$R_{AA}(p_T) \approx \frac{\frac{d\sigma_{pp}}{dp_T}(p_T + \Delta p(p_T))}{\frac{d\sigma_{pp}}{dp_T}(p_T)}$$

If good knowledge of both $\frac{d\sigma_{pp}}{dp_T}$ and $f(p,\ldots)$ (Theory Eloss)

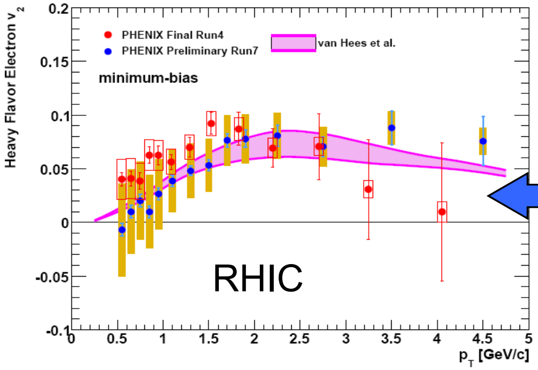
Extraction of $\int_0^L dl\, \rho$ or more precisely of $\sigma := \int_0^{+\infty} dt\, \rho(x(t),t)$

Integrated density

Other considerations ?

Why open heavy flavors in A-A ?

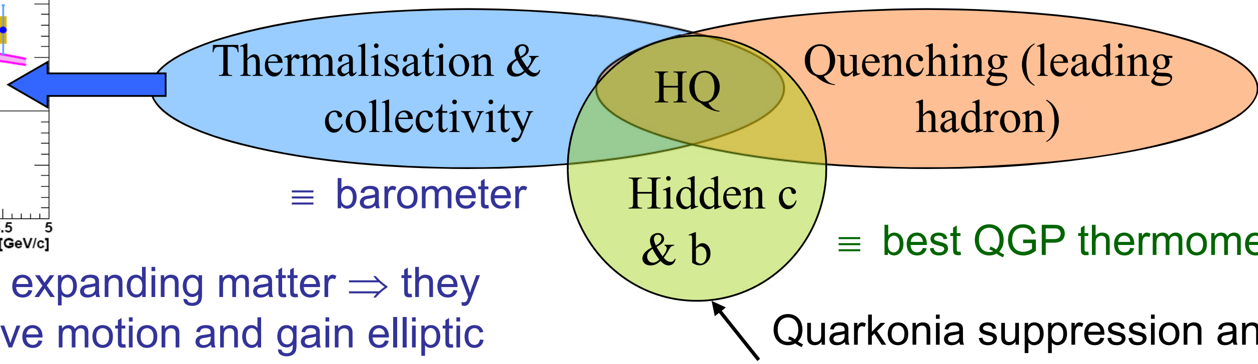
- Those are for sure sensitive to the early stages
- Much simpler then quarkonia and also sensitive to the medium properties (t_{equil} (α M_Q/T^2) $\Rightarrow$ clear hierarchy for s, c and b).
- Mandatory to understand Q-Qbar evolution in QGP & quarkonia production



HQ are imbedded in expanding matter $\Rightarrow$ they participate to collective motion and gain elliptic flow (v_2:azimutal asymmetry) at finite b… with additional inertia effect

The Trilogy:

≡ densimeter



≡ barometer

≡ best QGP thermometer ???

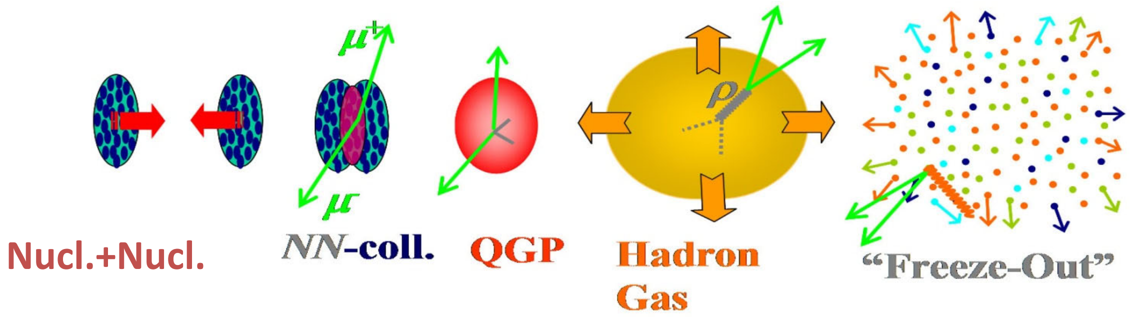
Quarkonia suppression and Dimuons product; heavy Q thermal production

Challenge:

Description of HQ E-loss / equilibration from fundamental theory. In fact we are at the same time probing the system but also using the results to better understand our probe (and the coupling to QGP) at the same time !

Some points of scientific method

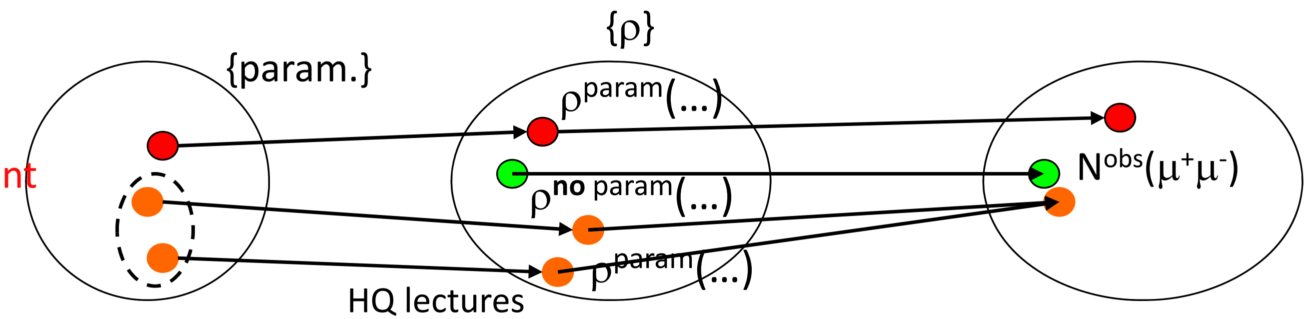
II. In principle, an inverse problem. Other example of dimuons production:



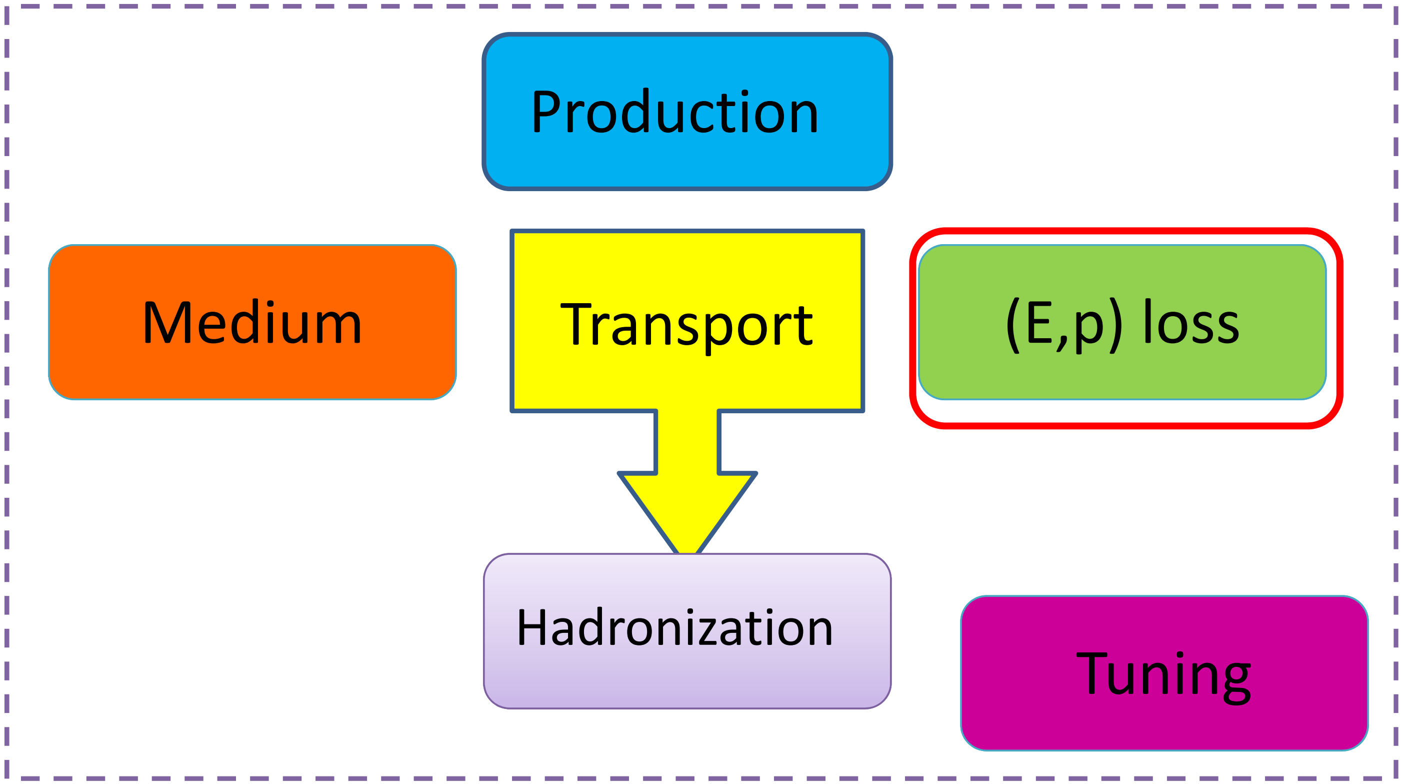
$$\text{Number}\ (\mu^+\mu^-) = \int_0^{+\infty} dt\ Rate(T(t),\ldots) = \int_0^{+\infty} dt \int dV\ \underbrace{\rho_1(t,M)\rho_2(t,M)}_{\text{Unknown densities}} \times \underbrace{v_{rel}\sigma_{prod}}_{\text{"known" 2-body process (depends on constituents)}}$$

In **practice**, one *never* extracts $\rho(t,M)$ from $R_{AA}(p_T)$, $N(\mu^+\mu^-)$,…; one considers some specific global scenario with free parameters (τ_0, EOS,…) and one performs the "inversion" in this subspace:

Answer might be not unique and/or not relevant (cf Batsouli 2002).

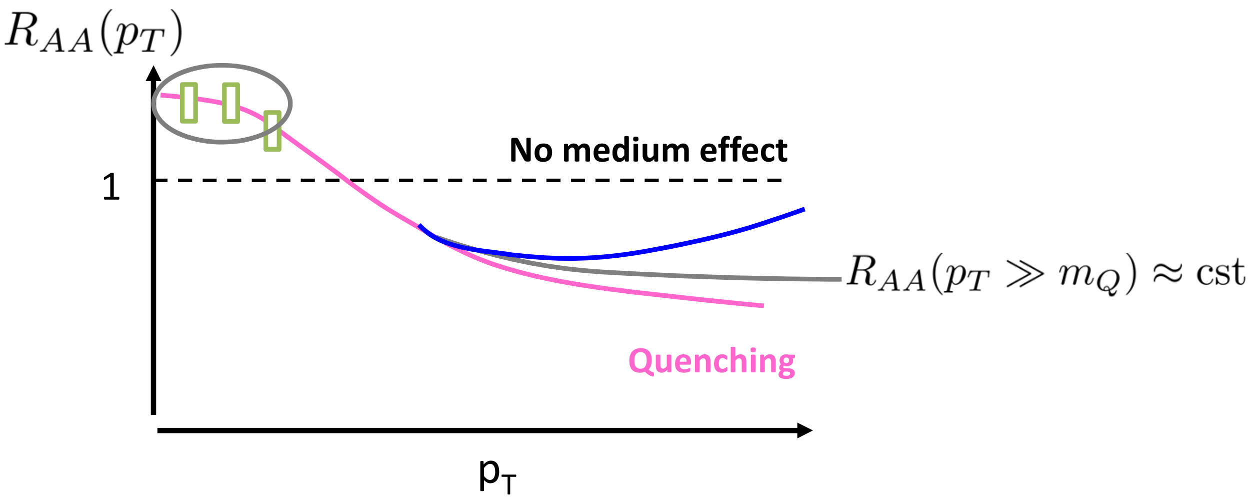


HQ observable (probe)



HQ (momentum-energy) loss (and gain) in QGP

Extracting density profiles with Tomography



Model

$$R_{AA}(p_T) \approx \frac{\frac{d\sigma_{pp}}{dp_T}(p_T + \Delta p(p_T))}{\frac{d\sigma_{pp}}{dp_T}(p_T)}$$

$$\sigma := \int_0^{+\infty} dt\, \rho(x(t), t)$$

$$\Delta p(p, \ldots) = f(p, m_Q, T, \mu_B, \ldots) \times \sigma$$

Crucial to master the dep. of Δp wrt all physical parameters (one big mandatory step for precision physics)

<u>In particular</u> wrt p (⇔ Lorentz boost $\gamma = p/(m_Q V)$)

Brain teaser : Assume $\frac{d\sigma_{pp}}{dp_T}(p_T) \propto p_T^{-n}$ at large p_T…. ⇨ $R_{AA}(p_T \gg m_Q) \approx \text{cst} \Leftrightarrow \Delta p \propto p^{?}$

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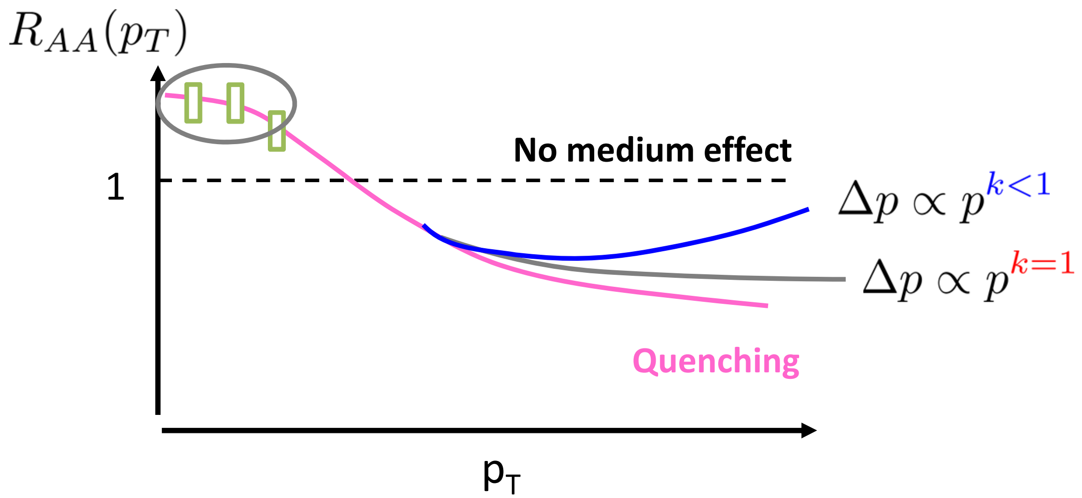
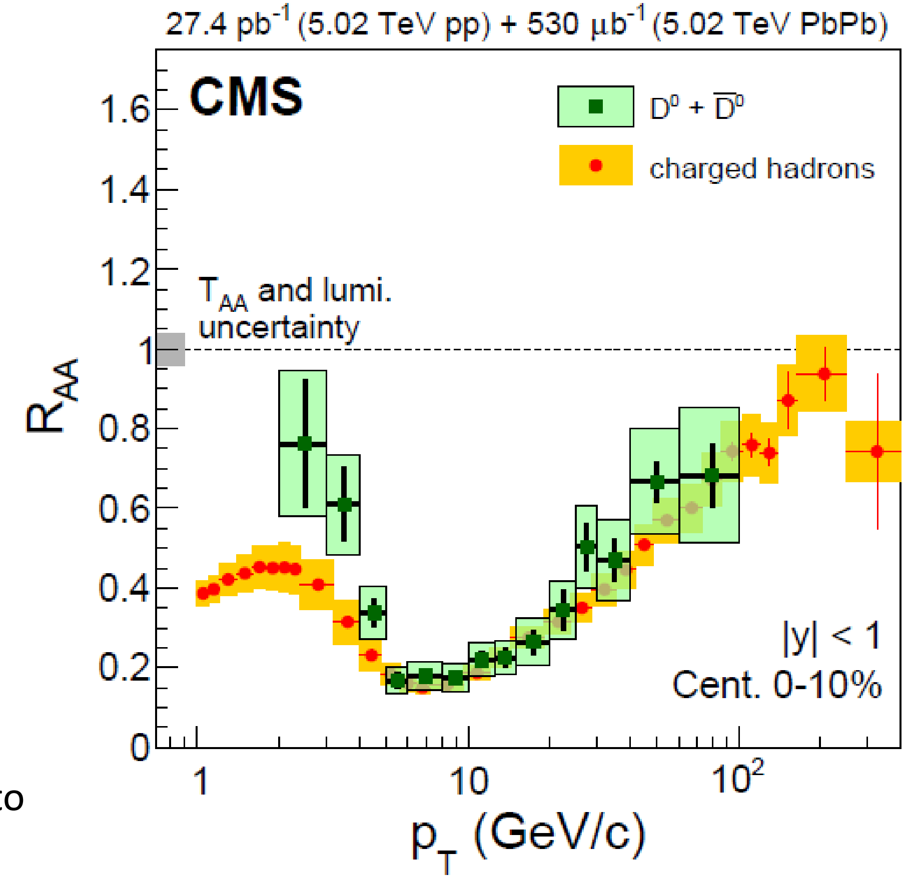


$$\Delta p(p,\ldots) = f(p, m_Q, T, \mu_B, \ldots) \times \sigma$$

Experimental judge : For p_T > 10 GeV: $\Delta p \propto p^{k<1}$

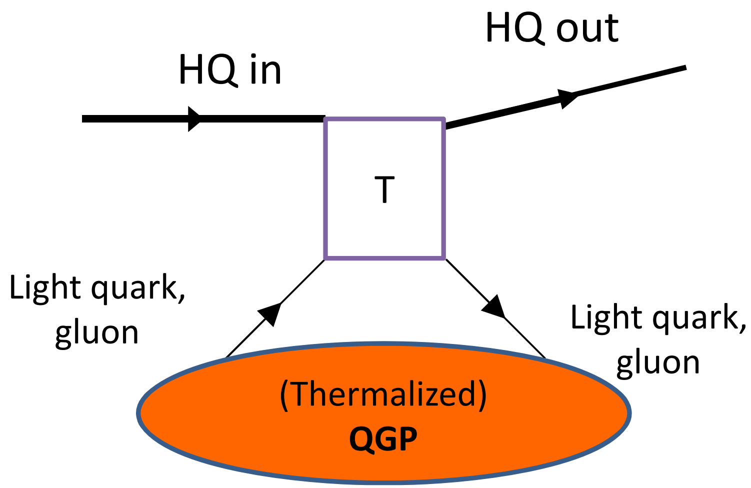
Theoretical view point : which method / object should we use to evaluate this quantity ?



HQ (momentum-energy) loss (and gain) in QGP

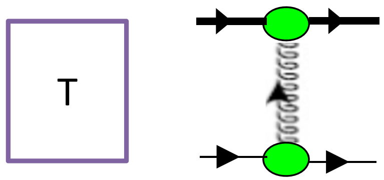
Strictly speaking: Both terms apply to pQCD processes (small and moderate coupling), or to pQCD-inspired processes for which quasi-particles still exists

Collisional (elastic): Q+x $\rightarrow$ Q+x

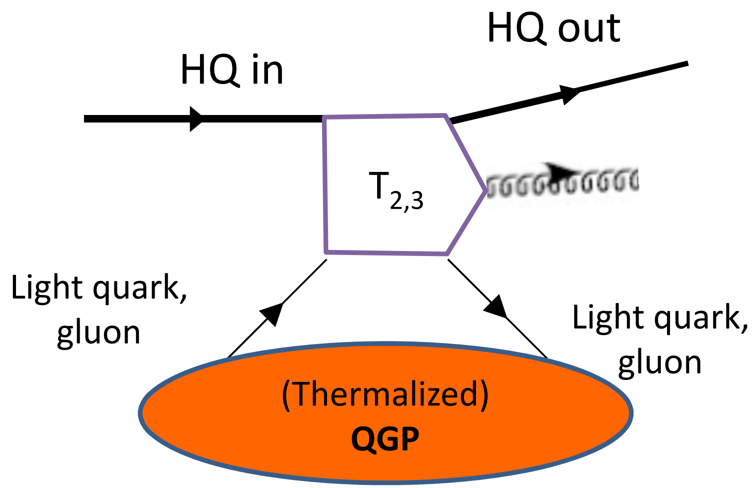


Energy flows from HQ $\leftrightarrow$ medium

e.g. :

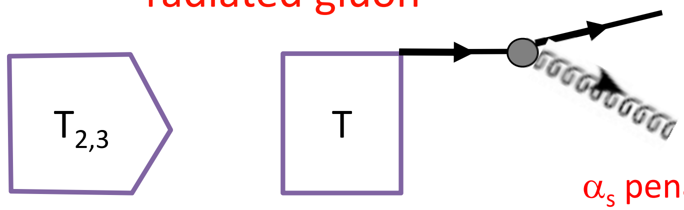


Radiative : Q+x $\rightarrow$ Q+x + **g**



Most of energy flows from HE HQ $\rightarrow$ radiated gluon

e.g. :



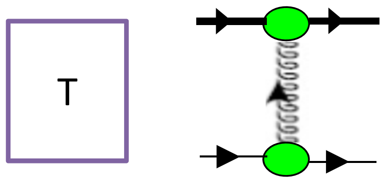
α_s penalty

Collisional (elastic) vs Radiative

Typical times

Collisional (elastic): 2 $\rightarrow$ 2

e.g. :



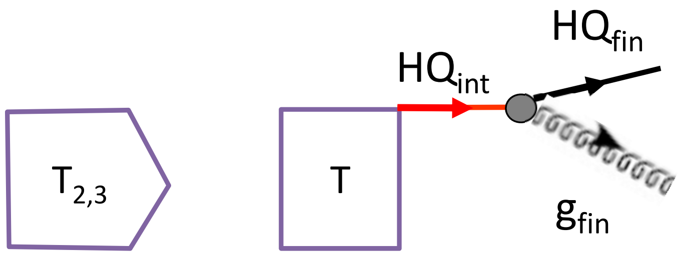
: space like; typical virtuality = μ (Debye mass)

=> Typical collision time = ...

Exo 1 : evaluate typical collision time for T = 2 T_c

Radiative : 2 $\rightarrow$ 3

e.g. :



Assuming (in light cone coordinates):

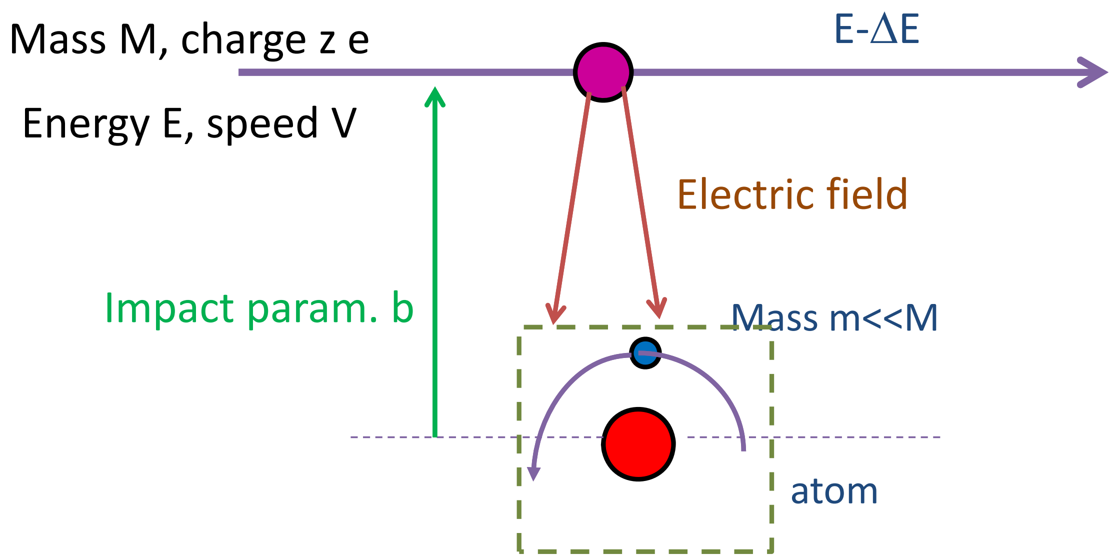
$$\begin{cases} HQ_{fin} \equiv ((1-z)P^+, P_Q^-, -\mathbf{k}_\perp) \\ g_{fin} \equiv (zP^+, P_g^-, \mathbf{k}_\perp) \end{cases}$$

Exo 2 : evaluate typical formation time

$$t_{\text{form}} \sim \frac{\text{HQ}^0_{\text{int}}}{\text{HQ}^2_{\text{int}} - M^2_{\text{HQ}}}$$

Collisional Energy loss in cold matter

a) The Bohr formula (classical, 1913-1915):



Impulse approximation (easy)

final momentum p_m = time-integration of the coulomb transverse force at a fixed point:

$$\Delta E(b) = \frac{2z^2(\alpha_{\rm QED}\hbar c)^2}{mV^2b^2}$$

$1/V^2$: comes from passing time

Collisional Energy loss in cold matter

Averaging on uniform electron density n_e:

$$\Delta E(b) = \frac{2z^2(\alpha_{\rm QED}\hbar c)^2}{mV^2b^2}$$

$$\frac{dE_{\rm col}}{dx} = 2\pi n_{\rm e}\int_{b_{\rm min}}^{b_{\rm max}} \Delta E(b)bdb$$

Need for some regularization **at both sides**

$b_{min} \Leftrightarrow$ large momentum transfer

Maximal ΔE from kinematics:

$$1) E \ll \frac{M^2}{2m} \Rightarrow \Delta E_{\rm max} \approx 2m\gamma^2V^2 \ll E$$

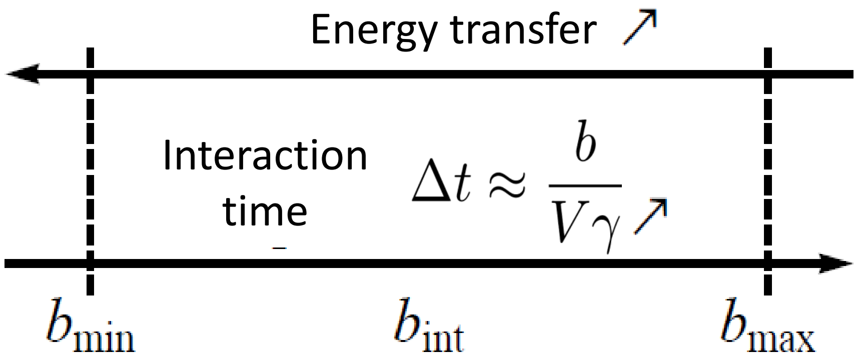
$$2) E \gg \frac{M^2}{2m} \Rightarrow \Delta E_{\rm max} \approx E - \frac{M^2}{2m^2} \approx E$$

Despite the boost

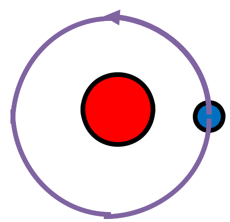
$$b_{\rm min} = \frac{z(\alpha_{\rm QED}\hbar c)}{V}\sqrt{\frac{2}{m\Delta E_{\rm max}}} \approx \frac{z(\alpha_{\rm QED}\hbar)c}{mV^2\gamma}$$

Collisional Energy loss in cold matter

b_{max}: far range ⇔ small momentum transfer



Δt Should stay smaller then the revolution time of the electron $1/\omega_{rev}$ in order to avoid adiabatic response from the atoms



$$\Delta t \sim \frac{b_{\rm max}}{\gamma V} \approx \frac{1}{\omega_{\rm rev}} \quad \Rightarrow \quad b_{\rm max} \approx \frac{\gamma V}{\omega_{\rm rev}}$$

Larger γ => transverse field more focused

Collisional Energy loss in cold matter

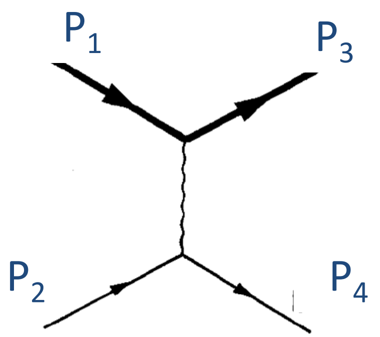
Result: $$\frac{dE_{\rm col}}{dx} = 4\pi n_{\rm e} \frac{z^2(\alpha_{\rm QED}\hbar c)^2}{m_{\rm e}c^2\beta^2}\left[\ln\left(\frac{1.123 m_{\rm e}c^2\beta^3\gamma^2}{z(\alpha_{\rm QED}\hbar)\langle\omega_{\rm rev}\rangle Z}\right) - \frac{\beta^2}{2}\right]$$

Remarks:

- Decrease $\alpha\ \beta^{-2}$ then rather mild logarithmic increase ./. γ (contribution from both b_{min} and b_{max})
- Energy loss $\alpha\ n_e$ (densitometer)
- Energy loss diverges when $m_e \to 0$?!
- Need to understand the close and the far collisions
- Charges show up in non algebraic way (through b_{min}) !!!

Rather troublesome; in pQED:

$$\frac{d\sigma_{\rm el}}{dt} \propto \frac{\alpha^2_{\rm QED}}{t^2}$$



Collisional Energy loss in cold matter

b) Quantum "corrections":

Bethe (1930), Bloch (1933), Moller and Foch, Fano (1964),...

More than 30 years

Main improvements:

- $\Delta E(b_{max}) >> \hbar\omega_{rev}$ for classical expression of b_{max}

Classical interpretation still holds ***on the average***... but quantum fluctuations (and corrections $\alpha\ z^4$)

$$\langle \omega_{\rm rev} \rangle_Z \to \frac{I(Z)}{\hbar}$$

← Average excitation energy

- b_{min} cannot be smaller than de Broglie wave length

$$\lambda = \frac{\hbar c}{\sqrt{mc^2 \Delta E_{\max}/2}}$$

$$b_{\rm min} \to \tilde{b}_{\rm min} = \hbar c \sqrt{\frac{2}{mc^2 \Delta E_{\max}}} \times \max\left(\underbrace{\frac{z\alpha_{\rm QED}}{\beta}}_{\text{classical}}, \underbrace{1}_{\text{quantum}} \right)$$

$$\frac{z\alpha_{\rm QED}}{\beta} \equiv$$ Ratio between classical size of the diffusion center and the de Broglie wave length

Collisional Energy loss in cold matter

Bethe's Result (valid for $\beta > z\,\alpha_{QED}$):

$$\frac{dE_{\rm col}}{dx} = 4\pi n_{\rm e}\frac{z^2\alpha_{\rm QED}^2\hbar c^2}{m_{\rm e}c^2\beta^2}\left[\frac{1}{2}\ln\left(\frac{2m_{\rm e}c^2\beta^2\gamma^2\Delta E_{\rm max}}{I^2(Z)}\right)-\beta^2\right]$$

Remarks:

- Hidden $\hbar$, but yes it is there (find it !), thanks to close collisions
- Quantum corrections do not affect the leading ln(γ)
- $dE_{col}/dx \propto \alpha_{QED}^2$ as it should be (?)

Collisional Energy loss in cold matter

Bloch's Result (Non Relativistic):

$$\frac{dE_{\rm col}}{dx} = 4\pi n_{\rm e}\frac{z^2\alpha_{\rm QED}^2\hbar c^2}{m_{\rm e}c^2\beta^2}\left[\ln\left(\frac{2m_{\rm e}c^2\beta^2}{I(Z)}\right) + \psi(1) - {\rm Re}\psi\left(1+i\frac{z\alpha_{\rm QED}}{\beta}\right)\right]$$

Digamma function

Remarks:

- Big difference ./. Bethe: asymptotic incoming and outgoing wave functions: not plane waves (distorded waves) ⇔ ladder ressummation
- $\alpha_{\rm QED}$ reenters in a non-algebraic way !

$$\frac{z\alpha_{\rm QED}}{\beta}$$

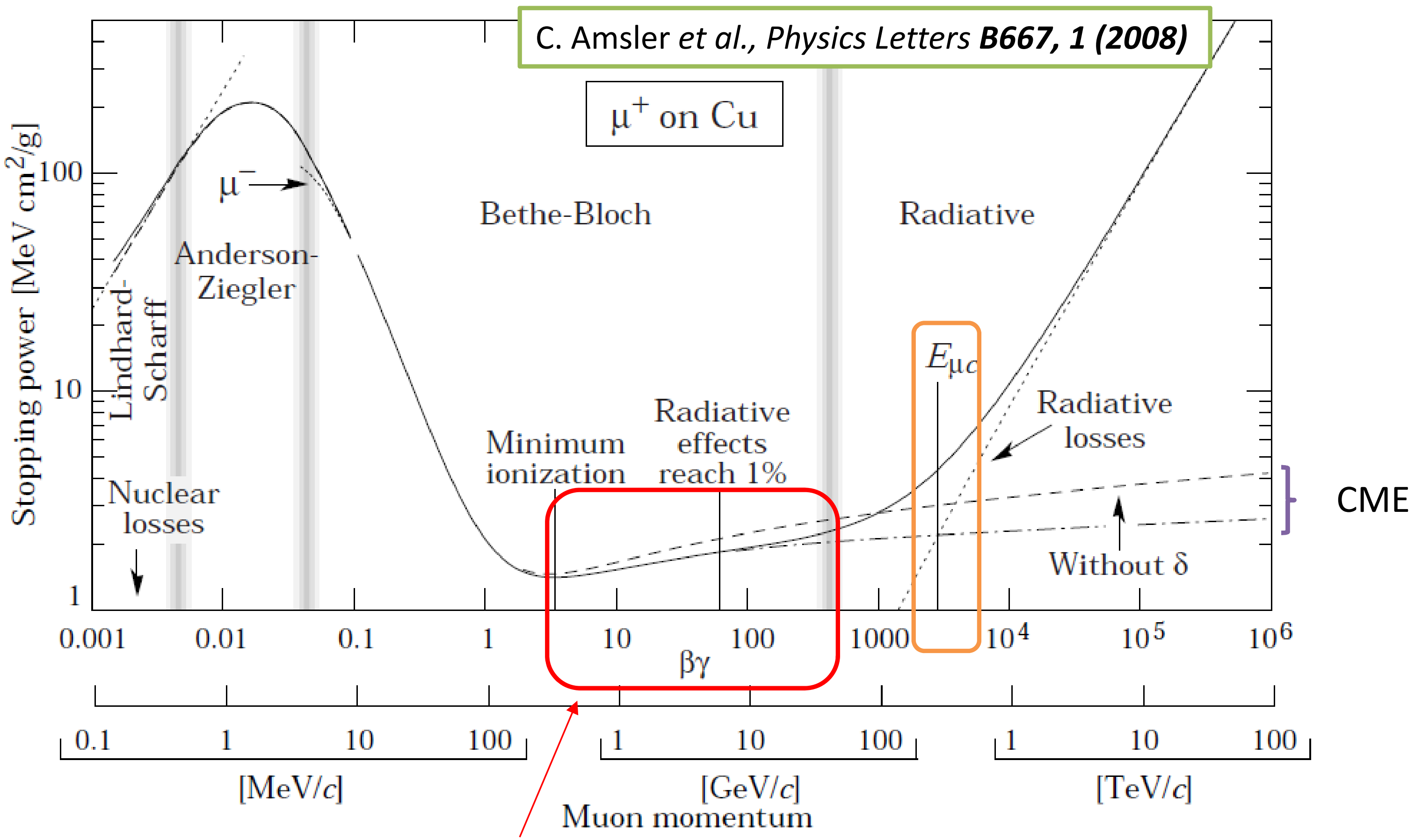
<<1 → Bethe

$$\frac{dE_{\rm col}}{dx} = 4\pi n_{\rm e}\frac{z^2\alpha_{\rm QED}^2\hbar c^2}{m_{\rm e}c^2\beta^2}\left[\frac{1}{2}\ln\left(\frac{2m_{\rm e}c^2\beta^2\gamma^2\Delta E_{\rm max}}{I^2(Z)}\right) - \beta^2\right]$$

>>1 → Bohr

$$\frac{dE_{\rm col}}{dx} = 4\pi n_{\rm e}\frac{z^2(\alpha_{\rm QED}\hbar c)^2}{m_{\rm e}c^2\beta^2}\left[\ln\left(\frac{1.123 m_{\rm e}c^2\beta^3\gamma^2}{z(\alpha_{\rm QED}\hbar)\langle\omega_{\rm rev}\rangle_Z}\right) - \frac{\beta^2}{2}\right]$$

Cold medium effects



The « good » regime for intermediate p_T HQ… but what is δ ? Lowers the increasing trend

Cold medium effects

Lesson from atomic physics : in dense matter, Electromagnetic field **does not** propagate like in vacuum !

Simple medium response (1 oscillator) : $\epsilon(\omega) = 1 - \chi(\omega)$ with $\chi(\omega) = \frac{\omega_p^2}{\omega^2 + i\omega\Gamma - \omega_0^2}$

Friction term ($i\omega\Gamma$); $\omega_0^2 \approx \omega_{\rm rev}$

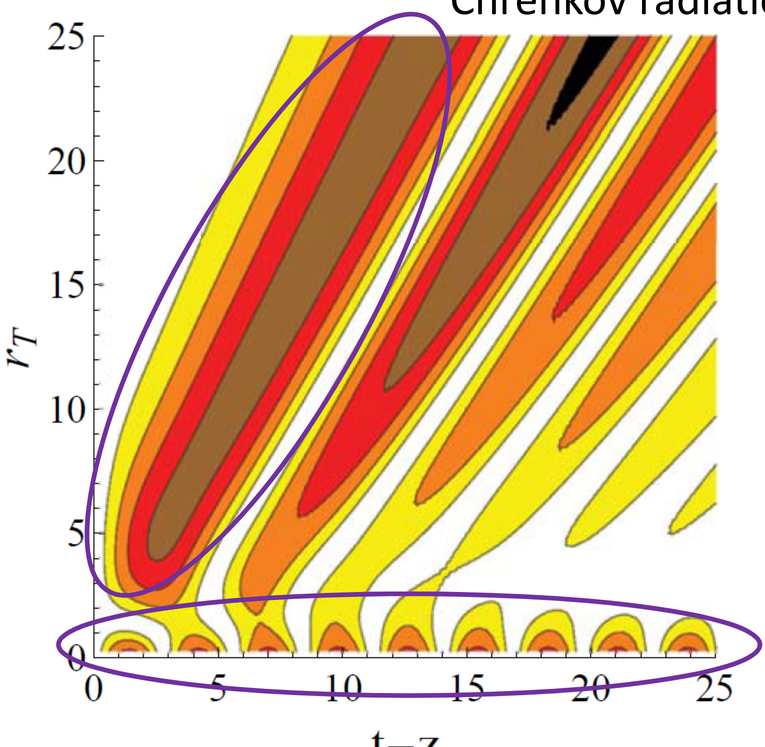
Field chart for r_T x E_T:

For $\gamma > \sqrt{\frac{\omega_0^2}{\omega_p^2} + 1}$

Transverse E field :

$$E_T \propto e^{-\frac{\omega_p r_T}{c}}$$

Screened !

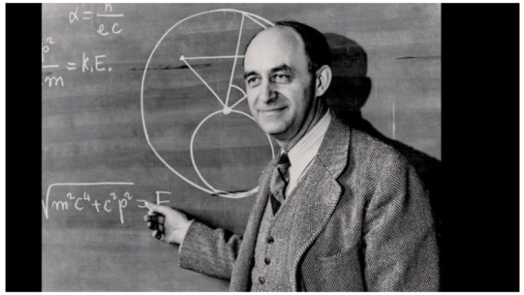


$$\omega_p = \sqrt{\frac{n_e e^2}{m_e \epsilon_0}}$$

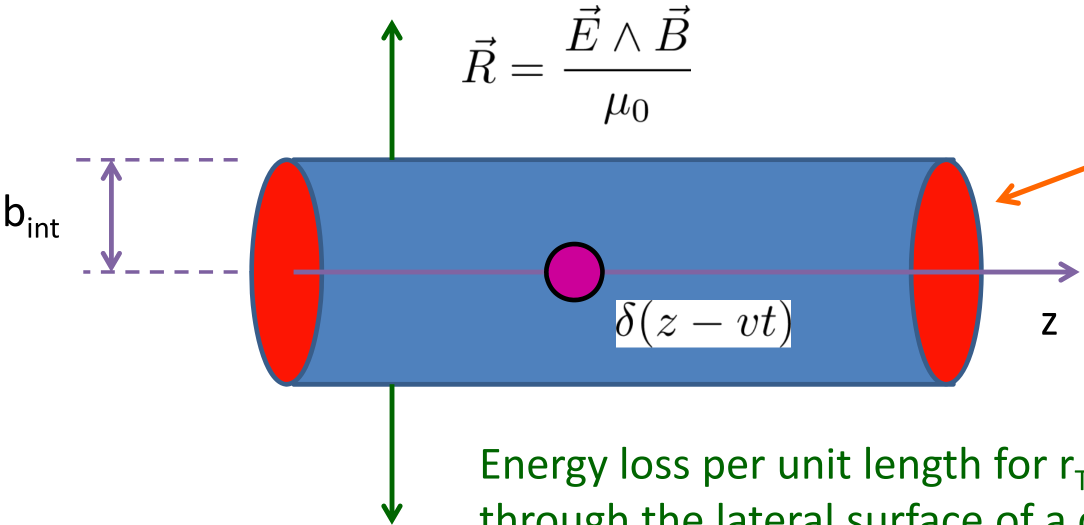
Classical plasmon frequency

Energy loss from a moving charge in a medium

First, need to know the (E,B) field due to the moving charge … what includes the medium response… then :



Fermi's art:



Energy loss inside cylinder is estimated as due to close collisions => Bethe – Bloch with UV cut-off on the momentum exchange ≈$1/b_{int}$

2 regimes !

Energy loss per unit length for $r_T > b_{int}$ = time integral of $\Phi(\vec{R})$ through the lateral surface of a cylinder

<u>Advantage</u> : allows for an easier identification of the various contributions

Cold medium effects: Summary

1

$$\frac{dE_{\rm col,Bohr}}{dx} = 2\pi n_e \frac{z^2(\alpha_{\rm QED}\hbar c)^2}{m_e c^2\beta^2}\left[\ln\left(\frac{1.123^2 v^2}{\omega_p^2 b_{\rm int}^2}\right) + \ln\left(\frac{\omega_p^2\gamma^2}{\omega_0^2}\right) - \beta^2\right]$$

Taking b_{min} -> b_{int}

Same !

$$\gamma_{crit} = \sqrt{\frac{\omega_0^2}{\omega_p^2}+1}$$

$$\left(\frac{dE_{\rm col}}{dx}\right)_{b>b_{\rm int}} = 2\pi n_e \frac{z^2\alpha_{\rm QED}^2\hbar c^2}{m_e c^2\beta^2} \times \left[\ln\left(\frac{1.123^2 v^2}{\omega_p^2 b_{\rm int}^2}\right) + \ln\left(\frac{\omega_p^2\gamma^2}{\omega_p^2+\omega_0^2}\right) - \beta^2\right]$$

Vanishing of the ln γ increase

Screened !

$$\left(\frac{dE_{\rm col}}{dx}\right)_{b>b_{\rm int}} = 2\pi n_e \frac{z^2\alpha_{\rm QED}^2\hbar c^2}{m_e c^2\beta^2} \times \left[\ln\left(\frac{1.123^2 v^2}{\omega_p^2 b_{\rm int}^2}\right) - \frac{\omega_0^2}{\omega_p^2\gamma^2}\right]$$

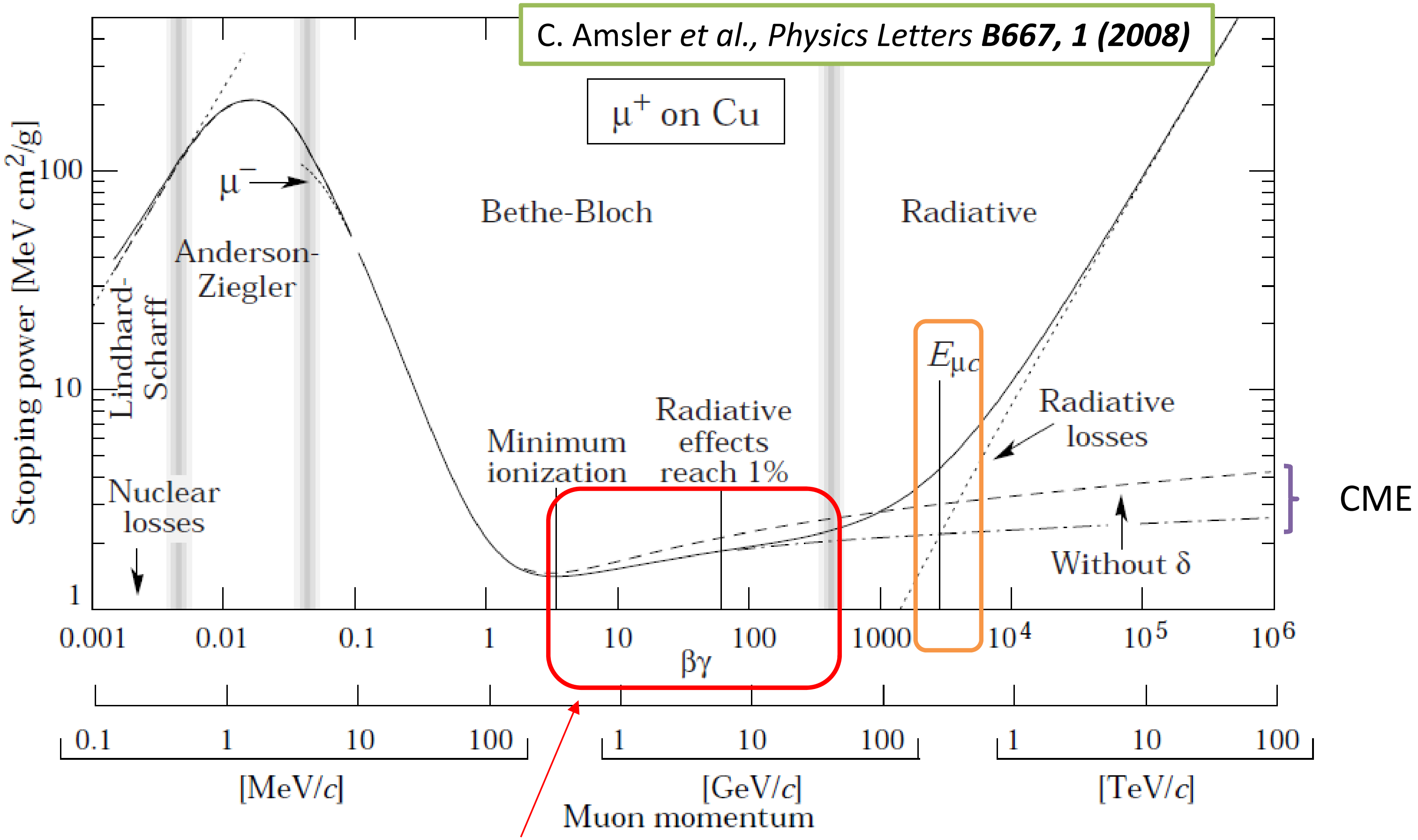
γ

2 explicit contributions:

$$\frac{dE_{\rm Cher.}}{dx} = 2\pi n_e \frac{z^2\alpha_{\rm QED}^2\hbar c^2}{m_e c^2\beta^2} \times \left[\ln\left(\frac{\omega_0^2+\omega_p^2}{\omega_p^2}\right) - \frac{\omega_0^2}{\omega_p^2\gamma^2}\right]$$

$$\left.\frac{dE_{\rightarrow{\rm mat.}}}{dx}\right|_{b>b_{\rm int}} = 2\pi n_e \frac{z^2\alpha_{\rm QED}^2\hbar c^2}{m_e c^2\beta^2} \times \ln\left(\frac{1.123^2 V^2}{(\omega_p^2+\omega_0^2) b_{\rm int}^2}\right)$$

Cold medium effects

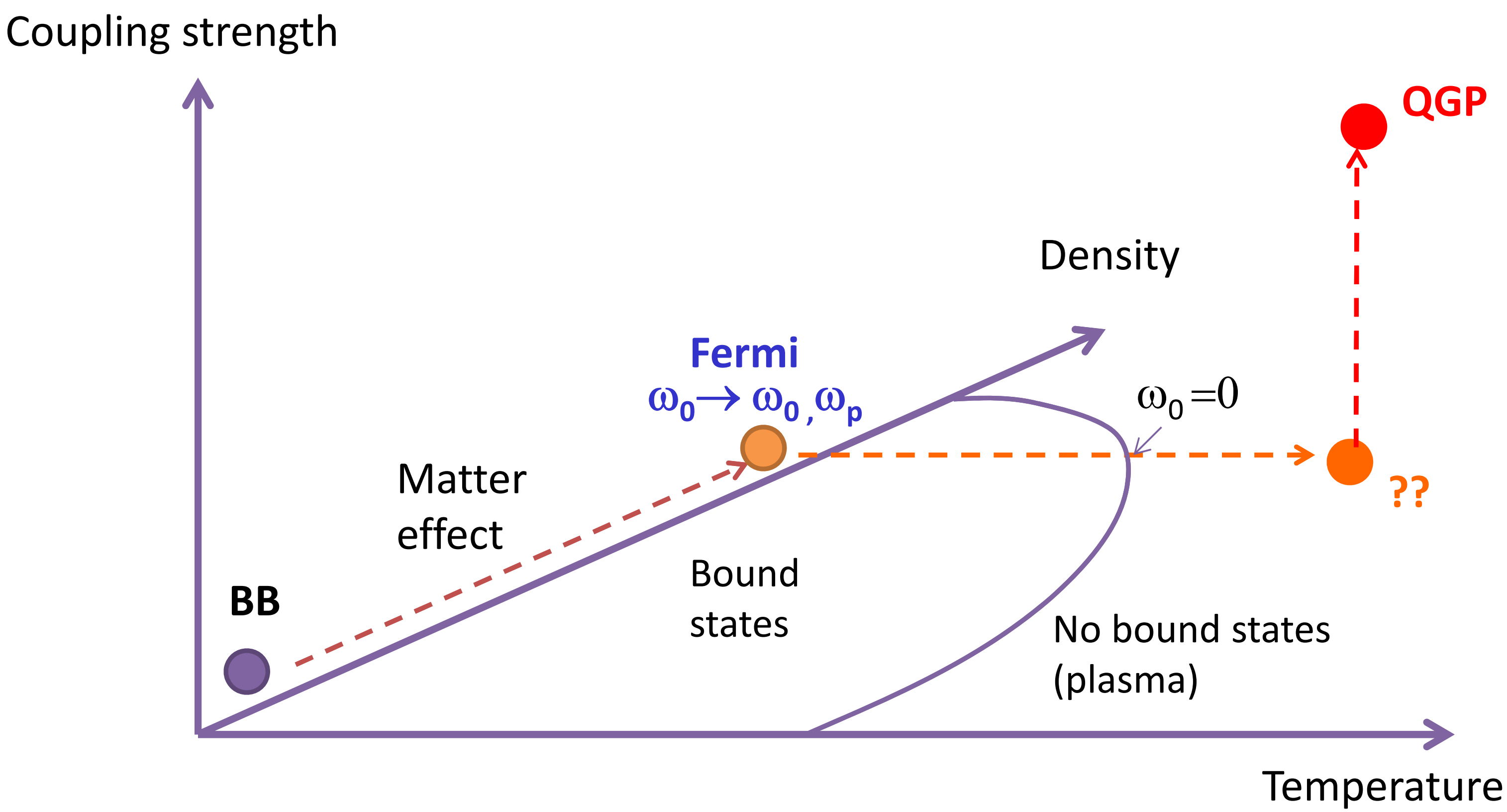


The « good » regime for intermediate p_T HQ… but what is δ ? Lowers the increasing trend

=> Lesson : important to understand the field propagation in the medium

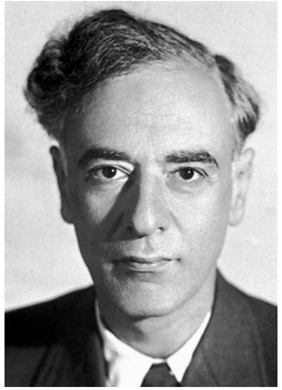
Intermediate summary

What do we want ? Acquire a global understanding of energy loss.



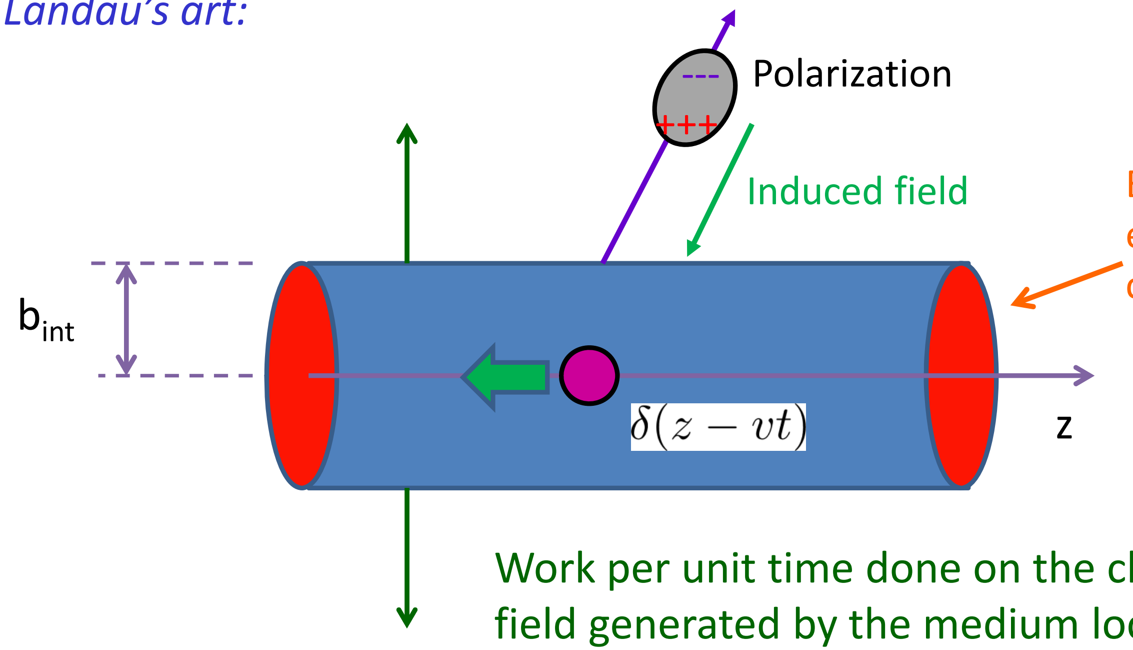
Next step : What is the effect of the temperature ?

Energy loss from a moving charge in a medium



First, need to know the (E,B) field due to the moving charge … what includes the medium response… then :

Landau's art:



Work per unit time done on the charge by the induced field generated by the medium located at $r_T > b_{int}$

Fermion Energy loss in a NR plasma

Landau's art: in evaluate the work done by the induced field on the charge itself; same as energy loss (up to a sign)

1. Start from Maxwell equations in Fourier space (ω,q):

$$\varepsilon_L \vec{E}_L + (\varepsilon_T - q^2/\omega^2)\,\vec{E}_T = \frac{4\pi}{i\omega}(\vec{j}_L + \vec{j}_T)$$

With ε_L and ε_T : dielectric functions of the plasma

With $\vec{j}_L$ $(\vec{j}_T)$: long . (transv.) projection of the current on $\vec{q}$

2. Solution for the *induced* field:

$$\vec{E}_{ind}(\vec{q},\omega) = \frac{4\pi}{i}\left[\frac{1}{\varepsilon_L}\times\frac{\vec{j}_L}{\omega} + \frac{1}{\varepsilon_T/\omega^2 - q^2}\times\frac{\vec{j}_T}{\omega}\right]_{\text{ind}}$$

Subtract the vacuum value (same with ε_L=ε_T=1)

Fermion Energy loss in a NR plasma

3. Evaluate the work done on the heavy fermion by the induced field, up to t=L/v, where L is the path length:

$$\Delta E(L) = i \int \frac{d^3 q}{4\pi^3} \int_{-\infty}^{+\infty} d\omega \left[\frac{1}{\varepsilon_L} \frac{\vec{j}_L(\omega,q)}{\omega} + \frac{1}{\varepsilon_L - q^2/\omega^2} \frac{\vec{j}_T(\omega,q)}{\omega} \right]_{\text{ind}} \cdot \int_0^{L/v} dt\, e^{-i(\omega - \vec{q}\cdot\vec{V})t} Q\, \vec{v}(t)$$

Energy loss = - Work on particle

Special case of stationary current:

$$j^\mu(t,\vec{x}) = Q\, v^\mu\, \delta^3(\vec{x} - \vec{V}t) \quad \text{with} \quad v^\mu = (1, \vec{V})$$

= L/V

$$\Rightarrow \quad j^\mu(\omega,\vec{q}) = 2\pi\, Q\, v^\mu\, \delta(\omega - \vec{q}\cdot\vec{V})$$

$$\frac{\Delta E(L)}{L} = i \frac{\alpha_{\text{QED}}}{V} \int \frac{d^3 q}{2\pi^2} \int_{-\infty}^{+\infty} d\omega \left[\frac{1}{\varepsilon_L} \frac{V_L^2(\vec{q})}{\omega} + \frac{1}{\varepsilon_T - q^2/\omega^2} \frac{V_T^2(\vec{q})}{\omega} \right]_{\text{ind}} \delta(\omega - \vec{q}\cdot\vec{V})$$

Only imaginary part contributes

In fact, no subtraction mandatory as the vacuum value is real… To proceed, need for concrete expressions of the permitivities.

Important digression : plasma response !!!

Assuming weak coupling (or more generally, the validity of linear response theory)

Vlasov equation

F: 1-body phase space distribution

$$\begin{cases} \left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\frac{\partial}{\partial \mathbf{x}} + \frac{\mathbf{G}}{m}\cdot\frac{\partial}{\partial \mathbf{v}}\right) F(\mathbf{x},\mathbf{v},t) = 0\,. \\ \text{with } \underbrace{\mathbf{G}(\mathbf{x},t) = \int n(\mathbf{x}',t)\mathbf{G}(\mathbf{x},\mathbf{x}')d\mathbf{x}'} \end{cases}$$

Normalization:

$$\int n(\mathbf{x}',t)d\mathbf{x}' = N$$

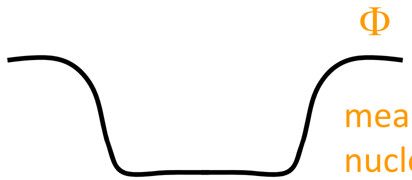
Mean force, derived from the mean potential energy

$$\Phi(\mathbf{x},t) = \int n(\mathbf{x}',t)\Phi(\|\mathbf{x}-\mathbf{x}'\|)d\mathbf{x}'$$

Compare with $\hat{L}_{N,s+1} = \sum_{l=s+1}^{N}\left(-\frac{\mathbf{p}_l}{m}\cdot\frac{\partial}{\partial \mathbf{x}_l} + \frac{\partial V(\mathbf{x}_l)}{\partial \mathbf{x}_l}\cdot\frac{\partial}{\partial \mathbf{p}_l}\right)$

for separable Hamiltonian (no 2 body forces)

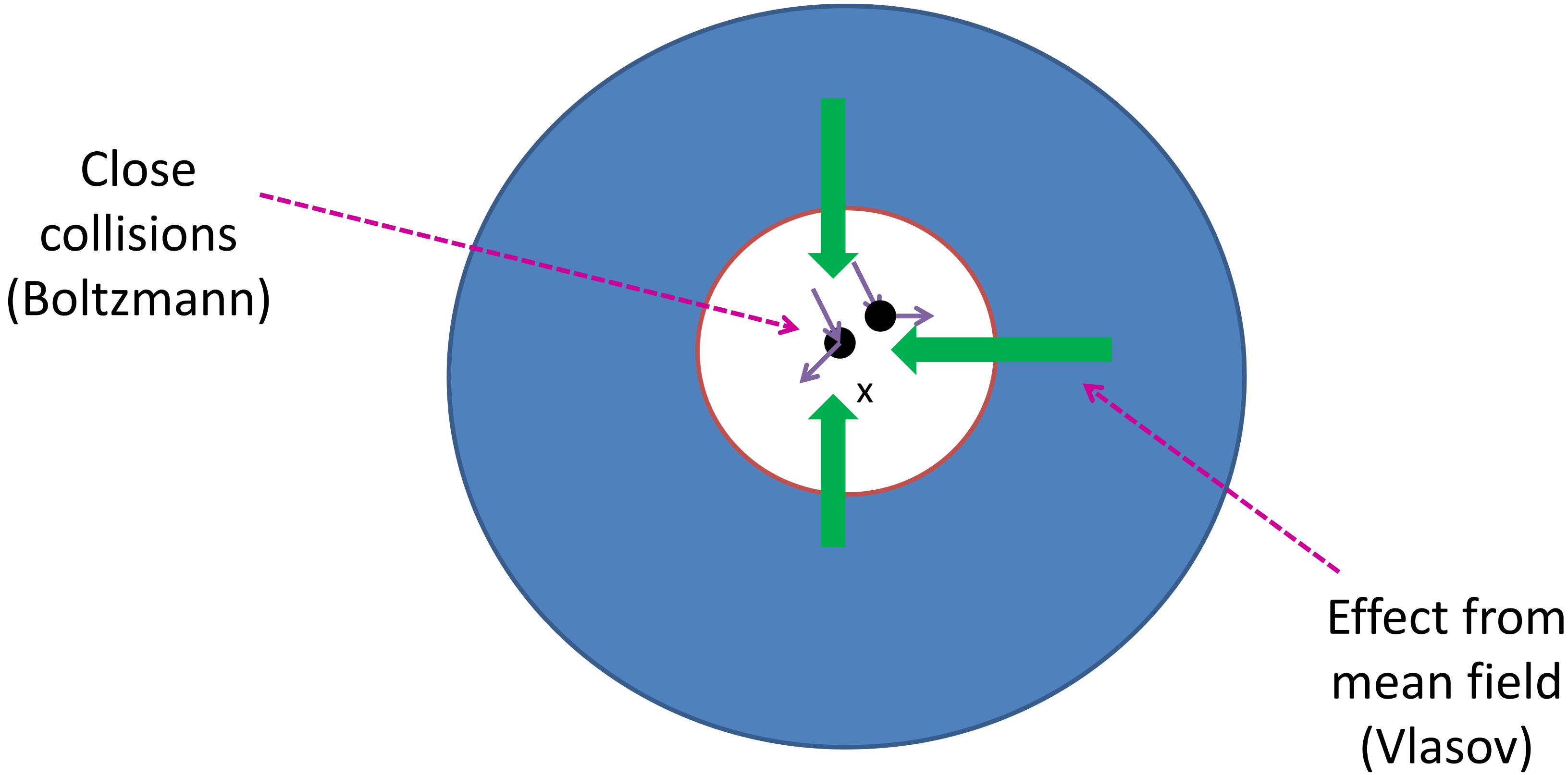
Interesting properties:



mean potential for a nucleon in a nucleus

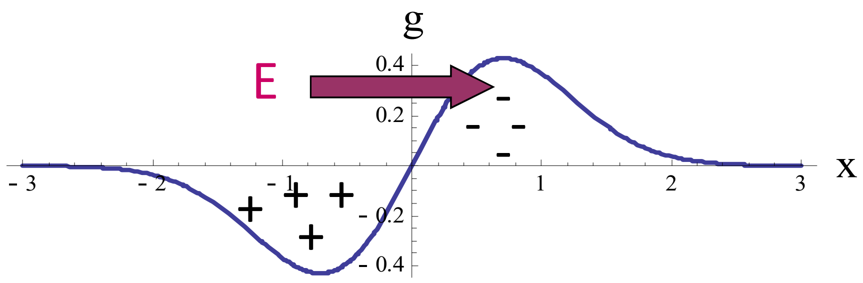
- The mean force=0 if the medium is uniform
- Non linear equation for the F_1 distribution ("hidden" in the mean force)
- Need for resolution algorithm for time evolution and stationary distributions

Concretely


Close
collisions
(Boltzmann)
x
Effect from
mean field
(Vlasov)

Important digression : plasma response !!!

- Ingredients: neutral plasma made of positive ions and electrons
- At equilibrium:
  - uniform spatial density
  - Maxwell-Boltzmann velocity-distribution
  - Charge density = 0 for all x => electric field E=0
- small perturbation
  - of electron density g(x,v) (ions: too heavy)
  - of electromagnetic field



Φ: coupling between the charges and the field

$\Rightarrow$ induced force and oscillations

Eigen-frequency ?
Damping ?

Application of Vlasov equation to weakly coupled plasma

- Basic coupled equations:

F_0: equilibirium 1-body phase space distribution

Vlasov equation
$$\frac{\partial g}{\partial t} + \mathbf{v} \cdot \frac{\partial}{\partial \mathbf{x}} g - \frac{e}{m} \mathbf{E} \cdot \frac{\partial F_0}{\partial \mathbf{v}} = 0$$

Gauss Law (c.g.s.)
$$\nabla \cdot \mathbf{E} = -4\pi e \underbrace{\int g \, d\mathbf{v}} + \ldots$$

Net volumic density of electrons n_p

Possible external source

- (Contrarily to the general Vlasov eq.), those equations are linear in *g* !!! One example of linear response theory (LRT)
- Standard method to pass from differential eq. to algebric equations: Fourier and Laplace transforms

Response from the plasma to the electrical field

- After Fourier transform:

$$\underbrace{\frac{\partial g}{\partial t} + \mathbf{v}\cdot\frac{\partial}{\partial \mathbf{x}}g}_{(\omega - \mathbf{k}\cdot\mathbf{v})\bar{g}} - \frac{e}{m}\mathbf{E}\cdot\frac{\partial F_0}{\partial \mathbf{v}} = 0$$

$$\bar{\mathbf{E}} = \frac{\mathbf{k}(\mathbf{k}\cdot\bar{\mathbf{E}})}{k^2} + \frac{(\mathbf{k}\times\bar{\mathbf{E}})\times\mathbf{k}}{k^2}$$

Transverse field: div (…)=0

Longitudinal field: rot (…)=0

Let us focus on longitudinal modes

$$\frac{\partial F_0}{\partial \mathbf{v}} = n_0\frac{\partial \tilde{f}_0}{\partial \mathbf{v}}$$

density in velocity space (Maxwell-Boltzmann)

- Plasma response :

(to the electric field)

$$\bar{g} = \frac{ien_0}{m}\frac{(\mathbf{k}\cdot\bar{\mathbf{E}})\mathbf{k}}{k^2(\omega - \mathbf{k}\cdot\mathbf{v})}\cdot\frac{\partial \tilde{f}_0}{\partial \mathbf{v}}$$

Gauss law not used yet

A bit of causality

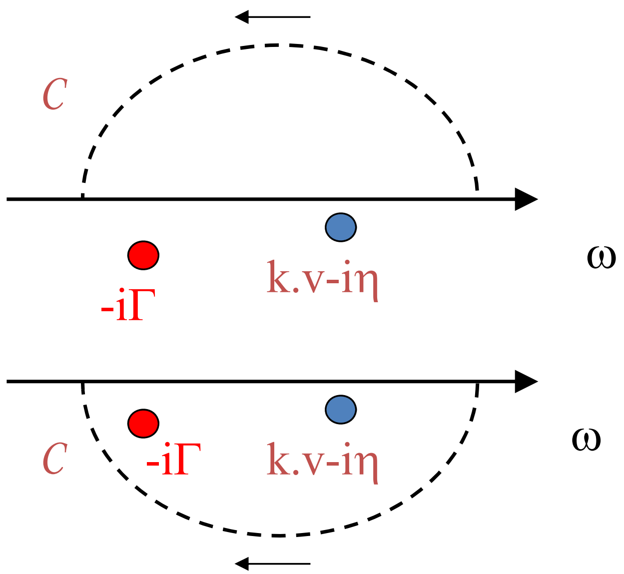
$$\bar{g} = \frac{i e n_0}{m} \frac{(\mathbf{k} \cdot \bar{\mathbf{E}})\mathbf{k}}{k^2(\omega - \mathbf{k} \cdot \mathbf{v})} \cdot \frac{\partial \tilde{f}_0}{\partial \mathbf{v}}$$

What if ω=k.v ? Some plasma particles accompany the (ω,k) component of the electromagnetic field=> amplification => in principle, limitation of LRT

Φ: Plasma response cannot precede the electric field. Take E=E(r)*$e^{-\Gamma t}$ Θ(t)

$$g(x,t,v) = \int d^3k \int_{-\infty}^{+\infty} \bar{g}(\omega,k,v) e^{i(\vec{k}\cdot\vec{x}-\omega t)} d\omega$$

$$\bar{E} \propto \int_0^{+\infty} e^{(i\omega-\Gamma)t} dt \propto \frac{1}{\omega+i\Gamma}$$



Cauchy contour for **t<0** => no poles inside closed contour => vanishing integral (Ok)

Cauchy contour for **t>0** => closed contour contains 2 poles => integral ≠ 0(Ok)

Prescription: ω –> ω + i η if any ambiguity shows up. Better treatment leading to same answer : Laplace transform

Plasma response + Gauss law => permittivity

$$\nabla \cdot \mathbf{E} = \underbrace{-4\pi e \int g\, d\mathbf{v}}_{4\pi\, \rho_{\mathbf{p}}(E)} + \rho_s$$

E field inducing the plasma polarisation has for origin the external source AND the polarisation itself. Usual electrodynamics:

(div E= -div P(E)+ ρ_s => div(E+P(E))=ρ_s)

Fourier

$$i\mathbf{k} \cdot \bar{\mathbf{E}} = \overline{\rho_s} + 4\pi\, \overline{\rho_{\mathbf{p}}}\, (\overline{E})$$

$$-i \frac{\omega_p^2 \mathbf{k} \cdot \bar{\mathbf{E}}}{k^2} \int \frac{\mathbf{k} \cdot \partial \tilde{f}_0 / \partial \mathbf{v}\, d\mathbf{v}}{\omega - \mathbf{k} \cdot \mathbf{v}} \quad \longleftarrow \quad \bar{g} = \frac{i e n_0}{m} \frac{(\mathbf{k} \cdot \bar{\mathbf{E}})\mathbf{k}}{k^2(\omega - \mathbf{k} \cdot \mathbf{v})} \cdot \frac{\partial \tilde{f}_0}{\partial \mathbf{v}}$$

where $\omega_p^2 = \frac{4\pi n e^2}{m_-}$ Plasmon frequency

Gathering terms α E:

$$\underbrace{\left(1 + \frac{\omega_p^2}{k^2} \int \frac{\mathbf{k} \cdot \partial \tilde{f}_0 / \partial \mathbf{v}\, d\mathbf{v}}{\omega - \mathbf{k} \cdot \mathbf{v}} \right)}_{\epsilon_L(k,\omega)} i\mathbf{k} \cdot \bar{\mathbf{E}} = 4\pi\, \overline{\rho_s}$$

$\epsilon_L(k,\omega)$ Long. Dielectric Permittivity [f_0]

Non local in time *and* space

Applications

$$\underbrace{\left(1+\frac{\omega_p^2}{k^2}\int\frac{\mathbf{k}\cdot\partial\tilde{f}_0/\partial\mathbf{v}\,d\mathbf{v}}{\omega-\mathbf{k}\cdot\mathbf{v}}\right)}_{\epsilon_L(k,\omega)} i\mathbf{k}\cdot\bar{\mathbf{E}} = 4\pi\,\overline{\rho_s}$$

$$\omega_p^2 = \frac{4\pi n e^2}{m_-}$$

Point-like charge –e at rest : $\Phi(r) = \frac{-e\exp(-k_d r)}{r}$ with $k_d^2 = \frac{\omega_p^2}{C^2}$

Debye screened potential

r.m.s. velocity of plasma particles (for C)

$$C^2 \sim \frac{T}{m_-}$$

Applications

$$\underbrace{\left(1 + \frac{\omega_p^2}{k^2}\int \frac{\mathbf{k}\cdot\partial \tilde{f}_0/\partial \mathbf{v}\, d\mathbf{v}}{\omega - \mathbf{k}\cdot\mathbf{v}}\right)}_{\epsilon_L(k,\omega)} i\mathbf{k}\cdot\bar{\mathbf{E}} = 4\pi\,\overline{\rho_s}$$

$$\omega_p^2 = \frac{4\pi n e^2}{m_-}$$

Point-like charge –e at rest : $\Phi(r) = \frac{-e\exp(-k_r r)}{r}$ with $k_d^2 = \frac{\omega_p^2}{C^2}$

Debye screened potential

r.m.s. velocity of plasma particles

Eigen modes : dispersion relation …

Longitudinal waves

No external source / perturbation:

$$\underbrace{\left(1+\frac{\omega_p^2}{k^2}\int\frac{\mathbf{k}\cdot\partial\tilde{f}_0/\partial\mathbf{v}\,d\mathbf{v}}{\omega-\mathbf{k}\cdot\mathbf{v}}\right)}_{\epsilon_L(k,\omega)}\;i\mathbf{k}\cdot\bar{\mathbf{E}}=0$$

In vacuum : no solution such that k.E=0 (no longitudinal wave, need a charge for creating a longitudinal field)

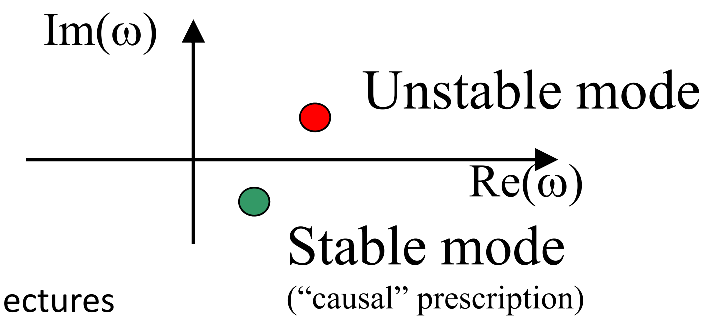
In a plasma : non trivial solutions if $\epsilon_L(k,\omega)=0$

Roots (*in complex plane !*): $\omega=\omega_L(k)$

Dispersion relation

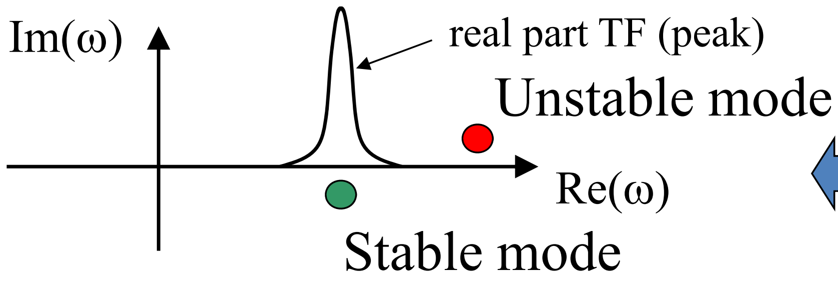
Generalized wave: $\exp(-i\,\omega_0\,t) * \exp(+/-\,\Gamma\,t) = \exp(-i(\omega_0 +/- i\,\Gamma)t)$

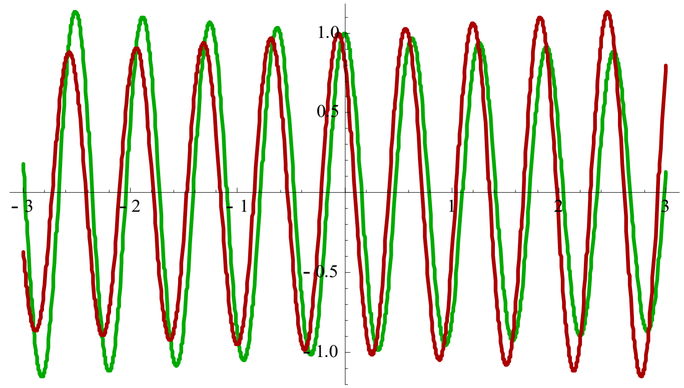
Naïve FT: $\frac{1}{\omega-(\omega_0+/-i\Gamma)}$



Longitudinal waves according to Fourier transform

Assuming that poles are close to the real ω axis





Quasi stationary wave

Let us first assume real ω_L (so neglecting possible damping or amplifications)

$$1+\frac{\omega_p^2}{k^2}\int\frac{\mathbf{k}\cdot\partial\tilde{f}_0/\partial\mathbf{v}\,d\mathbf{v}}{\omega-\mathbf{k}\cdot\mathbf{v}}=0$$

Integration taken in the sense of the principal part (PP)

Integrating on transverse composants $\mathbf{v}_\perp$ for MB

$\mu=\frac{\mathbf{k}\cdot\mathbf{v}}{Ck}$

$$1+\frac{\omega_p^2}{C^2k^2}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{d\mu\,\mu e^{-\mu^2/2}}{\mu-\beta}=0$$

where $\beta\equiv\frac{\omega}{Ck}=v_\varphi/C$ Not 1/T , not v/c !!!

Phase velocity

"Large-wavelength" (⇔ collective) modes

$k << \omega/C \;\Leftrightarrow\; \beta>>1$

$$1+\frac{\omega_p^2}{C^2k^2}\frac{1}{\sqrt{2\pi}}\,PP\!\int_{-\infty}^{\infty}\frac{d\mu\,\mu e^{-\mu^2/2}}{\mu-\beta}=0$$

Complicated integral

Expanding

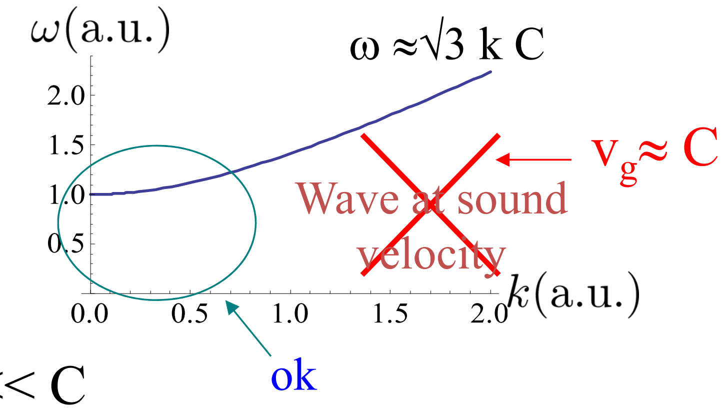
$$1=\frac{\omega_p^2}{C^2k^2}\frac{1}{\sqrt{2\pi}}\int d\mu\frac{\mu}{\beta}e^{-\mu^2/2}\left[1+\left(\frac{\mu}{\beta}\right)+\left(\frac{\mu}{\beta}\right)^2+\left(\frac{\mu}{\beta}\right)^3+\cdots\right]$$

Simple Integrals

Integrating

$$\omega_{(L)}^2=\omega_p^2+3C^2k^2$$

Dispersion relation



$k << \omega/C \Leftrightarrow k << \omega_p/C = k_D$

Ok, consistent with the picture of collective mode

Group velocity: $v_g = d\omega/dk = C^2k/\omega = C/v_\varphi \times C << C$

Collective **plasmonic modes** are subsonic (no shock waves)

Applications

$$\underbrace{\left(1+\frac{\omega_p^2}{k^2}\int\frac{\mathbf{k}\cdot\partial\tilde{f}_0/\partial\mathbf{v}\,d\mathbf{v}}{\omega-\mathbf{k}\cdot\mathbf{v}}\right)}_{\epsilon_L(k,\omega)}\; i\mathbf{k}\cdot\bar{\mathbf{E}} = 4\pi\,\overline{\rho_s}$$

$$\omega_p^2 = \frac{4\pi n e^2}{m_-}$$

- Point-like charge –e at rest : $\Phi(r) = \frac{-e\exp(-k_r r)}{r}$ with $k_d^2 = \frac{\omega_p^2}{C^2}$

  Debye screened potential

  r.m.s. velocity of plasma particles

- Eigen modes : dispersion relation … and attenuation

Extension to the complex plane

$u = \frac{\mathbf{k}\cdot\mathbf{v}}{k}$ $\qquad \epsilon(k,\omega) = 1 + \frac{e^2}{\epsilon_0\, m_e\, k}\int_{-\infty}^{\infty}\frac{\partial F_0/\partial u}{\omega - k\,u}\,du = 0,$ with $F_0(u) = \frac{n}{(2\pi\, T_e/m_e)^{1/2}}\exp(-m_e\, u^2/2\,T_e)$

$\omega_r + i\,\omega_i$ (for ω)

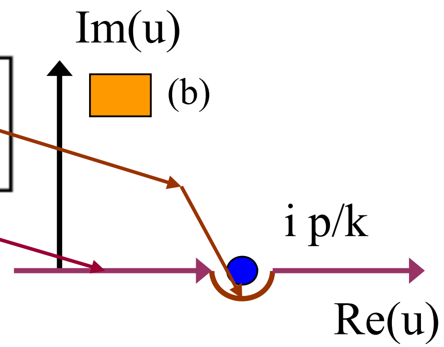
$W(\beta)$ $\qquad \tilde{f}_1(\mu)$

Take $u=\mu*(k_B\, T/m)^{1/2} = \mu*C$ => $\varepsilon(k,\omega) = 1 + \frac{\omega_p^2}{C^2k^2}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{d\mu\;\mu e^{-\mu^2/2}}{\mu-\beta}$ $\qquad$ with $\beta = \frac{\omega_r + i\,\omega_i}{k\,C}$

a) $\omega_i>0$: No unstable collective mode ! (status of a theorem, under some wide assumptions)

b) $\omega_i<0$ but small :

$$\varepsilon(k,\omega) = 1 - \frac{\omega_p^2}{C^2k^2}\frac{1}{\sqrt{2\pi}}\left[PP\int_{-\infty}^{\infty}\frac{d\mu\;\mu e^{-\mu^2/2}}{\beta-\mu} + \pi i\,\tilde{f}_1'(\beta)\right]$$



Large wavelength expansion ($k \ll \omega_r/C$)

$$\varepsilon(k,\omega) = 1 - \beta_p^2\left[\frac{1}{\beta^2} + \frac{3}{\beta^4} + \cdots + i\sqrt{\frac{\pi}{2}}\tilde{f}_1'(\beta)\right]$$

$\beta_p^2 = \frac{\omega_p^2}{C^2k^2}$ $\qquad \beta = \frac{\omega_r + i\,\omega_i}{k\,C}$

Small **but finite**

Damping of longitudinal waves

$\varepsilon(k,\omega) = 0 \Rightarrow 1 = \beta_p^2 \left[\frac{1}{\beta^2} + \frac{3}{\beta^4} + \cdots + i\sqrt{\frac{\pi}{2}}\tilde{f}_1'(\beta) \right]$

Search for some complex solution for fixed k

1) ω_r: $\beta \approx \beta_p >> 1 \Rightarrow$ $\approx \beta_p^4$

$\Rightarrow \bar{\beta}^2 = \beta_p^2 + 3 \Leftrightarrow \omega_{L,r}^2 = \omega_p^2 + 3C^2k^2$

2) ω_i: $1 \simeq \frac{\beta_p^2}{\beta^2} + i\beta_p^2 \sqrt{\frac{\pi}{2}}\tilde{f}_1'(\beta)$ with $\beta = \bar{\beta} + i\,\delta\beta$ ($<<\bar{\beta}$)

$\approx \beta_p^2 / (\bar{\beta}^2 + 2i\,\beta\,\bar{\delta}\beta)$

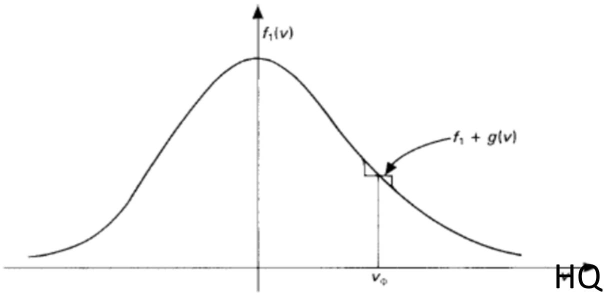
$\approx 1 - 2i\,\delta\beta/\bar{\beta}$

$\approx \tilde{f}_1'(\bar{\beta}) \times (1 + i\,a\,\delta\beta/\bar{\beta})$

$$\delta\beta = \Im\beta = \bar{\beta}^3 \sqrt{\tfrac{\pi}{8}}\tilde{f}_1'(\bar{\beta})$$

Which is <0 (consistent) provided $u\,\partial F_0/\partial u < 0$ (u is the longitudinal speed of the plasma particles along **k**)

Always satisfied for Maxwell-Boltzmann distribution:



Landau damping !

Energy of the plasmon is smoothly transfered -> plasma

Main facts from plasma physics according to Vlasov eq.

1. For a plasma in statistical equilibrium, roots of $\varepsilon(k,\omega)$ wrt ω are located in the lower half complex plane and correspong to (Landau) damped modes. There is no unstable mode!
2. For $k \ll k_D$,

$$\Gamma \approx \sqrt{\frac{\pi}{8}} e^{-3/2} \omega_p \beta_p^3 e^{-\beta_p^2/2} \quad \text{with} \quad \beta_p^2 = \frac{1}{k^2 \lambda_d^2} \gg 1$$

➡ $\Gamma \ll \omega_p$ so that collective modes have a lifetime (Γ^{-1}) much longer than their period ($1/\omega_p$) => well-defined quasi particle

3. Theorem: no unstable mode if the distribution of longitudinal velocities is simply peaked.

Applications

$$\underbrace{\left(1+\frac{\omega_p^2}{k^2}\int\frac{\mathbf{k}\cdot\partial\tilde{f}_0/\partial\mathbf{v}\,d\mathbf{v}}{\omega-\mathbf{k}\cdot\mathbf{v}}\right)}_{\epsilon_L(k,\omega)} i\mathbf{k}\cdot\bar{\mathbf{E}} = 4\pi\,\overline{\rho_s}$$

$$\omega_p^2 = \frac{4\pi n e^2}{m_-}$$

Point-like charge –e at rest : $\Phi(r) = \frac{-e\exp(-k_r r)}{r}$ with $k_d^2 = \frac{\omega_p^2}{C^2}$

Debye screened potential

r.m.s. velocity of plasma particles

Eigen modes : dispersion relation … and attenuation

Back to the Energy loss of a moving charge : permittivity is indeed a complex-valued function

The moving charge continuously induces some field propagating in the plasma that is damped => the lost energy is taken from the charge kinetic energy => Energy loss

Application: fermion Energy loss in a NR plasma

Physics question: what are the forces acting on a fermion at constant velocity in a plasma ?

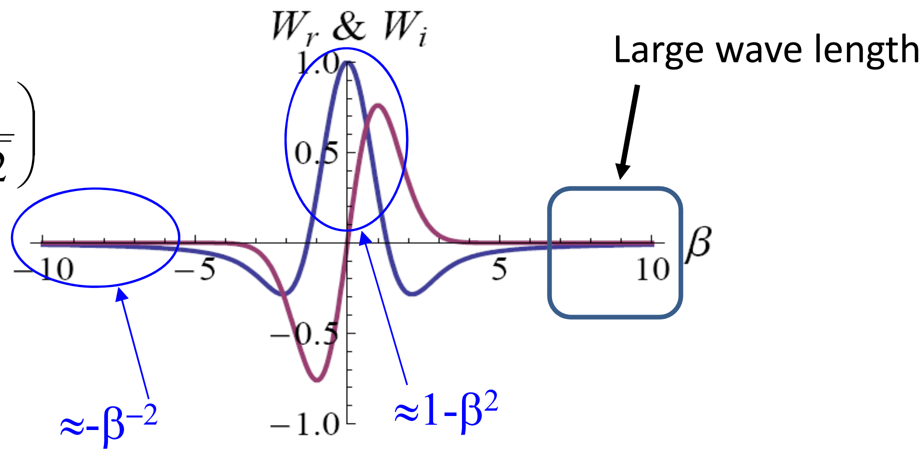
Need for a better insight on permittivity:

$$\varepsilon(k,\omega) = 1 + \frac{\omega_p^2}{C^2k^2}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{d\mu\,\mu e^{-\mu^2/2}}{\mu-\beta}$$

W(β) with $\beta := \frac{\omega_r + i\omega_i}{k\,C}$

For real β: $$W_r(\beta) = 1 - \sqrt{\frac{\pi}{2}}\beta\,\exp\left(-\frac{\beta^2}{2}\right)\mathrm{erfi}\left(\frac{\beta}{\sqrt{2}}\right)$$

$$W_i(\beta) = \sqrt{\frac{\pi}{2}}\beta\,\exp\left(-\frac{\beta^2}{2}\right)$$



$$1 - \beta_p^2\left[\frac{1}{\beta^2} + \frac{3}{\beta_4} + \cdots + i\sqrt{\frac{\pi}{2}}\tilde{f}_1'(\beta)\right]$$

Fermion Energy loss in a NR plasma

We will first concentrate on the "far" field contribution

$$\frac{dE^{\rm far}_{\rm col}}{dx} = -\frac{\alpha_{\rm QED}}{V}\,{\rm Im}\left(\int \frac{d^3q}{2\pi^2}\int_{-\infty}^{+\infty}\frac{d\omega}{\omega}\left[\frac{V_L^2(Q)}{\epsilon_L(Q)}+\frac{V_T^2(Q)}{\epsilon_T(Q)-q^2c^2/\omega^2}\right]_{\rm ind}\delta(\omega-\vec{q}\cdot\vec{V})\right)$$

Longitudinal part:

$$V_L = \vec{V}\cdot\frac{\vec{q}}{\|\vec{q}\|}$$

1. We integrate on cos θ

$$\int V^2\cos^2\theta\,\delta(\omega-qV\cos\theta) = \int\frac{V^2}{qV}\cos^2\theta\,\delta\left(\frac{\omega}{qV}-\cos\theta\right) = \frac{\omega^2}{q^3V}$$

provided

$$-qV < \omega < qV$$

2. We investigate $1/\varepsilon_L$

$$\epsilon_L(\omega,q) = 1+\frac{q_D^2}{q^2}W_r\left(\frac{\omega}{qv_{\rm rms}}\right)+i\frac{q_D^2}{q^2}W_i\left(\frac{\omega}{qv_{\rm rms}}\right)$$

$$v_{\rm rms} = C\ !!!$$

$$\Im\left(\frac{1}{\epsilon_L(\omega,q)}\right) = -\frac{\frac{q_D^2}{q^2}W_i\left(\frac{\omega}{qv_{\rm rms}}\right)}{\left[1+\frac{q_D^2}{q^2}W_r\left(\frac{\omega}{qv_{\rm rms}}\right)\right]^2+\left[\frac{q_D^2}{q^2}W_i\left(\frac{\omega}{qv_{\rm rms}}\right)\right]^2}$$

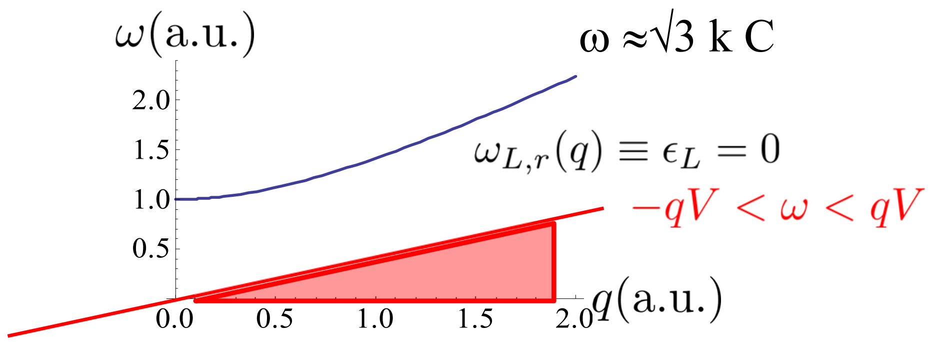
Fermion Energy loss in a NR plasma

$$\Im\left(\frac{1}{\epsilon_L(\omega,q)}\right) = -\frac{\frac{q_D^2}{q^2}W_i\left(\frac{\omega}{qv_{\mathrm{rms}}}\right)}{\left[1+\frac{q_D^2}{q^2}W_r\left(\frac{\omega}{qv_{\mathrm{rms}}}\right)\right]^2+\left[\frac{q_D^2}{q^2}W_i\left(\frac{\omega}{qv_{\mathrm{rms}}}\right)\right]^2}$$

$-qV < \omega < qV$

$\Re(\epsilon_L)$ Can it vanish for $-qV < \omega < qV$? If yes, then $\Im\left(\frac{1}{\epsilon_L(\omega,q)}\right) \propto \delta\left(\omega-\omega_{L,r}(q)\right)$

$V < \sqrt{3}C$



No intercept between $\omega_{L,r}(q)$

and $-qV < \omega < qV$ domain :

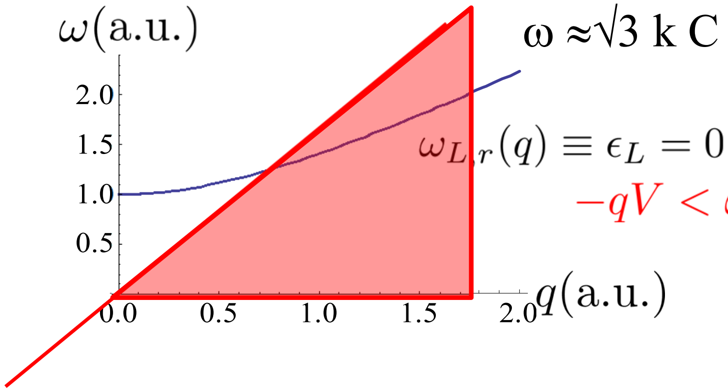
No plasmon can be excited

Fermion Energy loss in a NR plasma

$$\Im\left(\frac{1}{\epsilon_L(\omega,q)}\right) = -\frac{\frac{q_D^2}{q^2}W_i\left(\frac{\omega}{qv_{\rm rms}}\right)}{\underbrace{\left[1+\frac{q_D^2}{q^2}W_r\left(\frac{\omega}{qv_{\rm rms}}\right)\right]}_{\Re(\epsilon_L)}^2+\left[\frac{q_D^2}{q^2}W_i\left(\frac{\omega}{qv_{\rm rms}}\right)\right]^2}$$

$-qV < \omega < qV$

Can it vanish for $-qV < \omega < qV$? If yes, then $\Im\left(\frac{1}{\epsilon_L(\omega,q)}\right) \propto \delta\left(\omega - \omega_{L,r}(q)\right)$



$V < \sqrt{3}C$

No intercept between $\omega_{L,r}(q)$ and $-qV < \omega < qV$ domain : No plasmon can be excited

$V > \sqrt{3}C$

Intercepts between $\omega_{L,r}(q)$ and $-qV < \omega < qV$ domain : plasmon can be excited