Fluctuations and the QCD phase diagram

"A theory is something nobody believes, except the person who made it.
An experiment is something everybody believes, except the person who made it."

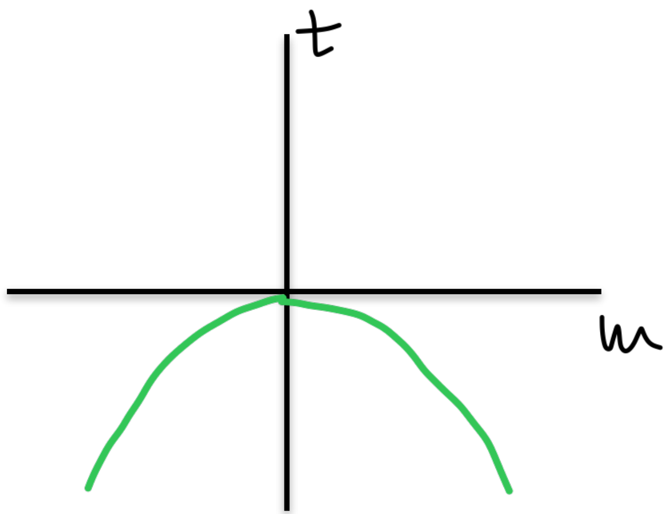
BEST COLLABORATION

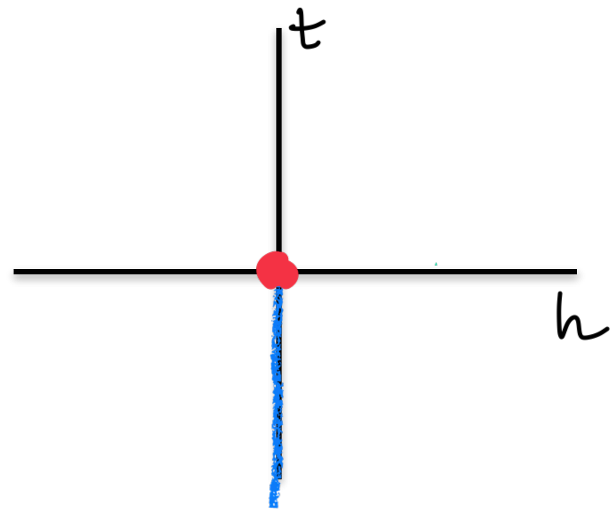
Lecture 1

- Introduction
- Phase transitions
- Ensembles
- Phase diagrams
- Ising model
- Cumulants are derivatives of $\ln Z \sim P$
-

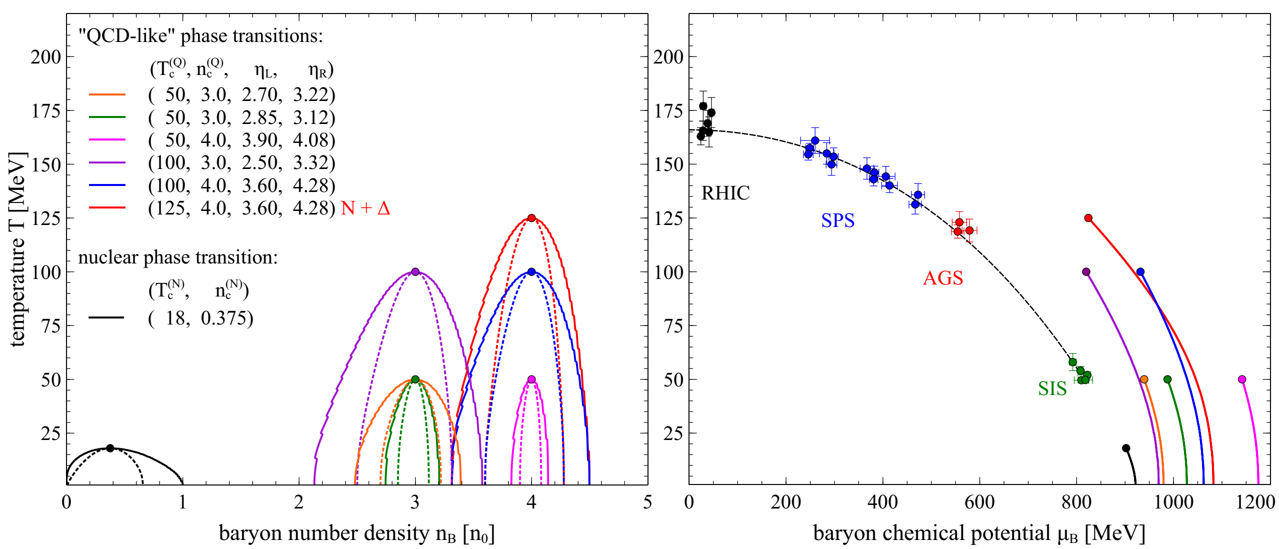
Phase diagrams

Ising



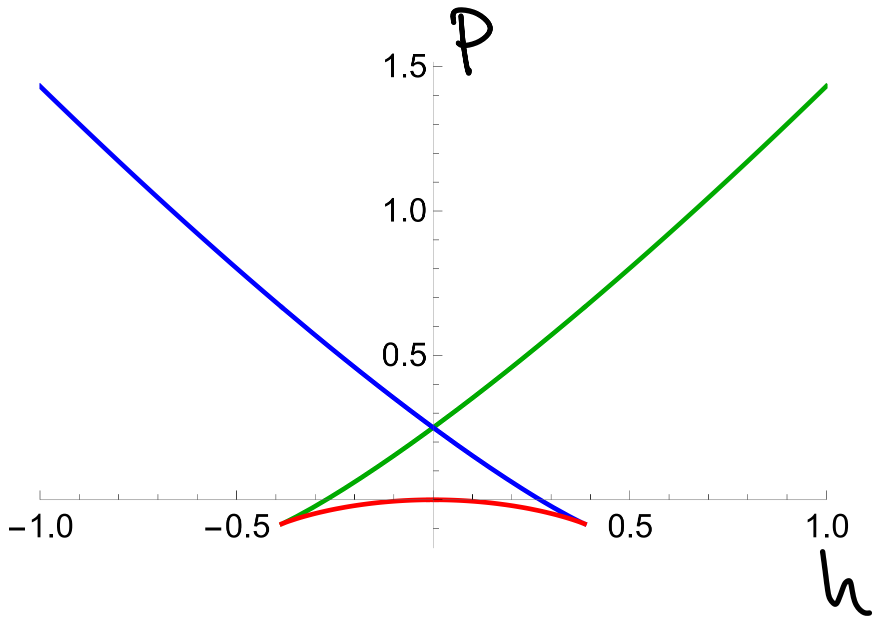


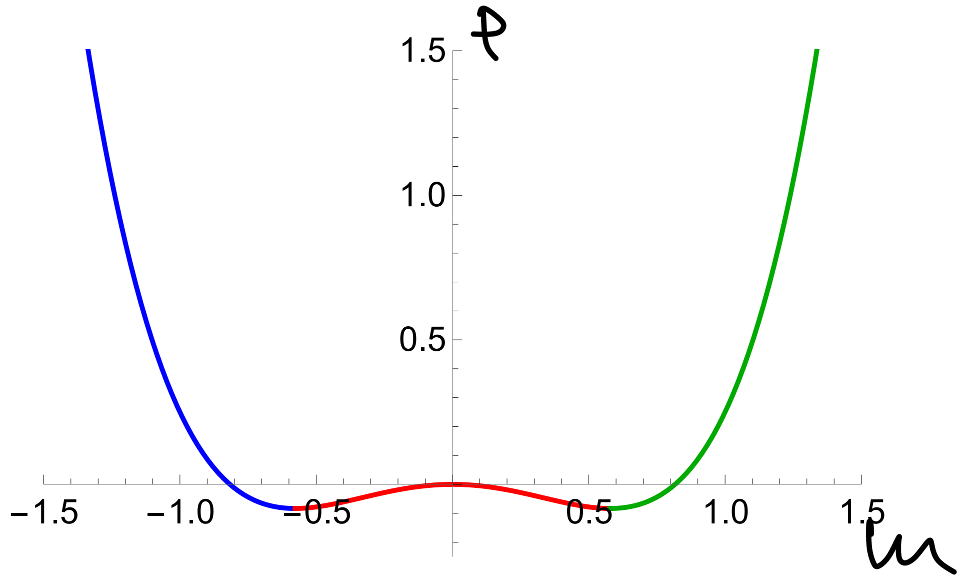
"QCD"



“Pressure”

$$t < 0; t = -1$$

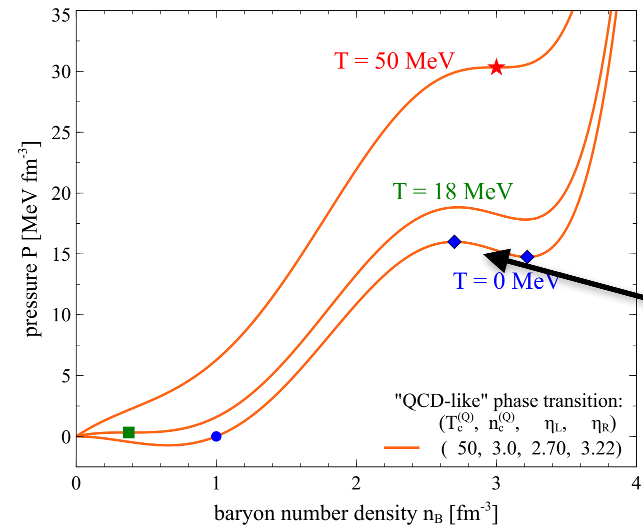




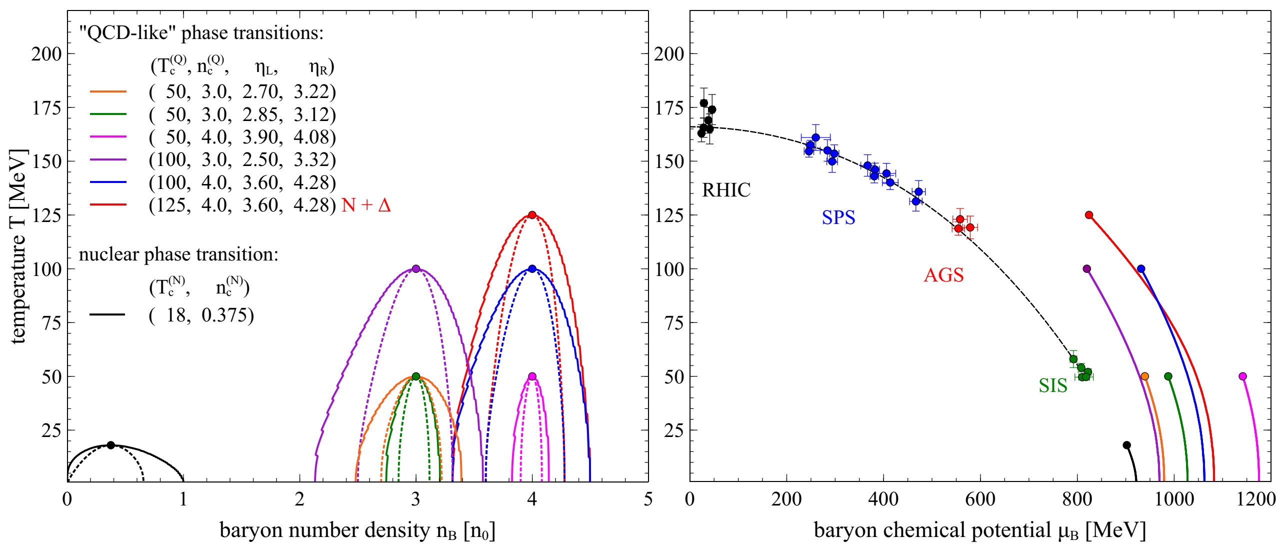
Lecture 2

- Spinodal instability
- Cumulants and correlations (factorial cumulants)
- Remarks phase diagram (liquid gas?)
- Measuring the phase diagram

Simple density functional model



$$\frac{dP}{d\rho} < 0$$



A. Sorensen and VK PRC **104**, 034904 (2021)

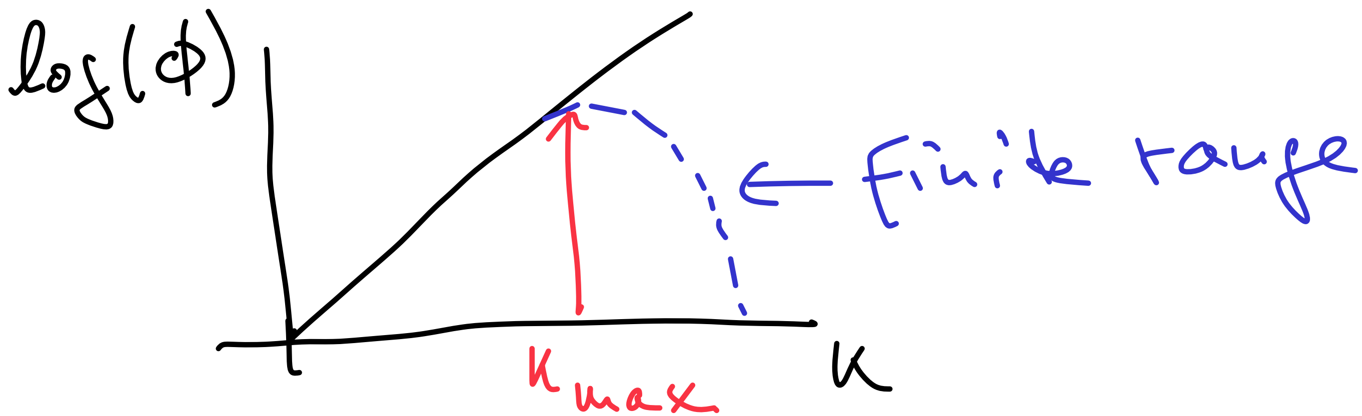
Spinodal instability

Sound Dispersion relation: $\omega = c_s k$

$$c_s^2 = \frac{dP}{d\rho} < 0 \quad \text{in Spinodal region}$$

$$\Rightarrow c_s = \sqrt{c_s^2} \equiv \pm i\gamma \quad \text{imaginary}$$

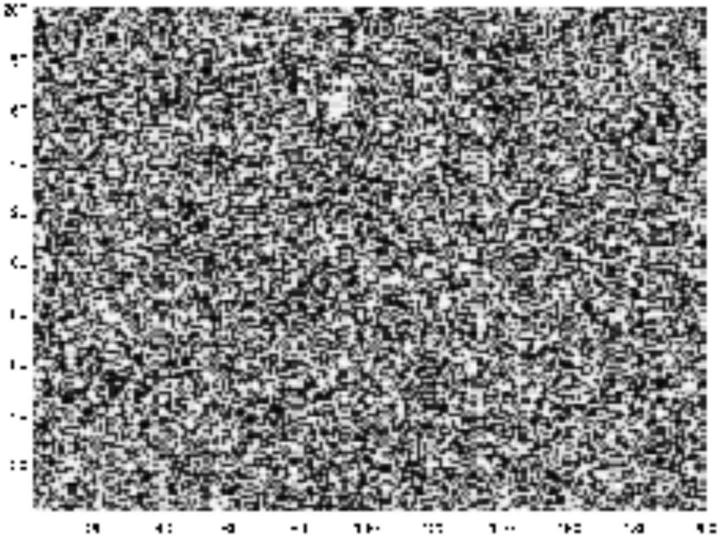
Small disturbance: $\phi = \phi_0 e^{i\omega t} = \phi_0 e^{\pm\gamma k t}$



Exponential growth!!!

$\gamma \sim k$ cut off by finite range interaction (see Randrup et al)

Spinodal decomposition



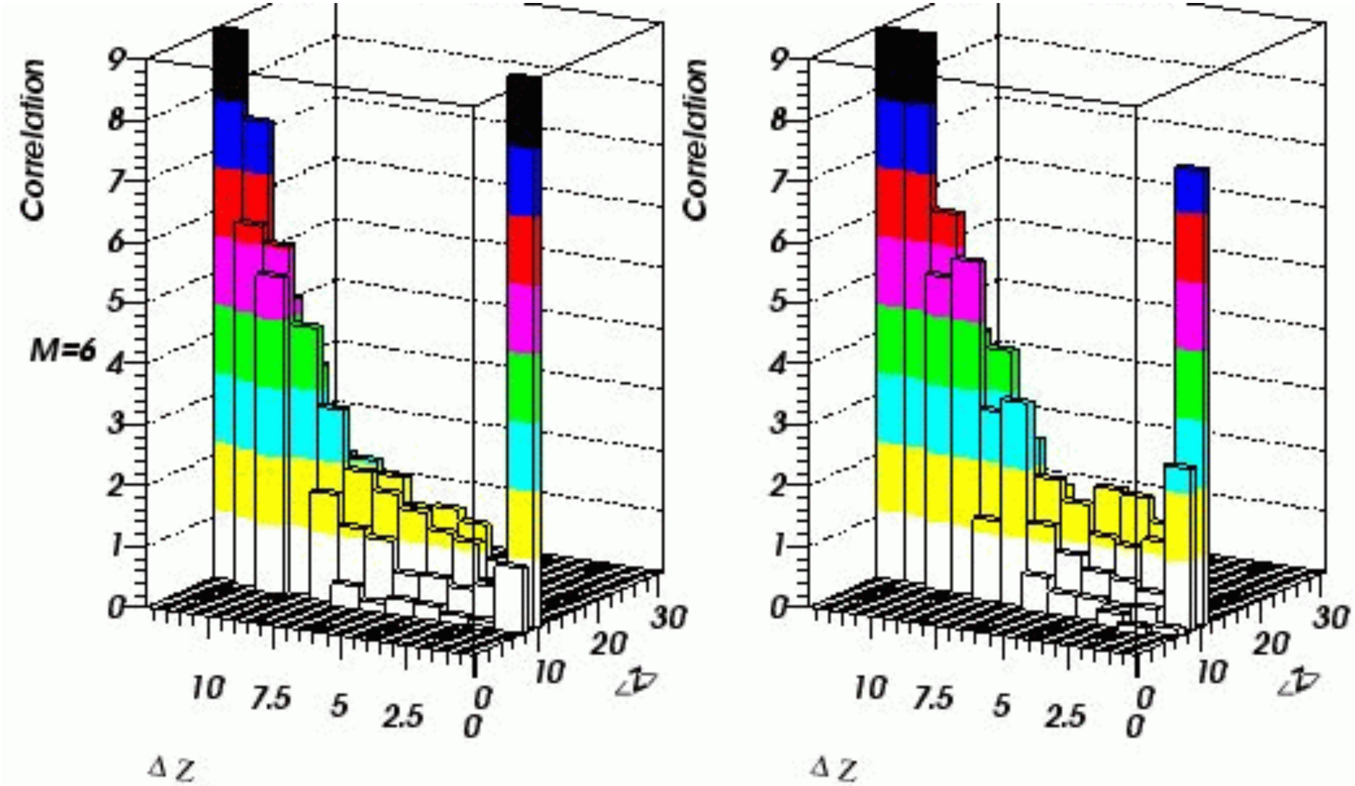
From Wikipedia

Spinodal decomposition in nuclear multifragmentation

32 MeV/A Xe + Sn (b=0)

(select events with 6 IMFs)

Bin wrt $\begin{cases} \langle Z \rangle : & \text{average IMF charge} \\ \Delta Z : & \text{dispersion in IMF charge} \end{cases}$



Experiment (*INDRA @ GANIL*)

Borderie *et al*, PRL 86 (2001) 3252

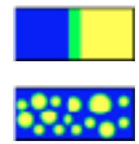
Theory (*Boltzmann-Langevin*)

Chomaz, Colonna, Randrup, …

J. Randrup

Phase-transition dynamics: Density clumping

Phase transition => { Phase coexistence: surface *tension*; Phase separation: *instabilities* }



Introduce a ***gradient*** term:

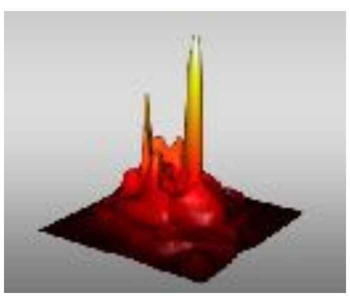
$$p(\mathbf{r}) = p_0(\varepsilon(\mathbf{r}), \rho(\mathbf{r})) - C\rho(\mathbf{r})\nabla^2\rho(\mathbf{r})$$

Insert the modified pressure into existing ideal finite-density fluid dynamics code

Use UrQMD for pre-equilibrium stage to obtain fluctuating initial conditions

Simulate central Pb+Pb collisions at ≈3 GeV/A beam kinetic energy on fixed target, using an Equation of State either ***with*** a phase transition or ***without*** (Maxwell partner):

With phase transition:



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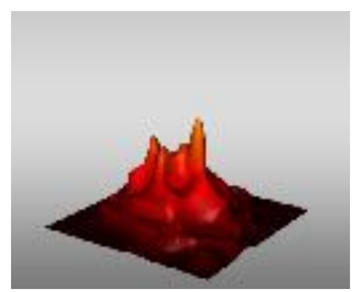
Without phase transition:



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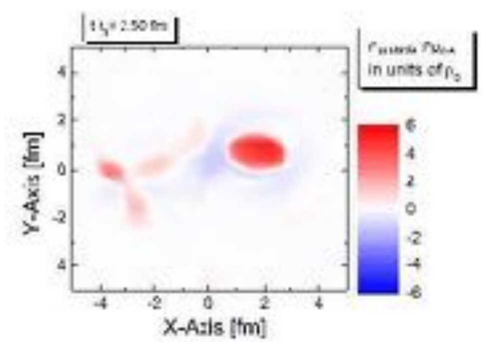
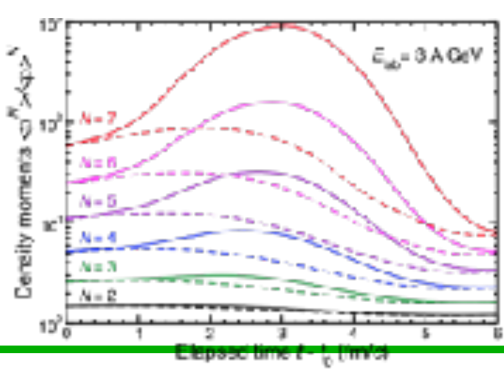
Density enhancement:

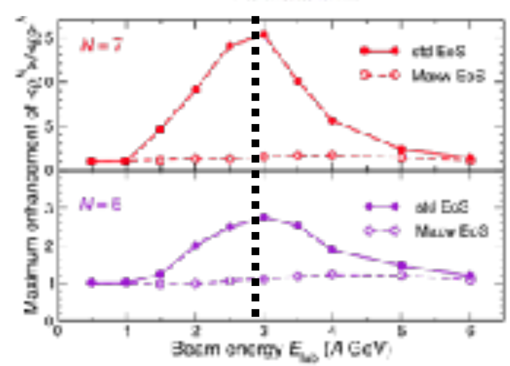




Evolution of density moments

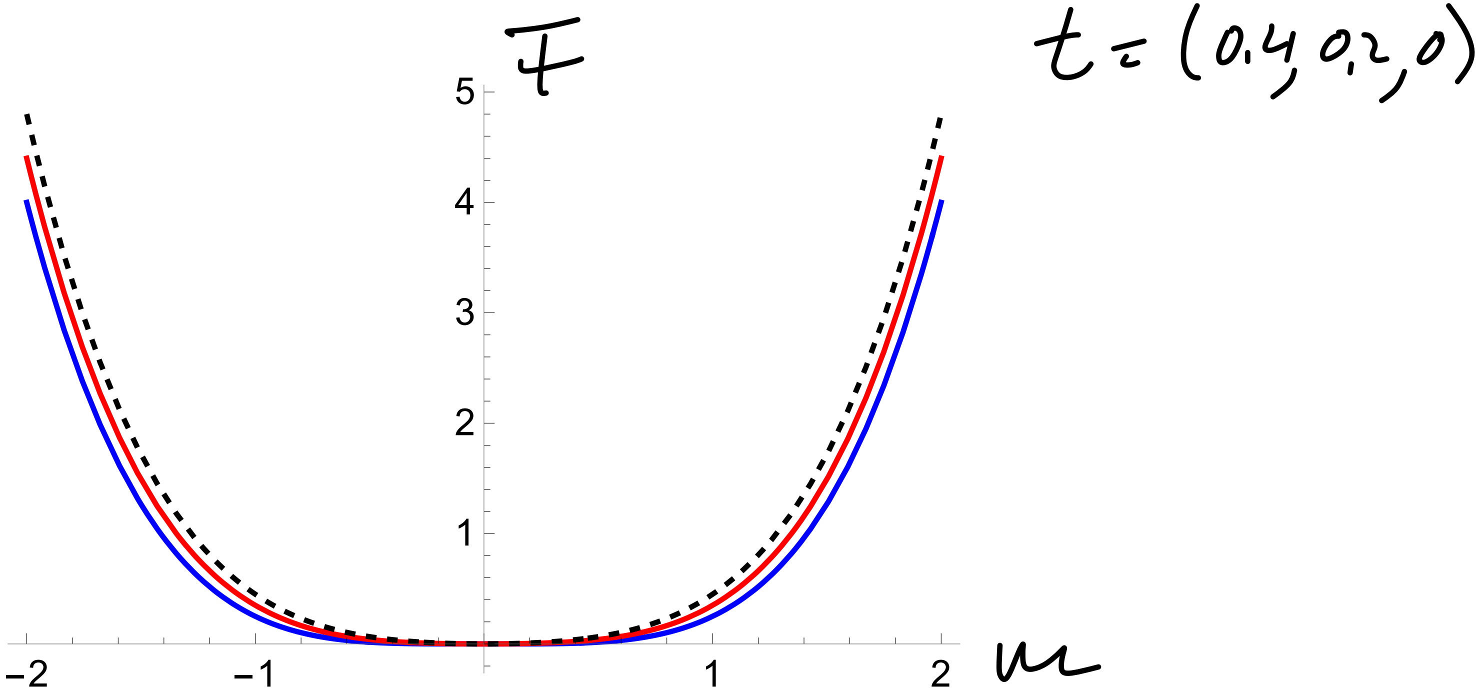
$$\langle \rho^N \rangle \equiv \frac{1}{A}\int \rho(\mathbf{r})^N \rho(\mathbf{r}) d^3\mathbf{r}$$

J. Steinheimer & J. Randrup,
PRL 109, 212301(2012)
PRC 87, 054903 (2013)



E_{Lab}=3 GeV

Approaching the critical point



Sloooow going! A.k.a critical slowing down

Critical point: Good luck?

VOLUME 37, NUMBER 1 PHYSICAL REVIEW LETTERS 5 JULY 1976

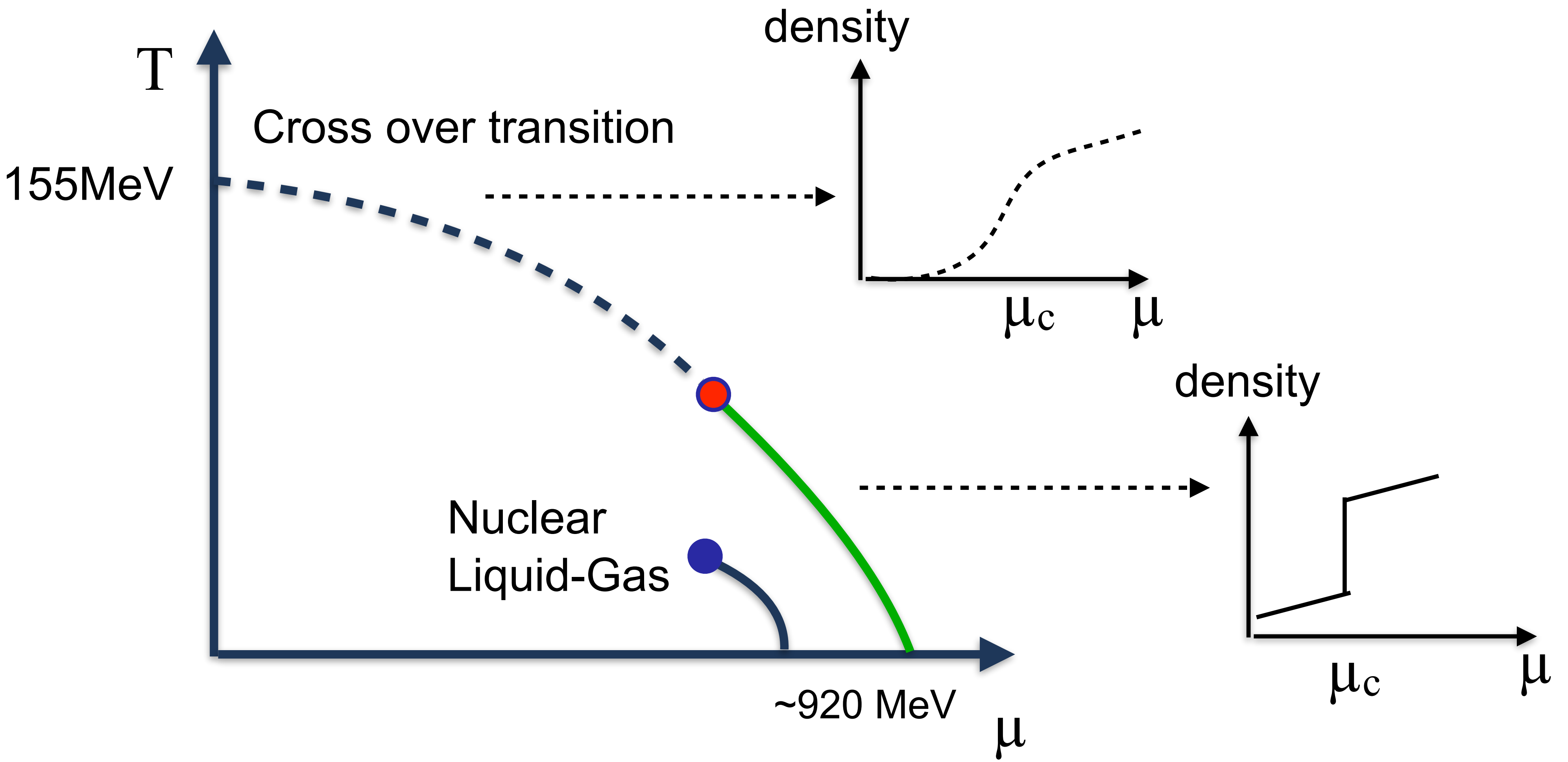
Ising Critical Exponents in Real Fluids: An Experiment

R. Hocken* and M. R. Moldover
Equation of State Section, Heat Division, National Bureau of Standards, Washington, D. C. 20234
(Received 1 March 1976)

We report precise optical measurements of the equations of state of Xe, SF_6, and CO_2 very near their critical points ($|T-T_c|/T_c < 5\times10^{-5}$). We find that the critical exponents of these fluids in this region are close to the exponents calculated from the three-dimensional Ising model.

The filled cell was enclosed in a seven-stage cylindrical thermostat, two stages of which were active and five passive. The cell was mechanically attached and thermally coupled to the innermost stage (a 25-kg cylinder of copper). This block was passive. Its temperature was controlled by controlling the temperature of the thermally decoupled heater shell which surrounded it. This inner stage was purposely isolated to reduce temperature gradients and integrate temperature oscillations. It has a time constant of six hours with respect to heater-shell temperature changes. The thermal equilibrium of the sample was assessed from the temporal stability of the Fraunhofer pattern. Two isotherms per day were taken far from T_c, but the rate became one per day or less as T_c was approached.

Looking for signs of a transition



Cumulants of (Baryon) Number

$$K_n = \frac{\partial^n}{\partial(\mu/T)^n} \ln Z = \frac{\partial^{n-1}}{\partial(\mu/T)^{n-1}} \langle N \rangle$$

$$K_1 = \langle N \rangle\,, \quad K_2 = \langle N - \langle N \rangle \rangle^2\,, \quad K_3 = \langle N - \langle N \rangle \rangle^3$$

Cumulants scale with volume (extensive): $K_n \sim V$

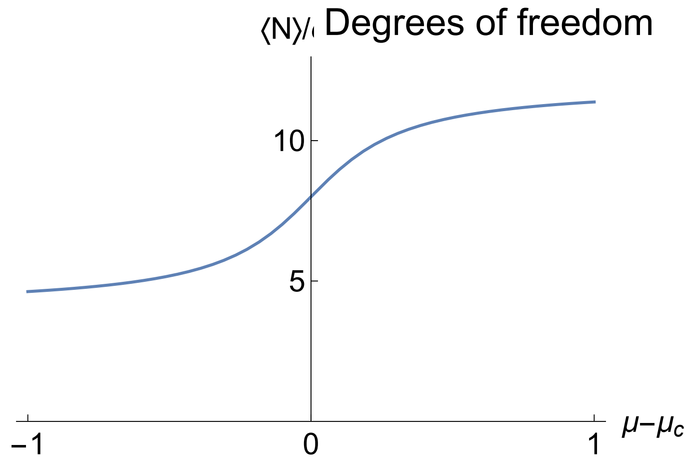
Volume not well controlled in heavy ion collisions

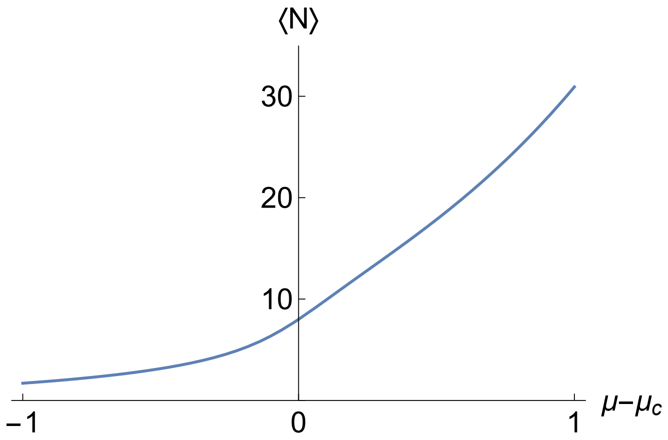
Cumulant Ratios: $\frac{K_2}{\langle N \rangle}, \quad \frac{K_3}{K_2}, \quad \frac{K_4}{K_2}$

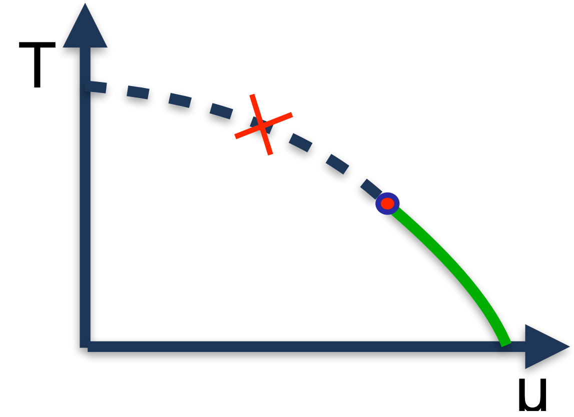
Simple model

Change degrees of freedom
at phase transition

$$\langle N \rangle = dof(\mu)\, e^{\mu/T} \int d^3p e^{-E/T}$$



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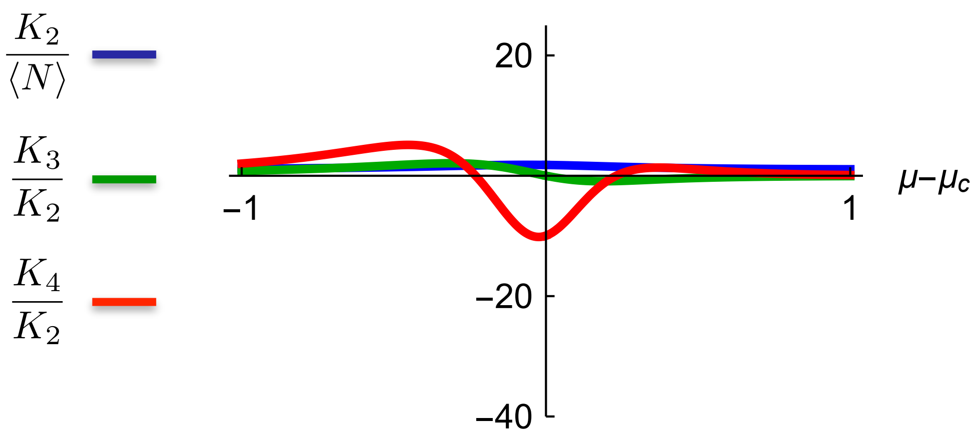
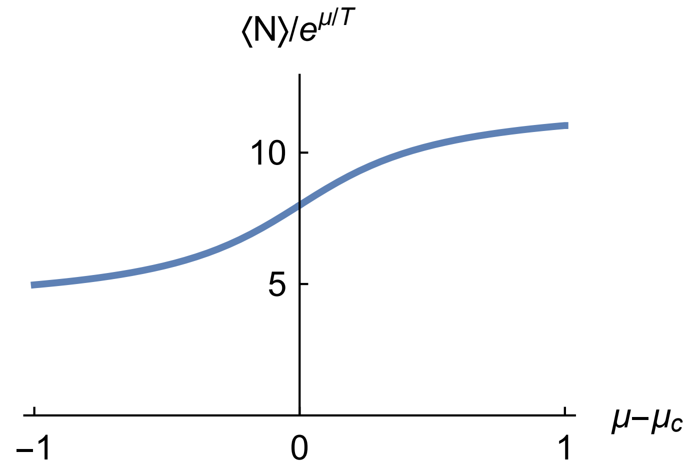
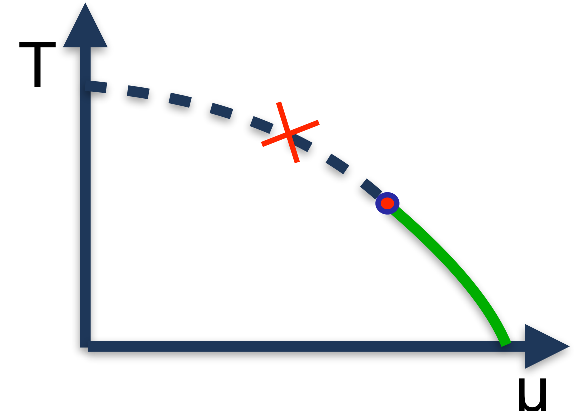

$\frac{K_2}{\langle N \rangle}$
$\frac{K_3}{K_2}$
$\frac{K_4}{K_2}$
20
−20
−40
−1
1
μ−μc


⟨N⟩/e^{μ/T}
10
5
−1
0
1
μ−μc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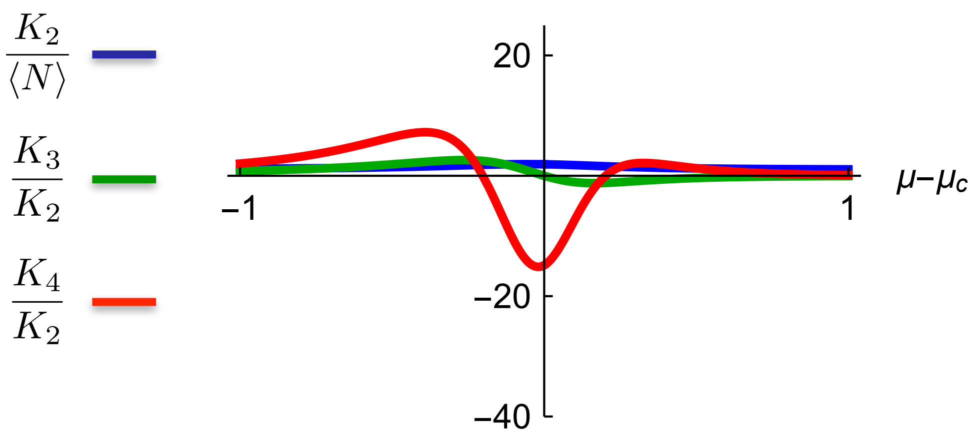
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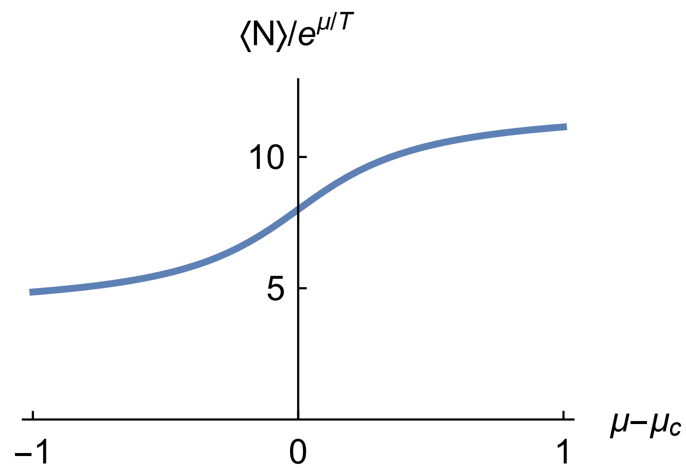
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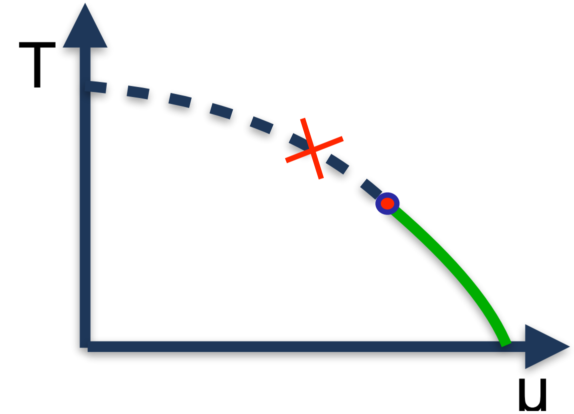
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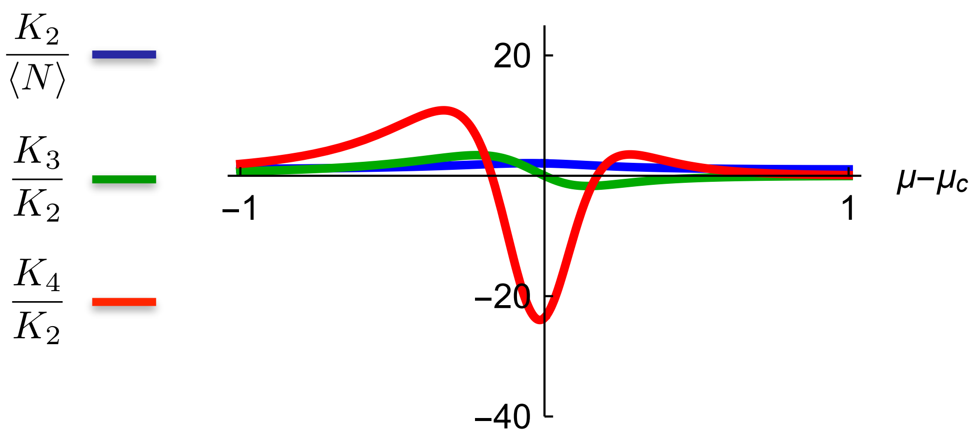
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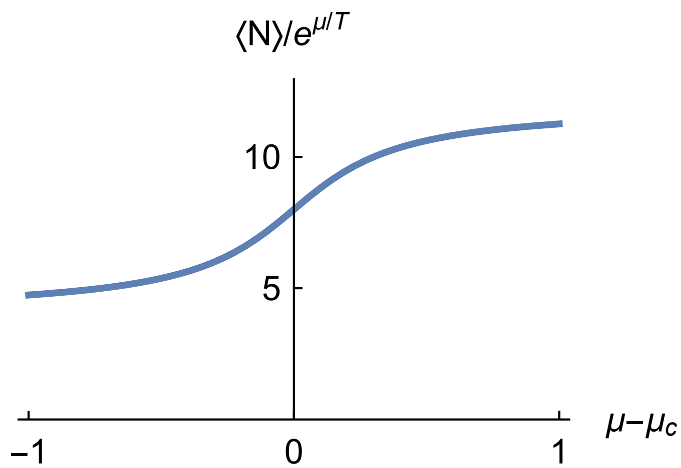
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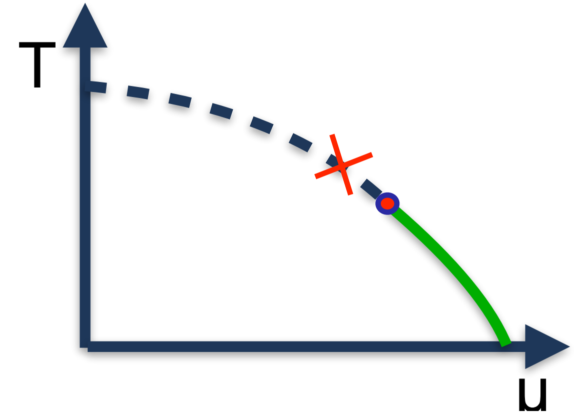
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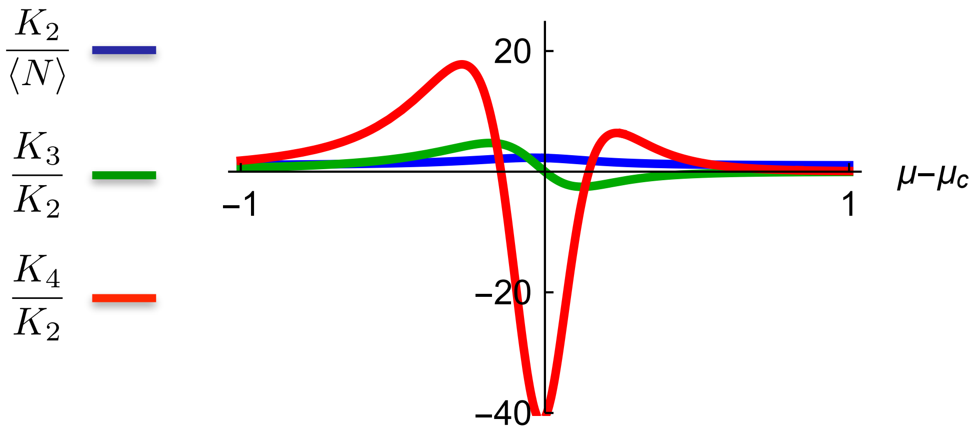
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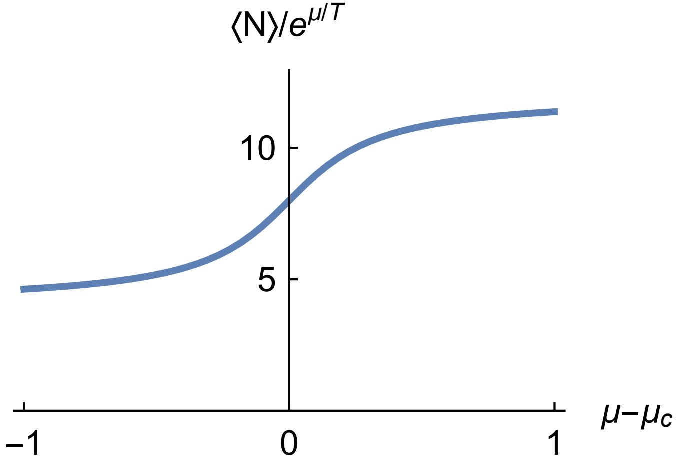
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Cumulants scale with volume (extensive): $K_n \sim V$

Volume not well controlled in heavy ion collisions

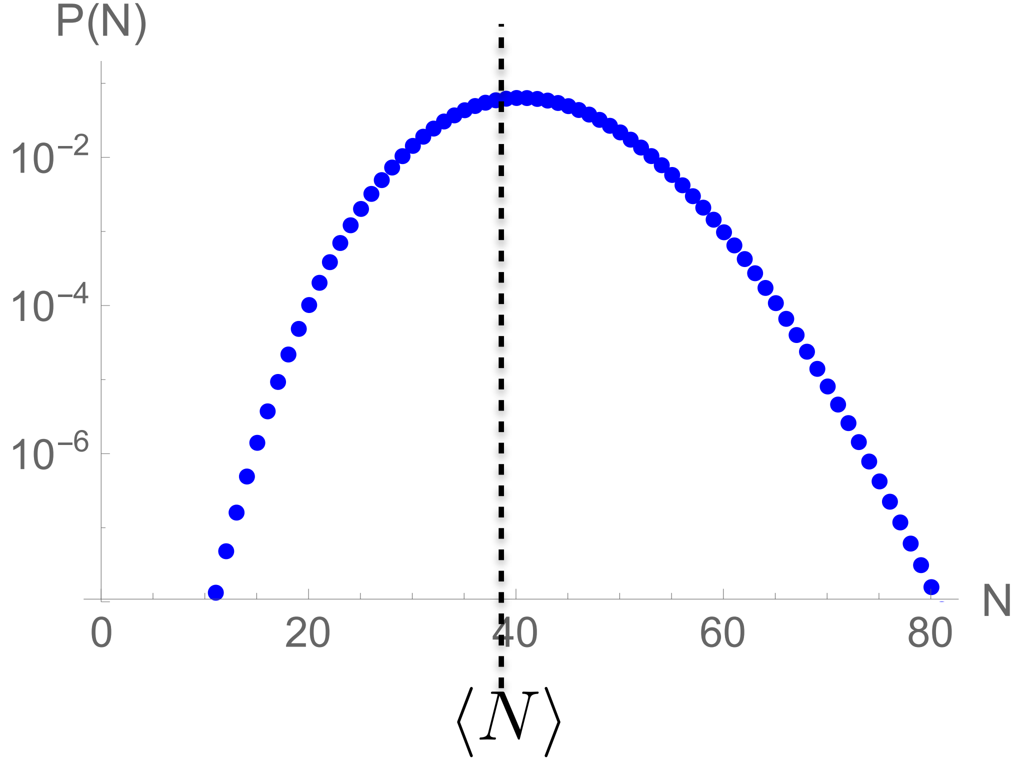
Cumulant Ratios: $\frac{K_2}{\langle N \rangle}, \quad \frac{K_3}{K_2}, \quad \frac{K_4}{K_2}$

Measuring cumulants (derivatives)

$$K_2 = \langle N - \langle N \rangle \rangle^2 = \sum_N P(N)(N - \langle N \rangle)^2$$

$$K_3 = \langle N - \langle N \rangle \rangle^3 = \sum_N P(N)(N - \langle N \rangle)^3$$

$$P(N) = \frac{N_{events}(N)}{N_{events}(total)}$$



Cumulants: a closer look

$$Z = tr\, e^{-\hat{E}/T + \mu/T \hat{N}_B}$$

$$K_n = \frac{\partial^n}{\partial(\mu/T)^n} \ln Z = \frac{\partial^{n-1}}{\partial(\mu/T)^{n-1}} \langle N \rangle$$

Cumulants are extensive: $K_n \sim V$

$$K_2 = \langle N - \langle N \rangle \rangle^2 = \int d^3x d^3y \, \langle \delta\rho(x) \delta\rho(y) \rangle \, ; \quad \delta\rho(x) = \rho(x) - \bar{\rho}$$

Susceptibility:

$$\chi_{(2)\;i,j} = \frac{1}{VT^3} \int d^3x d^3y \, \langle \delta\rho_i(x) \delta\rho_j(y) \rangle = \frac{1}{T^3} \bar{\rho}_i \delta_{i,j} + \frac{1}{T^3} \int d^3r C_{i,j}(r)$$

Correlation function (in configuration space!):

$$C_{i,j}(\vec{r}) = \langle \delta\rho_i(\vec{r}) \, \delta\rho_j(0) \rangle - \bar{\rho}_i \delta_{i,j} \delta(\vec{r}) \sim \frac{\exp\left[-r/\xi_{i,j}\right]}{r}$$

Correlation length (in configuration space!): $\xi_{i,j}$

Relation to cumulant: $K_2 = VT^3 \chi_{(2)\;i,i}$

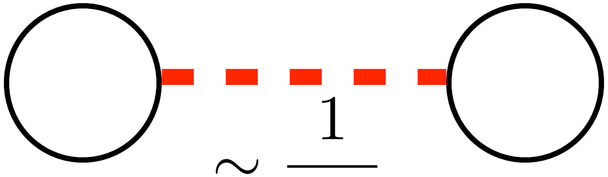
Correlation length

$$C(r) \sim \frac{\exp[-r/\xi]}{r}$$

Static correlation function;
"Yukawa" potential with mass: $m \sim \frac{1}{\xi}$

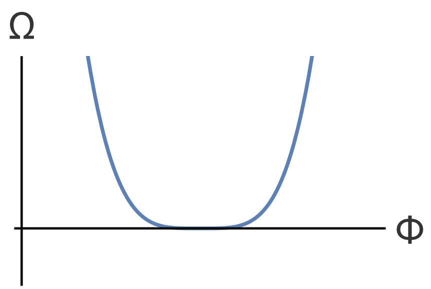
$$\chi \sim \int C(r) d^3 r \sim \xi^2 \sim \frac{1}{m^2}$$

simple "sigma" exchange



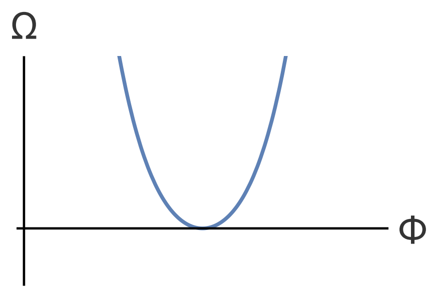
$$\sim \frac{1}{m_\sigma^2}$$

Critical point (second order)



$$m_\sigma \to 0, \quad \xi \to \infty$$

Cross over



m_σ, ξ finite

Correlation function in Ising Model ?

Correlation length: $\xi \sim \frac{1}{\sqrt{|t|}}$ diverges at critical point

second order cumulant: $K_2 \sim \chi_2 \sim \xi^2 \sim \frac{1}{|t|}$ diverges at critical point

Higher moments (cumulants) and ξ

- Consider probability distribution for the order-parameter field:

$$P[\sigma] \sim \exp\{-\Omega[\sigma]/T\},$$

$$\Omega = \int d^3x \left[\frac{1}{2}(\nabla\sigma)^2 + \frac{m_\sigma^2}{2}\sigma^2 + \frac{\lambda_3}{3}\sigma^3 + \frac{\lambda_4}{4}\sigma^4 + \dots\right]. \qquad \Rightarrow \qquad \xi = m_\sigma^{-1}$$

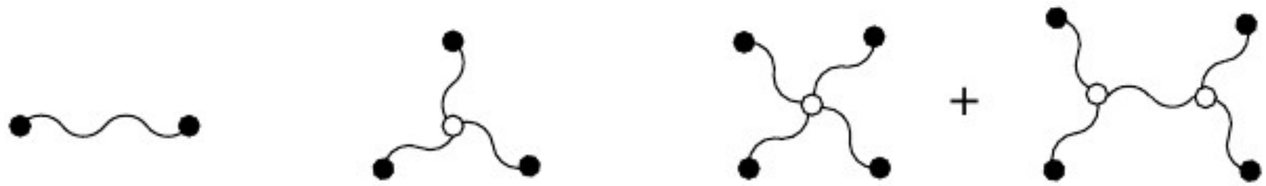
- Moments (connected) of $\boldsymbol{q} = 0$ mode $\sigma_V \equiv \int d^3x\, \sigma(x)$:

First approximation: count σ propagators

$$\kappa_2 = \langle\sigma_V^2\rangle = VT\,\xi^2; \qquad \kappa_3 = \langle\sigma_V^3\rangle = 2VT^2\,\lambda_3\,\xi^6;$$

$$\kappa_4 = \langle\sigma_V^4\rangle_c \equiv \langle\sigma_V^4\rangle - 3\langle\sigma_V^2\rangle^2 = 6VT^3\,[\,2(\lambda_3\xi)^2 - \lambda_4\,]\,\xi^8.$$

- Tree graphs. Each propagator gives ξ^2.

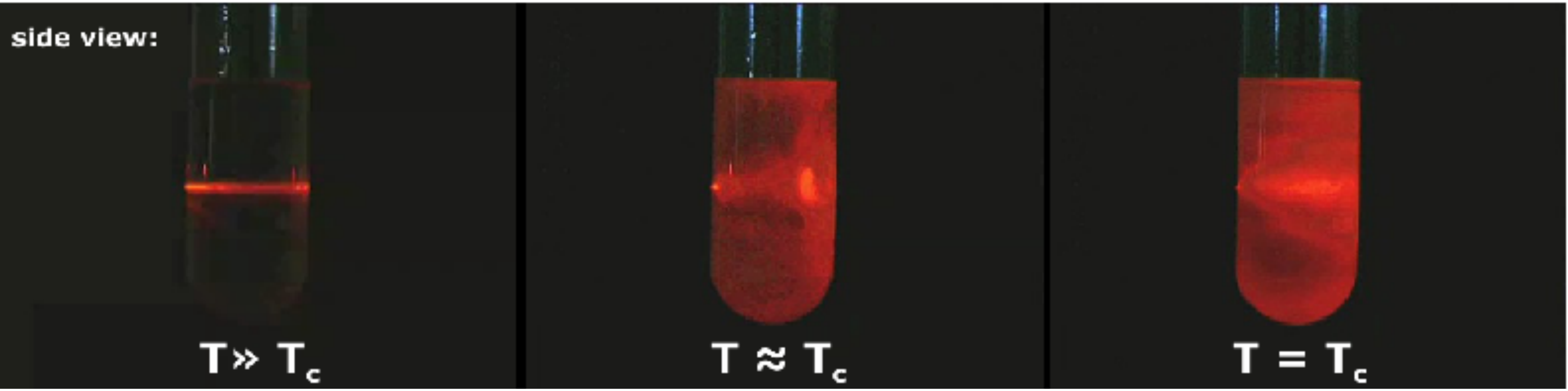


- Scaling requires "running": $\lambda_3 = \tilde\lambda_3 T(T\xi)^{-3/2}$ and $\lambda_4 = \tilde\lambda_4(T\xi)^{-1}$, i.e.,

$$\kappa_3 = \langle\sigma_V^3\rangle = 2VT^{3/2}\,\tilde\lambda_3\,\xi^{4.5}; \quad \kappa_4 = 6VT^2\,[\,2(\tilde\lambda_3)^2 - \tilde\lambda_4\,]\,\xi^7.$$

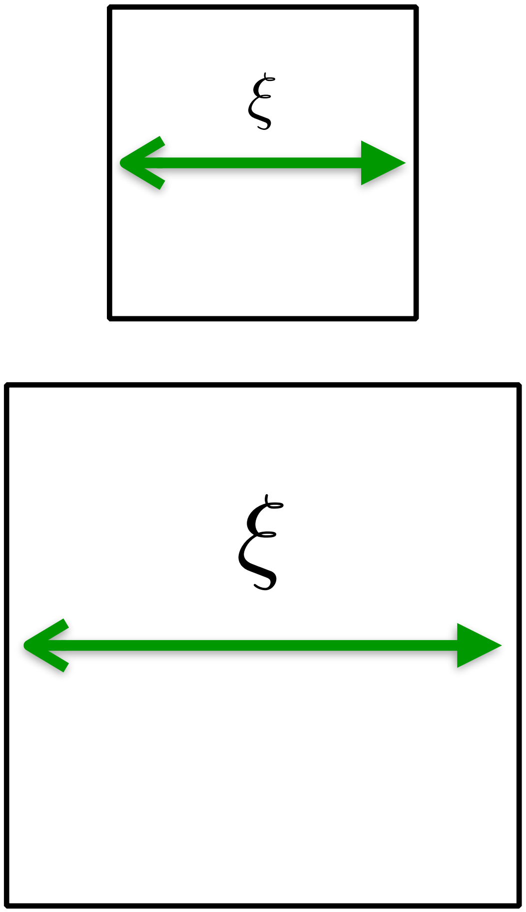
Critical point

- Second order phase transition
- Fluctuations at all length scales
    - Critical opalescence



Finite size scaling

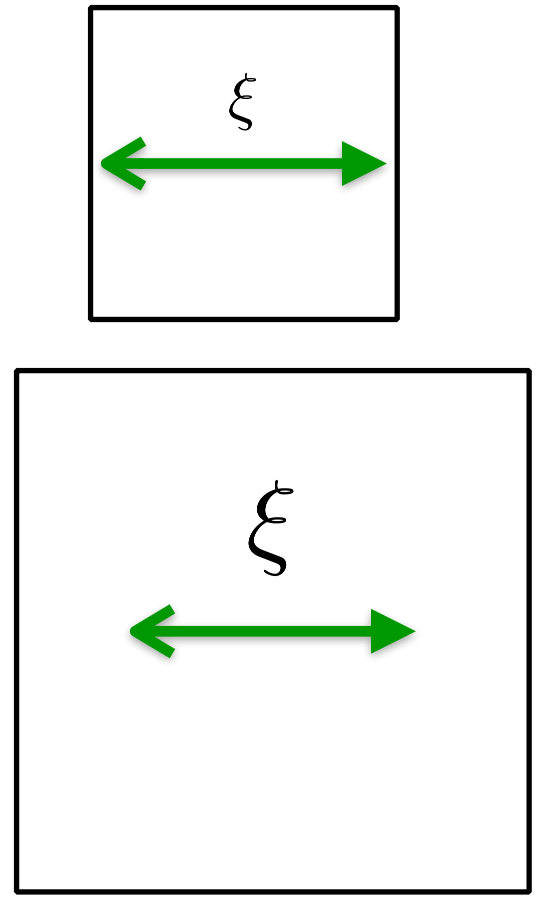
Second order (critical point)



$\xi \sim V^{2/3}, \;\; \chi \sim V^{4/3}$

(mean field)

Cross over

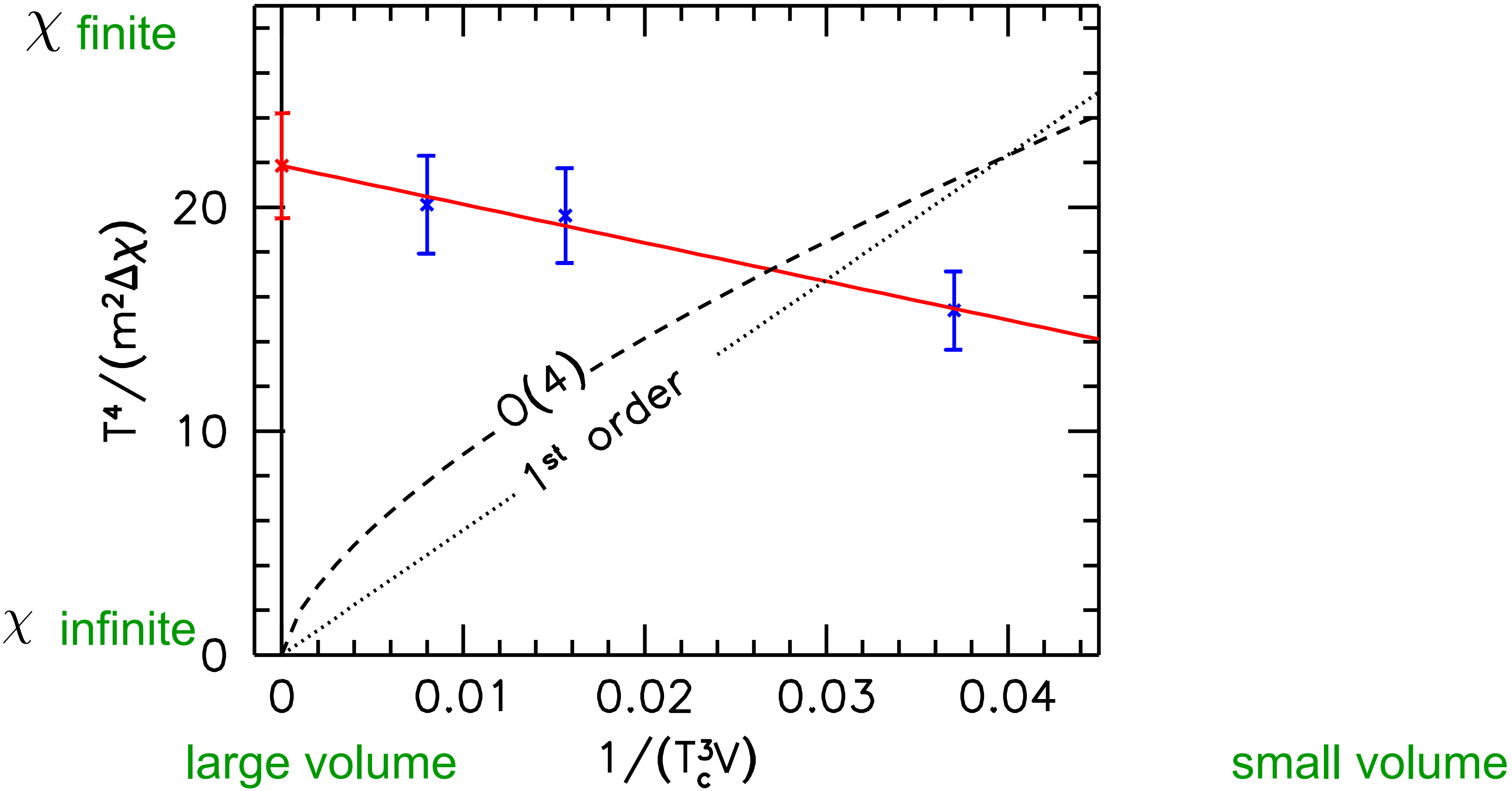


$\xi = \mathrm{const}, \;\; \chi = \mathrm{const}$

NB: 1st order: $\chi \sim V$

QCD at μ=0 is cross-over

Aoki et al, Nature 43:675-678,2006

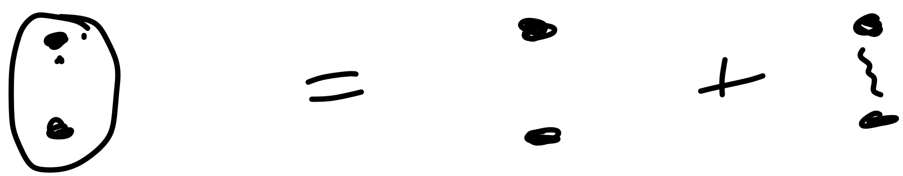


Particle Correlations

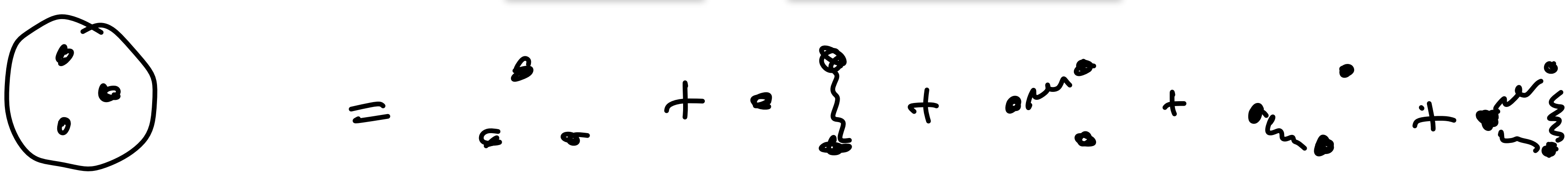
$$\frac{dN}{dp_1} \equiv \rho_1(p_1) \qquad \frac{d^2N}{dp_1 dp_2} \equiv \rho_2(p_1, p_2) \qquad \frac{d^3N}{dp_1 dp_2 dp_3} \equiv \rho_3(p_1, p_2, p_3)$$

$$\rho_2(p_1, p_2) = \rho_1(p_1)\rho_1(p_2) + C_2(p_1, p_2),$$

C_2: Correlation Function



$$\rho_3(p_1, p_2, p_3) = \rho_1(p_1)\rho_1(p_2)\rho_1(p_3) + \rho_1(p_1)\underline{C_2(p_2, p_3)} + \rho_1(p_2)\underline{C_2(p_1, p_3)} + \rho_1(p_3)\underline{C_2(p_1, p_2)} + \underline{C_3(p_1, p_2, p_3)}$$



Particle Correlations

$$\frac{dN}{dp_1} \equiv \rho_1(p_1) \qquad \frac{d^2N}{dp_1 dp_2} \equiv \rho_2(p_1, p_2) \qquad \frac{d^3N}{dp_1 dp_2 dp_3} \equiv \rho_3(p_1, p_2, p_3)$$

$$\int_{Acc} dp_1\, \rho_1(p_1) = < N > \qquad \int_{Acc} dp_1 dp_2\, \rho_2(p_1, p_2) = < N(N-1) >$$

$$\int_{Acc} dp_1 dp_2 dp_3\, \rho_3(p_1, p_2, p_3) = < N(N-1)(N-2) >$$

Integrate: $\rho_2(p_1, p_2) = \rho_1(p_1)\rho_1(p_2) + C_2(p_1, p_2);$

$$< N(N-1) > = < N >^2 + \int_{Acc} dp_1 dp_2\, C_2(p_1, p_2) \equiv < N >^2 + C_2$$

Relation to cumulant

$$K_2 = < N^2 > - < N >^2 = < N > + C_2$$

From Cumulants to Correlations (no anti-protons)

Defining integrated correlations function

$$C_n = \int dp_1 \ldots dp_n C_n(p_1, \ldots, p_n)$$

Factorial cumulant

Simple Algebra leads to relation between correlations C_n and K_n

$$C_2 = -K_1 + K_2,$$
$$C_3 = 2K_1 - 3K_2 + K_3,$$
$$C_4 = -6K_1 + 11K_2 - 6K_3 + K_4, .$$

or vice versa

$$K_2 = \langle N \rangle + C_2$$
$$K_3 = \langle N \rangle + 3C_2 + C_3$$
$$K_4 = \langle N \rangle + 7C_2 + 6C_3 + C_4$$

Factorial cumulants capture the leading divergencies

Interlude: generating functions

Moment generating function

$$h(t) = \sum_n P(n) e^{nt}; \quad h(0) = 1$$

$$\frac{d^k}{dt^k} h(t) = \sum_n P(n) n^k e^{nt} \xrightarrow[t=0]{} \sum_n P(n) n^k = \langle n^k \rangle$$

Cumulant generating function

$$g(t) = \ln[h(t)] = \ln\left[\sum_n P(n) e^{nt}\right]$$

$$K_n = \left.\frac{d^k}{dt^k} g(t)\right|_{t=0}$$

$$K_2 = \left.\frac{d^k}{dt^2} g(t)\right|_{t=0} = \frac{h''(0)}{h(0)} - \frac{h'(0)^2}{h(0)^2} = \langle n^2 \rangle - \langle n \rangle^2$$

Interlude: generating functions

Factorial moment generating function

$$\bar h(z) = \sum_n P(n) z^n; \quad \bar h(1) = 1$$

$$\frac{d^k}{dz^k}\bar h(z) = \sum_n P(n) n(n-1)\dots(n-k+1) z^{n-k} \xrightarrow[z=1]{} \sum_n P(n) n(n-1)\dots(n-k+1)$$

$$= \langle n(n-1)\dots(n-k+1)\rangle = f_k(n)$$

Factorial cumulant generating function

$$\bar g(z) = \ln[\bar h(z)] = \ln\left[\sum_n P(n) z^n\right]$$

$$C_n = \left.\frac{d^k}{dz^k}\bar g(z)\right|_{z=0}$$

$$C_2 = \left.\frac{d^2}{dz^2}\bar g(z)\right|_{z=0} = \frac{\bar h''(0)}{\bar h(0)} - \frac{\bar h'(0)^2}{\bar h(0)^2} = \langle (n(n-1)\rangle - \langle n\rangle^2$$

Interlude: generating functions

Relation between factorial cumulants and cumulants

$$h(t) = \sum_n P(n) e^{nt}$$

$$\bar{h}(z) = \sum_n P(n) z^n$$

$$h(t) = \bar{h}(z = e^t); \quad g(t) = \bar{g}(z = e^t)$$

$$\frac{d}{dt} g(t) = \frac{d}{dz} \bar{g}(z) \frac{d}{dt} z = e^t \frac{d}{dz} \bar{g}(z)$$

Cumulant   Factorial cumulant

and so on… Mathematica does this for you easily

Correlations ?

Assume we have exactly one particle in each event:

$$P(n) = \delta_{n,1}$$

$$K_1 = \langle N \rangle = 1$$
$$K_2 = \left\langle (N - \langle N \rangle^2) \right\rangle = 0$$
$$K_3 = \left\langle (N - \langle N \rangle^2) \right\rangle = 0$$
$$K_n = 0; \quad N > 1$$

$$C_2 = -K_1 + K_2 = -1$$
$$C_3 = 2K_1 - 3K_2 + K_3 = 2$$
$$C_4 = -6K_1 + 11K_2 - 6K_3 + K_4 = -6$$

In general: $$C_n = (-1)^{n-1}(n-1)!$$

(n>1)-particle correlations with one particle only!!!!

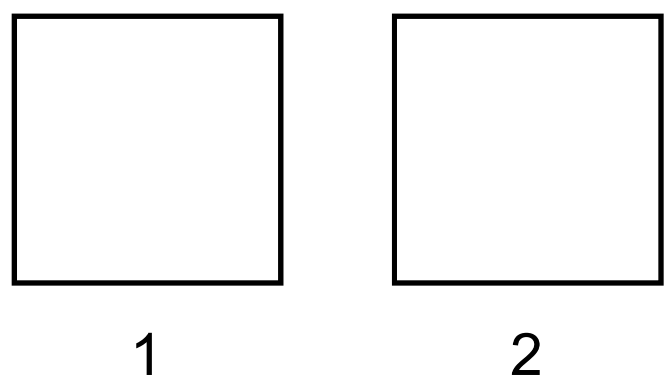
Factorial cumulants “measure” deviation from Poisson !

Common distributions

Poisson distribution: $$P(N) = e^{\Lambda}\frac{\Lambda^N}{N!}$$

Properties: Sum distribution of two Poissonian is again Poisson

$$P(N = N_1 + N_2) = \sum_{N_1,N_2} P_1(N_1)P(N_2)\delta_{N,N_1+N_2} = e^{\Lambda_1+\Lambda_2}\frac{(\Lambda_1 + \Lambda_2)^N}{N!}$$



No correlations; all factorial cumulants vanish $C_n = 0 \ n > 1$

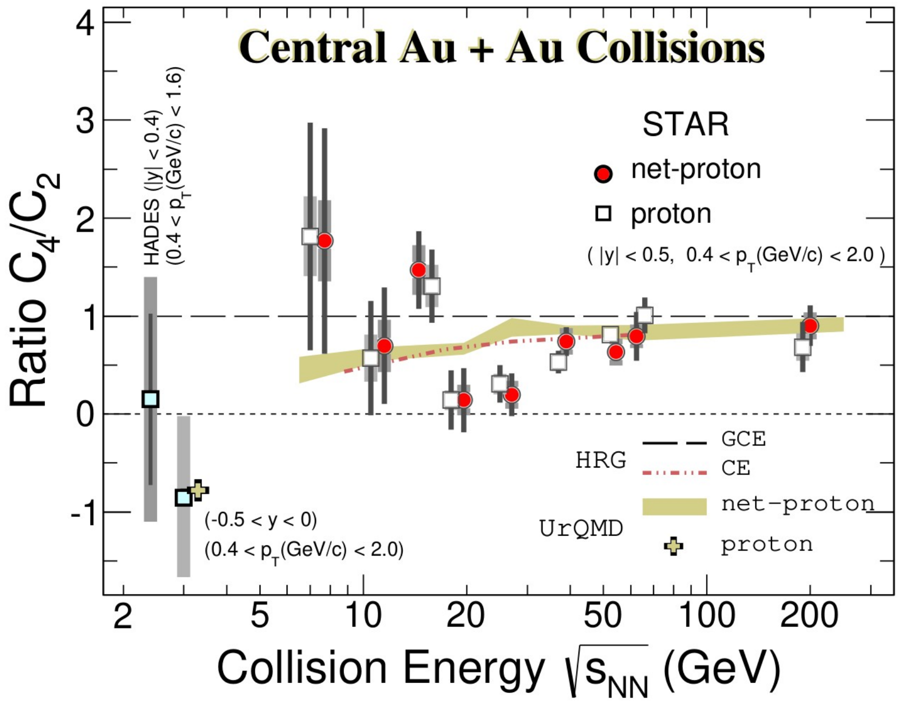
Cumulants: $K_n = < N >$

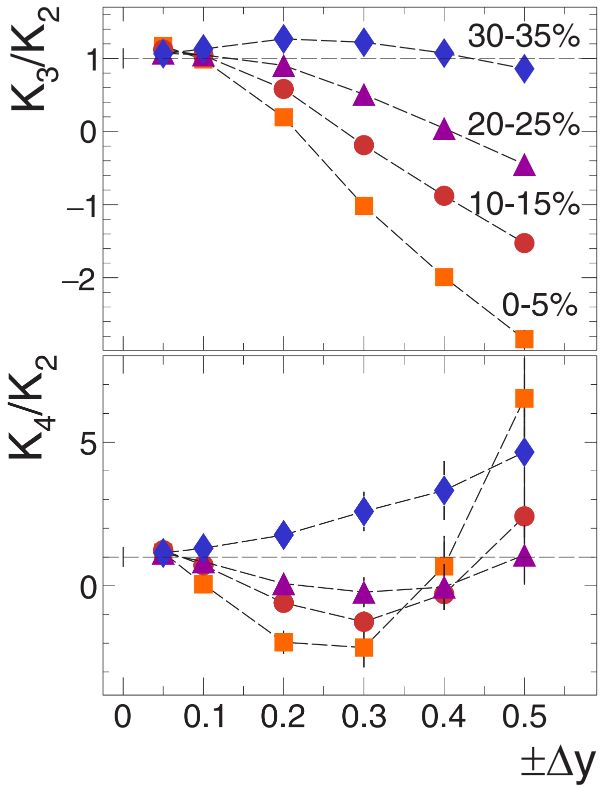
Common distributions

Binomial Distribution $$P(n;N) = \frac{N!}{n!(N-n)!} p^n (1-p)^{N-n}$$

p (Bernoulli) probability of success in one throw of "coin"

Data





STAR: arXiv:2112.00240

HADES: arXiv:2002.08701

$\sqrt{s} = 2.4$ GeV