- Convince yourself the the m-t has diagram for the Ising model is correct

Since we are looking at m-t diagram
$F(m,t)$ is the relevant potential

. Equilibrium: minima of $F(m,t)$
Co-existence: 2 minima.

Ising model.

$$F = \frac{1}{2}m^2 t + \frac{1}{4}m^4 - hm$$

Co-existence:
$h=0$

$$\frac{\partial F}{\partial m} = 0 \Rightarrow h = mt + m^3$$

$h=0$, $m=0$, $m = \pm\sqrt{-t}$

$t>0 \Rightarrow m=0$

$t<0 \Rightarrow m = \pm\sqrt{|t|}$; $m=0$.

$$\frac{\partial^2 F}{\partial m^2} = t + 2m^2$$

stable: $\frac{\partial^2 F}{\partial m^2} > 0$: $t>0$, $m=0$ ✓

$t<0$: $m=0 \Rightarrow$ unstable

$m = \pm\sqrt{|t|}$

$$\Rightarrow \frac{\partial^2 F}{\partial m^2} = -|t| + 2|t| > 0$$

stable

Calculate the magnetic susceptibility in Ising model, χ = dm

Calculate magnetic susceptibility.

$\chi = \frac{\partial m}{\partial h}$

$h = mt + m^3 \Rightarrow \frac{\partial h}{\partial m} = t + 3m^2$

$\Rightarrow \frac{\partial m}{\partial h} = \frac{1}{t+3m^2}$

$t > 0$: equilibrium: $m = 0 \Rightarrow \chi = \frac{1}{t}$

$t < 0$: equilibrium: $m = \pm\sqrt{|t|}$

$\Rightarrow \chi = \frac{1}{-|t| + 3|t|} = \frac{1}{2|t|}$

• Show that $\chi_2 \sim \left.\frac{d\rho}{d\mu}\right|_T = \frac{\rho}{dP/d\rho\big|_T}$. What does this imply?

$$\chi_2 \simeq \left.\frac{\partial \rho}{\partial \mu}\right|_T$$

$$dP = -\frac{S}{V}\,dT + \frac{N}{V}\,d\mu$$

$$= -s\,dT + \rho\,d\mu$$

$s =$ entropy density.

$$\left.\frac{dP}{d\rho}\right|_T = \rho\,\frac{d\mu}{d\rho} = \rho\,\frac{1}{\frac{\partial \rho}{\partial \mu}}$$

$$\Rightarrow \frac{d\rho}{\partial \mu} = \rho\,\frac{1}{\left.\frac{\partial P}{\partial \rho}\right|_T}$$

Simpler way: chain rule.

$$\underbrace{\frac{\partial P}{\partial \mu}}_{\rho} = \frac{\partial P}{\partial \rho}\underbrace{\frac{\partial \rho}{\partial \mu}}_{\chi_2} \Rightarrow \chi_2 = \frac{\rho}{\frac{\partial P}{\partial \rho}}$$

$$dE = -p\,dV + T\,dS + \mu\,dN$$

$$E = TS + \mu N - pV.$$

$$dE = T\,dS + S\,dT + \mu\,dN + N\,d\mu - p\,dV - V\,dP$$

$$\Rightarrow$$

$$S\,dT + N\,d\mu - V\,dP = 0$$

$$\Rightarrow dP = \frac{S}{V}\,dT + \rho\,d\mu$$

$$= s\,dT + \rho\,d\mu$$

Work out the first 2 or 3 (factorial) cumulants in terms of (factorial) moments. Do the same thing by using the generating functions

$$g(t) = \bar{g}(e^t) \;; \quad z(t) = e^t$$

$$u_1 \quad \frac{\partial g}{\partial t} = \frac{\partial \bar{g}}{\partial z}\frac{\partial z}{\partial t} = \frac{\partial \bar{g}}{\partial t} e^t \Big|_{t\to 0} = c_1$$

$$u_2 = \frac{\partial^2 g}{\partial t^2} = \frac{\partial^2 g}{\partial z^2}\frac{\partial z}{\partial t} + \frac{\partial g}{\partial z}\frac{\partial^2 z}{\partial t^2}$$

$$= \frac{\partial^2 g}{\partial z^2} e^t + \frac{\partial g}{\partial z} e^{2t}$$

$$= c_2 + c_1$$

$$u_3 = \frac{\partial^3 g}{\partial z^3}\frac{\partial z}{\partial t} + 3\frac{\partial^2 g}{\partial^2 z}\frac{\partial^2 z}{\partial t^2} + \frac{\partial g}{\partial z}\frac{\partial^3 z}{\partial t}$$

$$c_3 + 3c_2 + c_1$$

- Work out the factorial cumulant generating function for a Poisson and Binomial distribution and convince yourself that the factorial cumulants for Poisson vanish for n>1.

$$P(n,N) = \frac{N!}{n!(N-n)!} p^n (1-p)^{N-n}$$

$$g(z) = \sum_n P(n,N) z^n = \sum_n \frac{N!}{n!(N-n)!} (pz)^n (1-p)^{N-n}$$

$$= (pz + (1-p))^N$$

$$P(n,\lambda) = e^{-\lambda} \cdot \frac{\lambda^n}{n!}$$

$$g(z) = \sum_n z^n P(n) = e^{-\lambda} \sum_n \frac{(z\lambda)^n}{n!}$$

$$= e^{-\lambda} e^{z\lambda}$$

$$= e^{\lambda(z-1)}$$

- Work out factorial cumulant (Moment) generating function for $P(n) = \sum_N P_{binomial}(n; N) P(N)$ with $P(N)$ an arbitrary distribution function.

$$h(z) = \sum_n P(n) = \sum_n \sum_N z^n P_{Bin.}(n, N) P(N)$$

$$= \sum_N P(N) \underbrace{\sum_n z^n P_{Bin}(n, N)}_{\text{generating function for Binomial}}$$

$$h_B(z) = (1 - p + zp)^N$$

$$h(z) = \sum_N (1 - p + zp)^N P(N)$$

define $\tilde{z}(z)$: $\tilde{z}(z) = (1 - p + zp)$; $\tilde{z}(z=1) = \underline{1}$

$$h(z) = \sum_N \tilde{z}(z) P(N)$$

$$= \underset{\substack{\uparrow \\ \text{generating function for } P(N)}}{h_p}(\tilde{z})$$

$$\Rightarrow h(z) = h_p(\tilde{z})$$

$$\Rightarrow \frac{\partial h}{\partial z} = \frac{\partial h_p}{\partial \tilde{z}} \frac{\partial \tilde{z}}{\partial z} = p \frac{\partial h_p(\tilde{z})}{\partial(\tilde{z})}$$

$\Rightarrow$
q^{th} order Factorial moment

$$f_q = \frac{\partial^q h(z)}{\partial z^q}\Big|_{z=1} = p^q \frac{\partial^q h_p(\tilde{z}(z))}{\partial \tilde{z}(z)^q}\Big|_{z=0}$$

$$= p^q \frac{\partial^q g_p(\tilde{z})}{\partial \tilde{z}}\Big|_{\tilde{z}=0}$$

$$= p^q F_q^{(P)}$$

where $F_q^{(P)}$ is factorial moment of $P(N)$