Problems

Contents

I. BASIC THERMODYNAMIC PROPERTIES

Problem:
The pressure for non-interacting fermions (upper sign) and bosons (lower sign) is given by

$$P = \pm T \int \frac{d^3\mathbf{k}}{(2\pi)^3} \ln\left[1 \pm e^{-(E_k-\mu)/T}\right] , \qquad E_k = \sqrt{k^2+m^2} \, . \tag{1}$$

1. Show that for fermions

$$s = -\int \frac{d^3\mathbf{k}}{(2\pi)^3} \left[(1-f_k)\ln(1-f_k) + f_k \ln f_k\right] \tag{2}$$

   and derive the analogous expression for bosons

2. Derive expressions for the specific heat for bosons and fermions,

$$c_V = T\frac{\partial s}{\partial T} \tag{3}$$

   and evaluate them

   (a) for $T \gg m, \mu$ (fermions and bosons), using

$$\int_0^\infty dx\, \frac{x^4}{\cosh x + 1} = \frac{7\pi^4}{15} \, , \qquad \int_0^\infty dx\, \frac{x^4}{\cosh x - 1} = \frac{8\pi^4}{15} \tag{4}$$

   (b) for $T \ll \mu$ and $m = 0$ (only fermions), using

$$\int_0^\infty dx\, \frac{x^2}{\cosh x + 1} = \frac{\pi^2}{3} \tag{5}$$

II. NON-INTERACTING NUCLEAR MATTER

Problem:

1. Show that electrically neutral, non-interacting nuclear matter (n,p,e) at zero temperature and in β-equilibrium (assuming $\mu_\nu \simeq 0$)

   (a) must contain protons in general, $n_p \neq 0$

   (b) has a proton fraction $\frac{n_p}{n_B} = \frac{1}{9}$ in the ultra-relativistic limit

   (c) obeys $\frac{n_p}{n_B} < \frac{1}{9}$ except for very small densities (requires numerical evaluation)

2. Show that non-interacting, pure neutron matter in the non-relativistic limit has a following "polytropic" equation of state,

$$P(\epsilon) = K\epsilon^p \, , \tag{6}$$

   and compute K and p.