Solutions to problems

Contents

I. BASIC THERMODYNAMIC PROPERTIES

Problem:
The pressure for non-interacting fermions (upper sign) and bosons (lower sign) is given by

$$P = \pm T \int \frac{d^3\mathbf{k}}{(2\pi)^3} \ln\left[1 \pm e^{-(E_k-\mu)/T}\right] , \qquad E_k = \sqrt{k^2+m^2} \, . \tag{1}$$

1. Show that for fermions

$$s = -\int \frac{d^3\mathbf{k}}{(2\pi)^3} \left[(1-f_k)\ln(1-f_k) + f_k \ln f_k\right] \tag{2}$$

   and derive the analogous expression for bosons

2. Derive expressions for the specific heat for bosons and fermions,

$$c_V = T\frac{\partial s}{\partial T} \tag{3}$$

   and evaluate them

   (a) for $T \gg m, \mu$ (fermions and bosons), using

$$\int_0^\infty dx \, \frac{x^4}{\cosh x + 1} = \frac{7\pi^4}{15} \, , \qquad \int_0^\infty dx \, \frac{x^4}{\cosh x - 1} = \frac{8\pi^4}{15} \tag{4}$$

   (b) for $T \ll \mu$ and $m = 0$ (only fermions), using

$$\int_0^\infty dx \, \frac{x^2}{\cosh x + 1} = \frac{\pi^2}{3} \tag{5}$$

Solution:

1. We abbreviate

$$x \equiv \frac{E_k - \mu}{T} \, , \qquad \int_k \equiv \int \frac{d^3\mathbf{k}}{(2\pi)^3} \, . \tag{6}$$

   Then, for fermions, we have

$$f_k = \frac{1}{e^x + 1} \, , \tag{7}$$

   from which we obtain the useful relation $e^x = (1-f_k)/f_k$. Then we find

$$s_{\text{fermions}} = \frac{\partial P}{\partial T} = \int_k \left[\ln(1+e^{-x}) + x f_k\right] = -\int_k \left[(1-f_k)\ln(1-f_k) + f_k \ln f_k\right] \, . \tag{8}$$

For bosons, we have

$$f_k = \frac{1}{e^x - 1}\,, \tag{9}$$

and thus $e^x = (1+f_k)/f_k$. This yields

$$s_{\rm bosons} = \frac{\partial P}{\partial T} = -\int_k \left[\ln(1-e^{-x}) - x f_k\right] = \int_k \left[(1+f_k)\ln(1+f_k) - f_k \ln f_k\right]\,. \tag{10}$$

2. We compute for fermions (upper sign) and bosons (lower sign)

$$c_V = T\frac{\partial s}{\partial T} = T\int_k x\frac{\partial f_k}{\partial T} = \int_k \frac{x^2 e^x}{(e^x \pm 1)^2} = \int_k \frac{x^2}{e^x + e^{-x} \pm 2} = \frac{1}{2}\int_k \frac{x^2}{\cosh x \pm 1}\,, \tag{11}$$

(a) For sufficiently large temperatures, we can neglect m and μ, such that

$$c_V \simeq \frac{1}{4\pi^2}\int_0^\infty dk\, k^2 \frac{k^2}{T^2}\frac{1}{\cosh\frac{k}{T} \pm 1} = \frac{T^3}{4\pi^2}\int_0^\infty dy\,\frac{y^4}{\cosh y \pm 1} = \begin{cases} \dfrac{7\pi^2 T^3}{60} & \text{(fermions)} \\ \dfrac{2\pi^2 T^3}{15} & \text{(bosons)} \end{cases} \tag{12}$$

(b) For small temperatures we use the fact that the main contribution to the integral comes from the Fermi surface (the Fermi momentum for massless fermions simply is μ),

$$c_V = \frac{1}{4\pi^2}\int_0^\infty dk\, k^2 \frac{(k-\mu)^2}{T^2}\frac{1}{\cosh\frac{k-\mu}{T} + 1} \simeq \frac{\mu^2}{4\pi^2}\int_0^\infty dk\,\frac{(k-\mu)^2}{T^2}\frac{1}{\cosh\frac{k-\mu}{T} + 1}\,. \tag{13}$$

Introducing the new integration variable $y = (k-\mu)/T$ yields

$$c_V \simeq \frac{\mu^2 T}{4\pi^2}\int_{-\mu/T}^\infty dy\,\frac{y^2}{\cosh y + 1} \simeq \frac{\mu^2 T}{4\pi^2}\int_{-\infty}^\infty dy\,\frac{y^2}{\cosh y + 1} = \frac{\mu^2 T}{2\pi^2}\int_0^\infty dy\,\frac{y^2}{\cosh y + 1} = \frac{\mu^2 T}{6}\,. \tag{14}$$

II. NON-INTERACTING NUCLEAR MATTER

Problem:

1. Show that electrically neutral, non-interacting nuclear matter (n,p,e) at zero temperature and in β-equilibrium (assuming $\mu_\nu \simeq 0$)
   (a) must contain protons in general, $n_p \neq 0$
   (b) has a proton fraction $\frac{n_p}{n_B} = \frac{1}{9}$ in the ultra-relativistic limit
   (c) obeys $\frac{n_p}{n_B} < \frac{1}{9}$ except for very small densities (requires numerical evaluation)
2. Show that non-interacting, pure neutron matter in the non-relativistic limit has a following "polytropic" equation of state,

$$P(\epsilon) = K\epsilon^p \,, \tag{15}$$

   and compute K and p.

Solution:

1. (a) Neutrality requires $n_e = n_p$ and thus

$$k_{F,e} = k_{F,p} \tag{16}$$

   With $\mu = \sqrt{k_F^2 + m^2}$ and the condition from β-equilibrium $\mu_e + \mu_p = \mu_n$ we have

$$\sqrt{k_{F,e}^2 + m_e^2} + \sqrt{k_{F,p}^2 + m_p^2} = \sqrt{k_{F,n}^2 + m_n^2} \,. \tag{17}$$

   Suppose the system contains no protons (and then, because of neutrality, no electrons either), $k_{F,p} = 0$. Then, this equation becomes,

$$k_{F,n}^2 = (m_e + m_p)^2 - m_n^2 \,. \tag{18}$$

   The right-hand side is negative, because the neutron is slightly heavier than electron and proton together [that's why a neutron in vacuum decays into a proton and an electron (and an anti-neutrino)]. Hence there is no solution for $k_{F,n}$ and we conclude that protons must be present.

   (b) In the ultra-relativistic limit, $m_e \simeq m_n \simeq m_p \simeq 0$, Eq. (17) becomes

$$2k_{F,p} = k_{F,n} \tag{19}$$

   Since $n \propto k_F^3$, this is equivalent to

$$8n_p = n_n \,, \tag{20}$$

   and thus $\frac{n_p}{n_B} = \frac{1}{9}$ with $n_B = n_n + n_p$. That's why dense nuclear matter is neutron rich and hence the name neutron star.

   (c) For the numerical solution we replace $k_{F,e}$ and $k_{F,p}$ in Eq. (17) by $(3\pi^2 n_p)^{1/3}$ and $k_{F,n}$ by $[3\pi^2(n_B - n_p)]^{1/3}$, and solve the resulting equation numerically for n_p for given n_B. The result for a large range of n_B is shown in Fig. 1, and we see that $n_p \leq n_B/9$ with the upper limit approached asymptotically for $n_B \to \infty$. We also see that there is an onset density for neutrons below which the system only contains electrons and protons.

2. The non-relativistic limit is given by $m \gg k_F$. We can thus approximate the energy density as

$$\epsilon = \frac{1}{\pi^2}\int_0^{k_F} dk\, k^2 \sqrt{k^2 + m^2} \simeq \frac{m}{\pi^2}\int_0^{k_F} dk\, k^2 \left(1 + \frac{k^2}{2m}\right) = \frac{m k_F^3}{3\pi^2} + \mathcal{O}(k_F^5) \,. \tag{21}$$

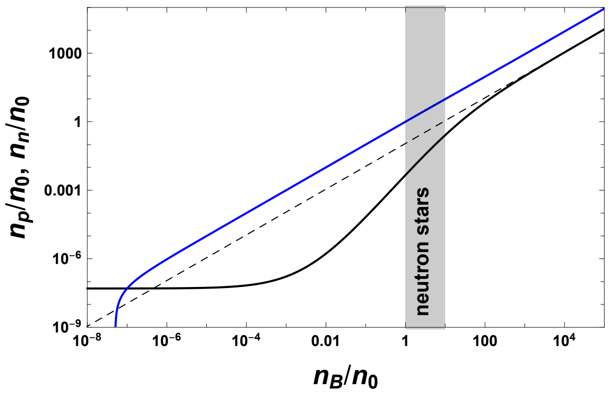

FIG. 1: Proton and neutron densities as a function of the total baryon number density, all given in units of nuclear saturation density, $n_0 \simeq 0.15\,\mathrm{fm}^{-3} \simeq 1.15 \times 10^6\,\mathrm{MeV}^3$. Dense matter in neutron stars only covers a small part of this logarithmic plot, $n_B \sim (1-10)n_0$. The dashed line is $n_B/9$.

The pressure becomes

$$P = \frac{1}{\pi^2}\int_0^{k_F} dk\, k^2(\mu - \sqrt{k^2+m^2}) \simeq \frac{1}{\pi^2}\int_0^{k_F} dk\, k^2 \left[m\left(1+\frac{k_F^2}{2m}\right) - m\left(1+\frac{k^2}{2m}\right)\right]$$

$$= \frac{1}{2m\pi^2}\int_0^{k_F} dk\, k^2(k_F^2 - k^2) = \frac{1}{2m\pi^2}\left(\frac{k_F^5}{3} - \frac{k_F^5}{5}\right) = \frac{k_F^5}{15m\pi^2}\,, \tag{22}$$

where $\mu = \sqrt{k_F^2+m^2} \simeq m\left(1+\frac{k_F^2}{2m}\right)$ has been used.

Putting these two results together yields the equation of state given in Eq. (15) with

$$p = \frac{5}{3}\,, \qquad K = \left(\frac{3\pi^2}{m}\right)^{5/3}\frac{1}{15m\pi^2}\,. \tag{23}$$