Including interactions and Cooper pairing

- including interactions between (unpaired) quarks perturbatively
  $\rightarrow$ corrections in powers of α_s
  G. Baym and S. A. Chin, PLB 62, 241 (1976)
  B. A. Freedman and L. D. McLerran, PRD 16, 1169 (1977)

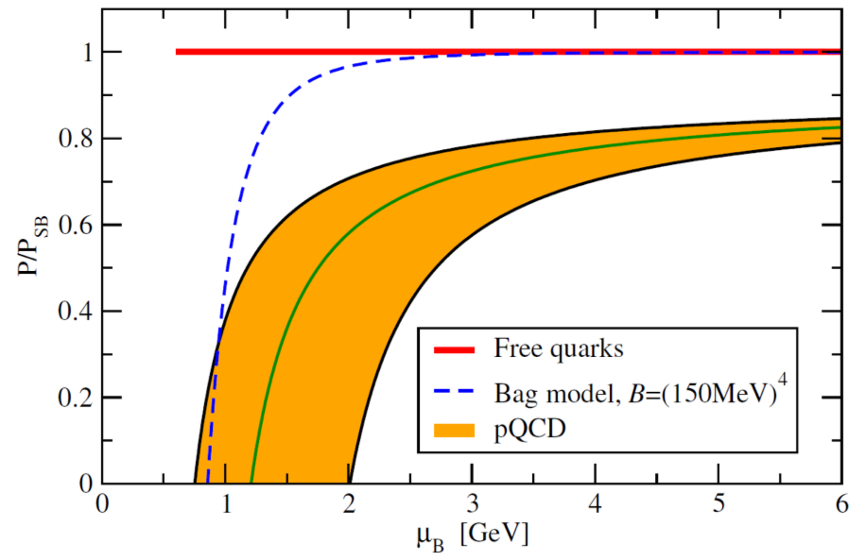
$$k_F = \mu\left(1 - \frac{2\alpha_s}{3\pi}\right)$$

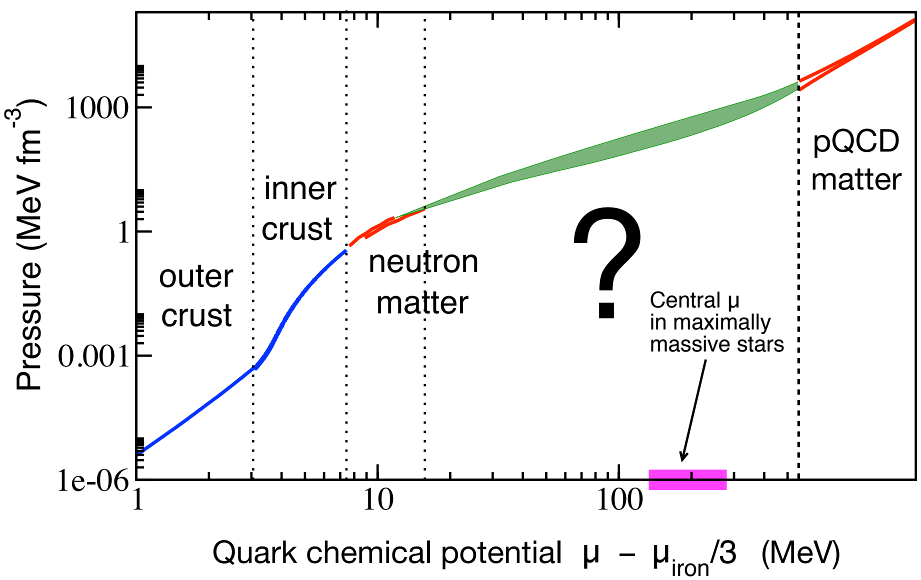
- include energy gap Δ from Cooper pairing

$$P \simeq \frac{3\mu^4}{4\pi^2}\left(1 - \frac{2\alpha_s}{\pi}\right) - \frac{3\mu^2}{4\pi^2}(m_s^2 - 4\Delta^2) - B$$

Improved perturbative studies of dense quark matter

- second-order corrections in α_s
  A. Kurkela, P. Romatschke, A. Vuorinen
  PRD 81, 105021 (2010)
- large corrections to bag model at all relevant densities!





- connect nuclear matter (low density) to perturbative QCD (high density)
  A. Kurkela, E. S. Fraga,
  J. Schaffner-Bielich, A. Vuorinen,
  Astrophys. J. 789, 127 (2014)

Summary: unpaired quark matter

- zero quark masses:

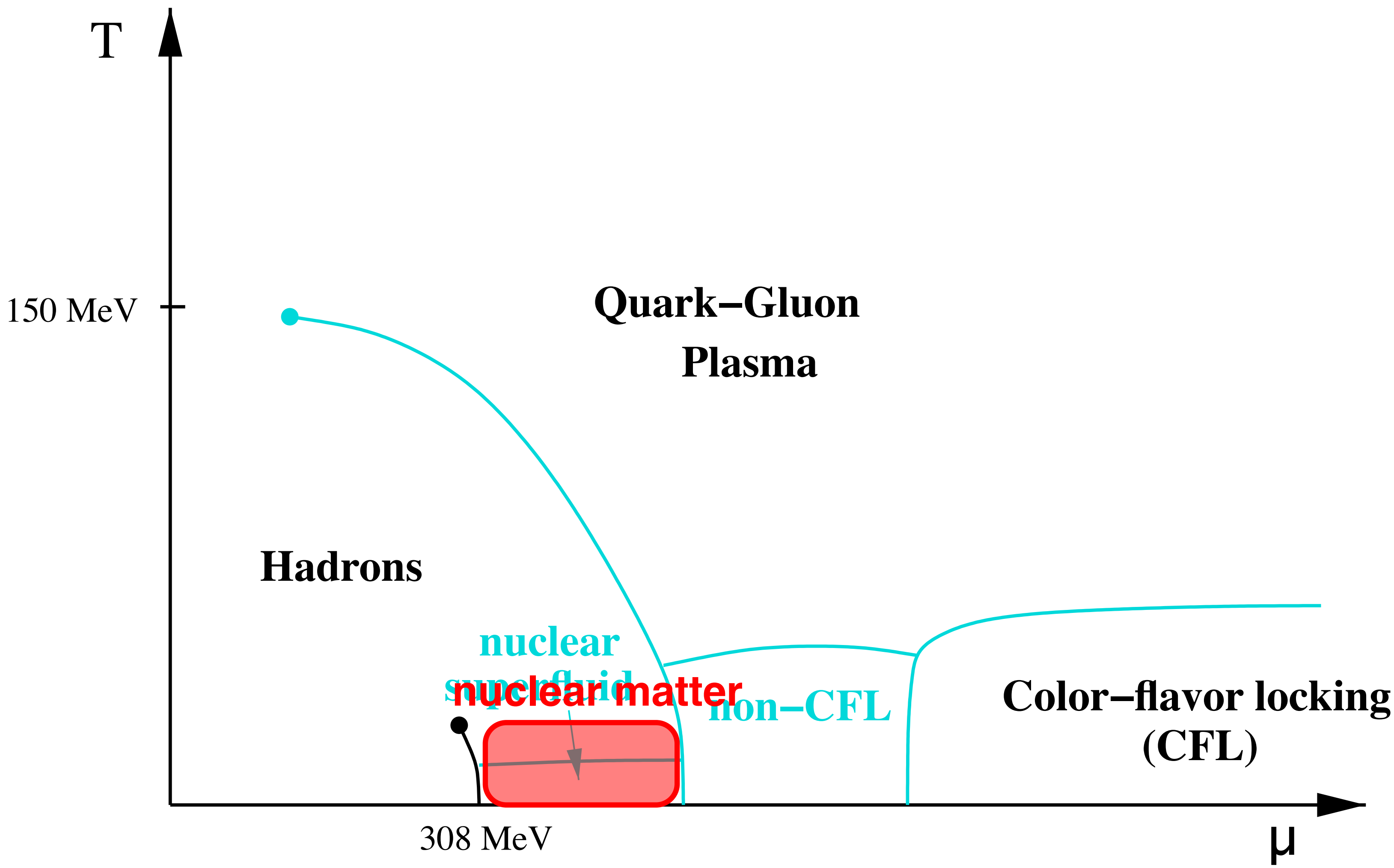
  quark matter is particularly symmetric:
  $n_u = n_d = n_s$ (and no electrons)

- nonzero strange quark mass:

  β-equilibrated, electrically neutral quark matter has $n_d > n_u > n_s$ (and nonzero n_e)

- perturbative results can be used to constrain equation of state at moderate densities

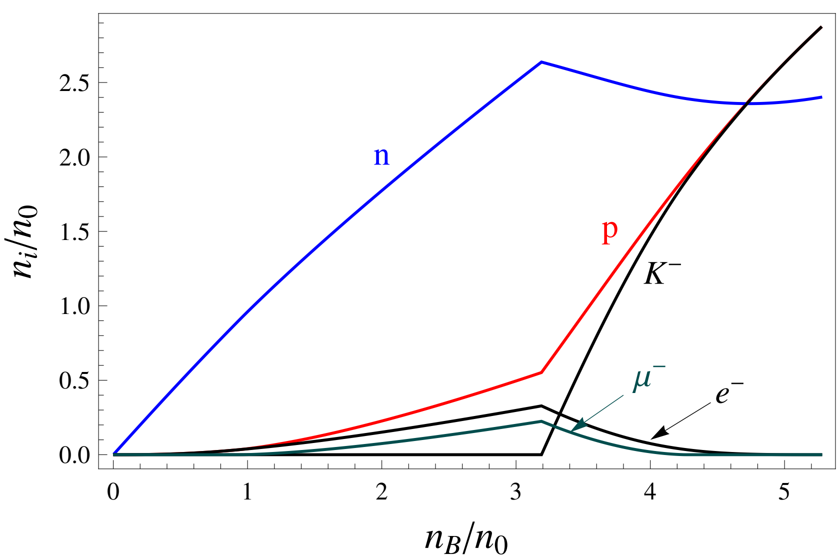
Nuclear matter



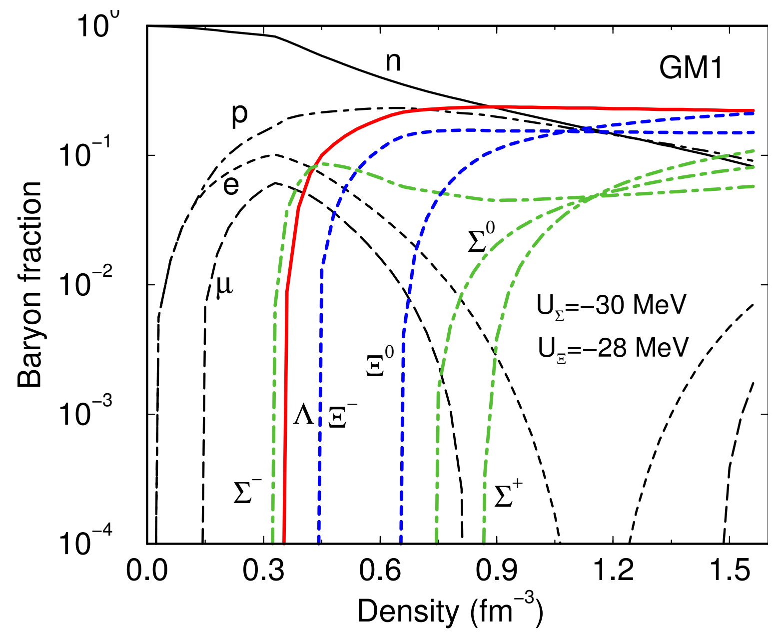
Nuclear matter

- "ordinary" nuclear matter: neutrons (n), protons (p), electrons (e)
  β-equilibrium + charge neutrality → neutron rich matter
  non-interacting nuclear matter → **Problems II**

- more exotic phases possible at high density:
  kaon condensation, hyperons, ...



A. Schmitt, Lect. Notes Phys. 811, 1 (2010)



J. Schaffner-Bielich, NPA 835, 279 (2010)

Problems II: non-interacting nuclear matter

1. Show that electrically neutral, non-interacting nuclear matter (n,p,e) at zero temperature and in β-equilibrium (assuming $\mu_\nu \simeq 0$)

   (a) must contain protons in general, $n_p \neq 0$

   (b) has a proton fraction $\frac{n_p}{n_B} = \frac{1}{9}$ in the ultra-relativistic limit

   (c) obeys $\frac{n_p}{n_B} < \frac{1}{9}$ except for very small n_B (requires numerical evaluation)

2. Show that non-interacting, pure neutron matter in the non-relativistic limit has a "polytropic" equation of state,

$$P(\epsilon) = K\epsilon^p \,,$$

   and compute K and p.

Basic properties of (interacting) nuclear matter

see Sec. 3.1 in A. Schmitt, Lect. Notes Phys. 811, 1 (2010)

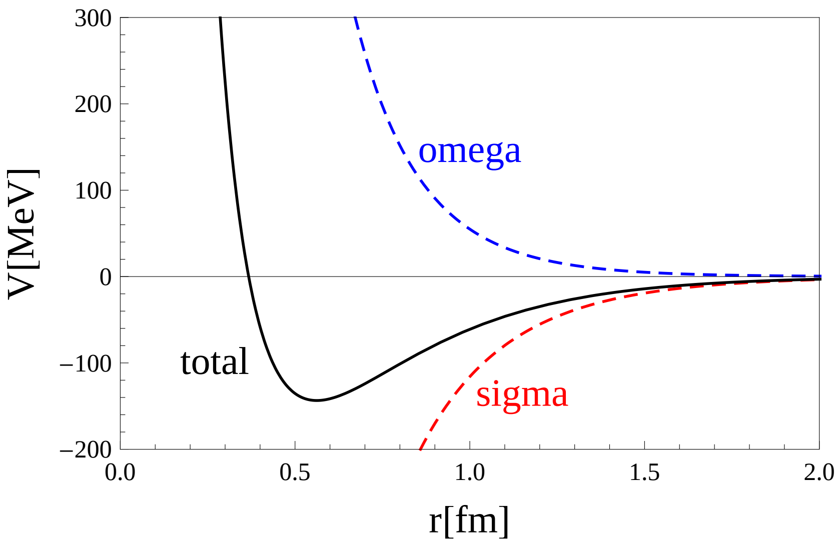
- relativistic, symmetric nuclear matter ("Walecka model")

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu\partial_\mu - m_N + \mu\gamma^0)\psi + g_\sigma\bar{\psi}\sigma\psi - g_\omega\bar{\psi}\gamma^\mu\omega_\mu\psi$$

$$+\frac{1}{2}\left(\partial_\mu\sigma\partial^\mu\sigma - m_\sigma^2\sigma^2\right) - \frac{1}{4}\omega_{\mu\nu}\omega^{\mu\nu} + \frac{1}{2}m_\omega^2\omega_\mu\omega^\mu$$

(with μ introduced through $\mathcal{H} - \mu\mathcal{N}$)

- two parameters (to be fitted later): g_σ, g_ω
- attractive and repulsive interaction through sigma and omega exchange



Mean-field approximation

- replace meson fields by their vevs (space-time independent)

$$\sigma \to \langle\sigma\rangle\,, \qquad \omega_\mu \to \langle\omega_0\rangle\delta_{0\mu}$$

- mean-field Lagrangian

$$\mathcal{L}_{\rm mean-field} = \bar\psi\left(i\gamma^\mu\partial_\mu - m_N^* + \mu^*\gamma_0\right)\psi - \frac{1}{2}m_\sigma^2\langle\sigma\rangle^2 + \frac{1}{2}m_\omega^2\langle\omega_0\rangle^2$$

with

$$m_N^* \equiv m_N - g_\sigma\langle\sigma\rangle\,, \qquad \mu^* \equiv \mu - g_\omega\langle\omega_0\rangle$$

→ looks like non-interacting Lagrangian: interaction absorbed in effective mass m_N^* and effective chemical potential μ^*

Pressure from partition function

- partition function

$$Z = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}\sigma\mathcal{D}\omega \exp \int_X \mathcal{L}$$

$$= e^{\frac{V}{T}\left(-\frac{1}{2}m_\sigma^2\langle\sigma\rangle^2+\frac{1}{2}m_\omega^2\langle\omega_0\rangle^2\right)} \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp \int_X \bar{\psi}\left(i\gamma^\mu\partial_\mu - m_N^* + \mu^*\gamma_0\right)\psi$$

- perform functional integral (see thermal field theory intermezzo) & ignore "vacuum contribution" & neglect anti-baryons

$$P = \frac{T}{V}\ln Z = -\frac{1}{2}m_\sigma^2\langle\sigma\rangle^2+\frac{1}{2}m_\omega^2\langle\omega_0\rangle^2 + \underbrace{4T\int \frac{d^3\mathbf{k}}{(2\pi)^3}\ln\left(1+e^{-(E_k-\mu^*)/T}\right)}_{P_N}$$

with $E_k = \sqrt{k^2 + (m_N^*)^2}$

Intermezzo: Thermal field theory (page 1/6)

- partition function and grand-canonical potential density from statistical physics

$$Z = \mathrm{Tr}\, e^{-\beta(\hat{H}-\mu\hat{N})}\,, \qquad \Omega(T,\mu) = -\frac{T}{V}\ln Z$$

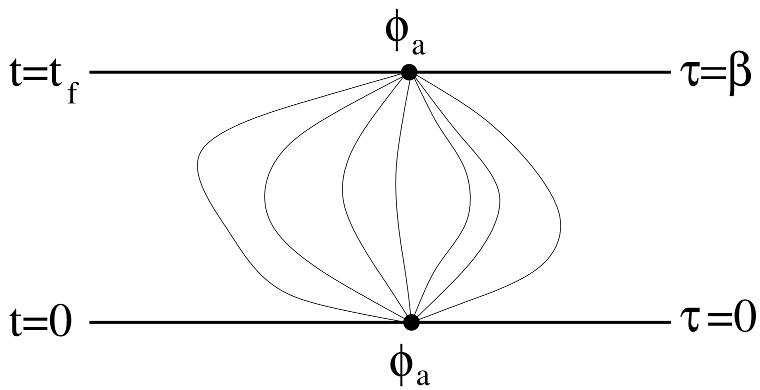
- functional integral (scalar field ϕ with conjugate momentum π)

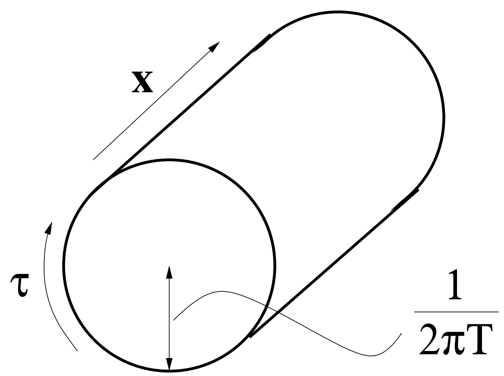
$$Z = \int d\phi\,\langle\phi|e^{-\beta(\hat{H}-\mu\hat{N})}|\phi\rangle = \int \mathcal{D}\pi \int_{\text{periodic}} \mathcal{D}\phi\, \exp\left[\int_X (i\pi\partial_\tau\phi - \mathcal{H} + \mu\mathcal{N})\right]$$

$$= N\int \mathcal{D}\phi\, \exp\int_X \mathcal{L}$$

with

$$\int_X \equiv \int_0^\beta d\tau \int d^3x\,, \qquad \tau = it \;\;\text{“imaginary time”}\,, \qquad \beta = 1/T$$





Intermezzo: Thermal field theory (page 2/6)

- fermionic version:

$$Z = \int_{\text{antiperiodic}} \mathcal{D}\psi^\dagger \mathcal{D}\psi \, \exp\left[\int_X \bar{\psi}(i\gamma^\mu \partial_\mu + \gamma^0 \mu - m)\psi\right]$$

with antiperiodicity $\psi(0,\mathbf{x}) = -\psi(\beta,\mathbf{x})$

- Fourier transform

$$\psi(X) = \frac{1}{\sqrt{V}} \sum_K e^{-iK\cdot X} \psi(K)$$

with

$$X^\mu = (-i\tau, \mathbf{x})\,, \qquad K^\mu = (-i\omega_n, \mathbf{k})\,, \qquad K \cdot X = -(\omega_n \tau + \mathbf{k}\cdot\mathbf{x})$$

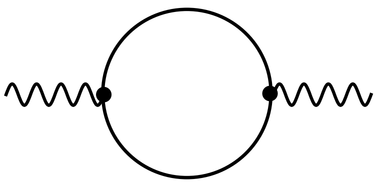
and Matsubara frequencies

$$\omega_n = (2n+1)\pi T\,, \qquad n \in \mathbb{Z} \qquad (\text{bosonic:} \quad \omega_n = 2n\pi T)$$

Intermezzo: Thermal field theory (page 3/6)

- interacting field theories: Feynman rules similar to $T = 0$ quantum field theory

- for instance: photon propagator in a QED plasma $D^{-1}_{\mu\nu} = D^{-1}_{0,\mu\nu} + \Pi_{\mu\nu}$
  with photon polarization tensor



$$= \Pi_{\mu\nu}(Q) = e^2 \frac{T}{V} \sum_K \mathrm{Tr}[\gamma_\mu G_0(K) \gamma_\nu G_0(K-Q)]$$

$$\rightarrow e^2 \frac{T}{V} \sum_n \int \frac{d^3k}{(2\pi)^3} \mathrm{Tr}[\gamma_\mu G_0(K) \gamma_\nu G_0(K-Q)]$$

$$\equiv F(Q) P_{L,\mu\nu} + G(Q) P_{T,\mu\nu}$$

- in "Hard Thermal Loop" approximation [$k \sim T$ ("hard"), $q_0, q \sim eT$ ("soft")]
  E. Braaten and R. D. Pisarski, Nucl. Phys. B 337, 569 (1990)

$$F(Q) = -2m^2 \frac{Q^2}{q^2} \left(1 - \frac{q_0}{2q} \ln \frac{q_0 + q}{q_0 - q}\right), \qquad G(Q) = \frac{m^2 q_0}{q} \left(\frac{q_0}{q} - \frac{Q^2}{2q^2} \ln \frac{q_0 + q}{q_0 - q}\right)$$

with $m^2 \equiv \frac{e^2 T^2}{6}$

Intermezzo: Thermal field theory (page 4/6)

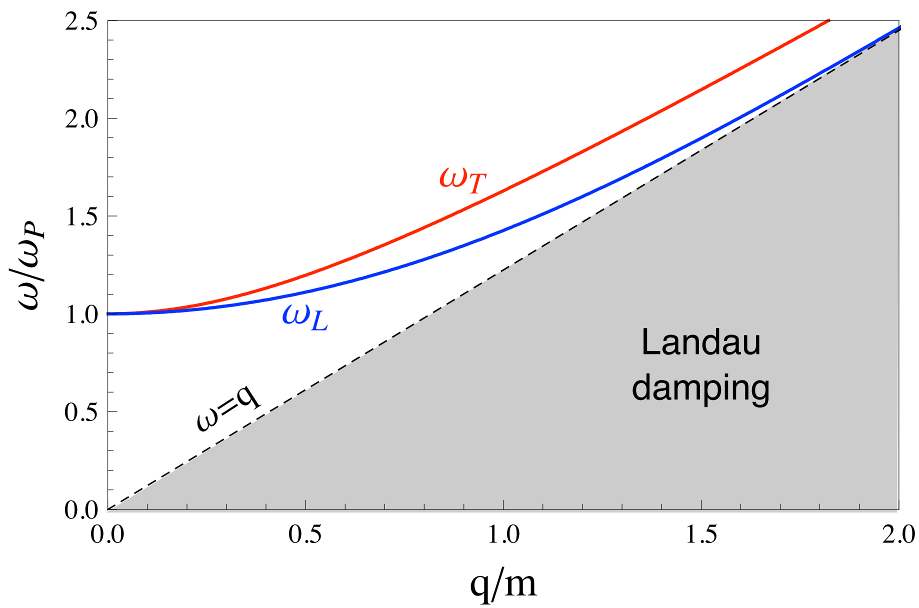
- spectral density

$$\rho_{L,T}(Q) \equiv \frac{1}{\pi}\mathrm{Im}\, D_{L,T}(Q)$$

with

$$D_L(Q) = \frac{Q^2}{q^2}\frac{1}{F(Q) - Q^2}$$

$$D_T(Q) = \frac{1}{G(Q) - Q^2}$$



$$\rho_L(Q) = \frac{\omega_L(\omega_L^2 - q^2)}{q^2(3\omega_P^2 + q^2 - \omega_L^2)}[\delta(q_0 - \omega_L) - \delta(q_0 + \omega_L)] - \frac{1}{\pi}\frac{\Theta(q^2 - q_0^2)\,\mathrm{Im}\,\Pi_{00}}{(\mathrm{Re}\,\Pi_{00} - q^2)^2 + (\mathrm{Im}\,\Pi_{00})^2}$$

$$\rho_T(Q) = \frac{\omega_T(\omega_T^2 - q^2)}{3\omega_P^2\omega_T^2 - (\omega_T^2 - q^2)^2}[\delta(q_0 - \omega_T) - \delta(q_0 + \omega_T)] - \frac{1}{\pi}\frac{\Theta(q^2 - q_0^2)\,\mathrm{Im}\,G}{(\mathrm{Re}\,G - Q^2)^2 + (\mathrm{Im}\,G)^2}$$

Intermezzo: Thermal field theory (page 5/6)

- free fermions: can perform the functional integral exactly

$$\int_X \bar{\psi}\left(i\gamma^\mu\partial_\mu + \gamma^0\mu - m\right)\psi = -\sum_K \bar{\psi}(K)\frac{G_0^{-1}(K)}{T}\psi(K)\,,$$

with the free inverse fermion propagator in momentum space

$$G_0^{-1}(K) = -\gamma^\mu K_\mu - \gamma^0\mu + m$$

- integration over Grassmann variables

$$Z = \int_{\text{antiperiodic}} \mathcal{D}\psi^\dagger\mathcal{D}\psi\,\exp\left[-\sum_K \psi^\dagger(K)\gamma^0\frac{G_0^{-1}(K)}{T}\psi(K)\right]$$

$$= \det\frac{G_0^{-1}(K)}{T} = \prod_K\left[\frac{k^2+m^2-(k_0+\mu)^2}{T^2}\right]^2$$

(determinant over Dirac and momentum space)

Intermezzo: Thermal field theory (page 6/6)

$$\Rightarrow \ln Z = \sum_K \ln\left[\frac{\epsilon_k^2 - (k_0+\mu)^2}{T^2}\right]^2 = \sum_K \left[\ln\frac{\omega_n^2 + (\epsilon_k-\mu)^2}{T^2} + \ln\frac{\omega_n^2 + (\epsilon_k+\mu)^2}{T^2}\right]$$

with $\epsilon_k \equiv \sqrt{k^2+m^2}$

- sum over fermionic Matsubara frequencies
  (for instance via contour integration in the complex ω plane)

$$\sum_n \ln\frac{\omega_n^2+\epsilon_k^2}{T^2} = \frac{\epsilon_k}{T} + 2\ln\left(1+e^{-\epsilon_k/T}\right) + \text{const}$$

$$\Omega = -\frac{T}{V}\ln Z = -2\int\frac{d^3k}{(2\pi)^3}\left[\epsilon_k + T\ln\left(1+e^{-(\epsilon_k-\mu)/T}\right) + T\ln\left(1+e^{-(\epsilon_k+\mu)/T}\right)\right]$$

Pressure from partition function

- partition function

$$Z = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}\sigma\mathcal{D}\omega \, \exp \int_X \mathcal{L}$$

$$= e^{\frac{V}{T}\left(-\frac{1}{2}m_\sigma^2\langle\sigma\rangle^2+\frac{1}{2}m_\omega^2\langle\omega_0\rangle^2\right)} \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \, \exp \int_X \bar{\psi}\left(i\gamma^\mu\partial_\mu - m_N^* + \mu^*\gamma_0\right)\psi$$

- perform functional integral (see thermal field theory intermezzo) & ignore "vacuum contribution" & neglect anti-baryons

$$P = \frac{T}{V}\ln Z = -\frac{1}{2}m_\sigma^2\langle\sigma\rangle^2+\frac{1}{2}m_\omega^2\langle\omega_0\rangle^2 + \underbrace{4T\int\frac{d^3\mathbf{k}}{(2\pi)^3}\ln\left(1+e^{-(E_k-\mu^*)/T}\right)}_{P_N}$$

with $E_k = \sqrt{k^2 + (m_N^*)^2}$

Stationarity equations

- compute meson vevs from

$$0 = \frac{\partial P}{\partial\langle\sigma\rangle} = -m_\sigma^2\langle\sigma\rangle - g_\sigma\frac{\partial P_N}{\partial m_N^*} \equiv -m_\sigma^2\langle\sigma\rangle + g_\sigma n_s$$

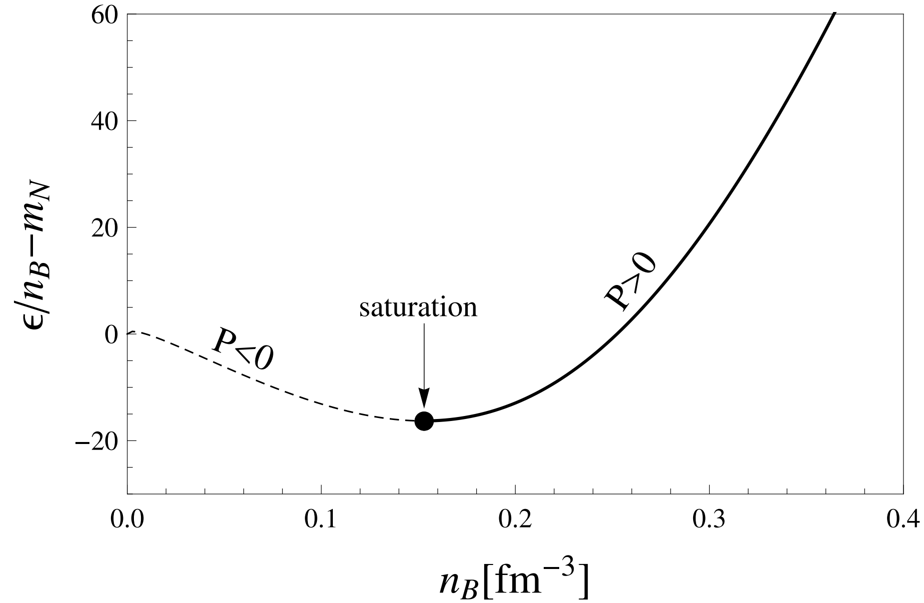
$$0 = \frac{\partial P}{\partial\langle\omega_0\rangle} = m_\omega^2\langle\omega_0\rangle - g_\omega\frac{\partial P_N}{\partial \mu^*} \equiv m_\omega^2\langle\omega_0\rangle - g_\omega n_B$$

- for given n_B the equations decouple and we need to solve

$$m_N^* = m_N - \frac{g_\sigma^2}{m_\sigma^2} n_s$$

for m_N^*

Saturation density and binding energy



- $\exists$ minimum of $\epsilon/n_B = E/A$ at "saturation density"

$$n_0 \simeq 0.15\,\mathrm{fm}^{-3}$$

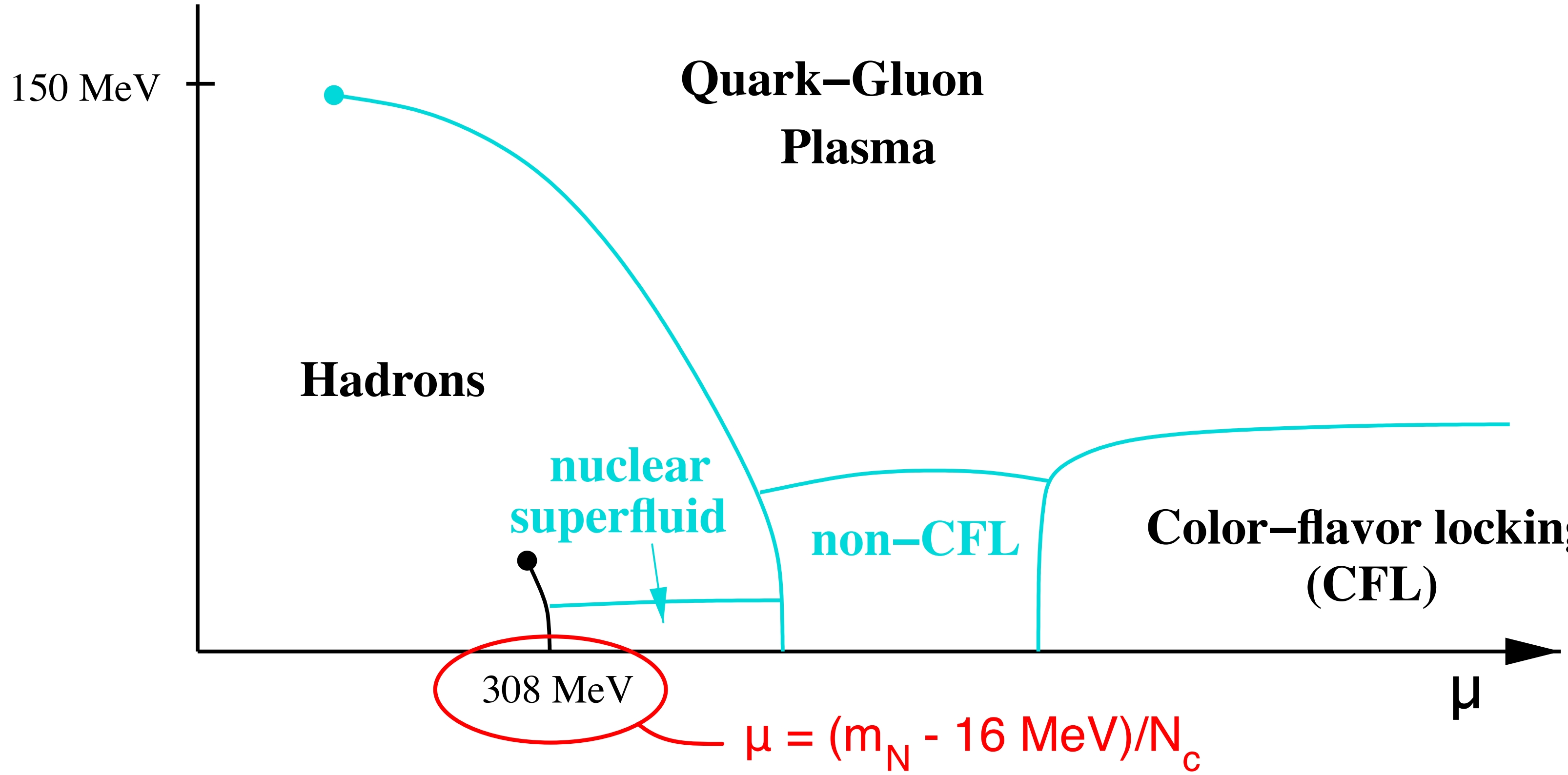
- semi-empirical energy

$$E = -a_1 A + \underbrace{a_2 A^{3/2}}_{\text{surface}} + \underbrace{a_3 \frac{Z^2}{A^{1/3}}}_{\text{Coulomb}} + \underbrace{a_4 \frac{(A-2Z)^2}{A}}_{\text{(a)symmetry}}$$

- symmetric, infinite nuclear matter without EM has binding energy $E_0 \equiv E/A = -a_1 = -16\,\mathrm{MeV}$

- g_σ and g_ω fitted to reproduce n_0 and E_0

Saturation density in the QCD phase diagram

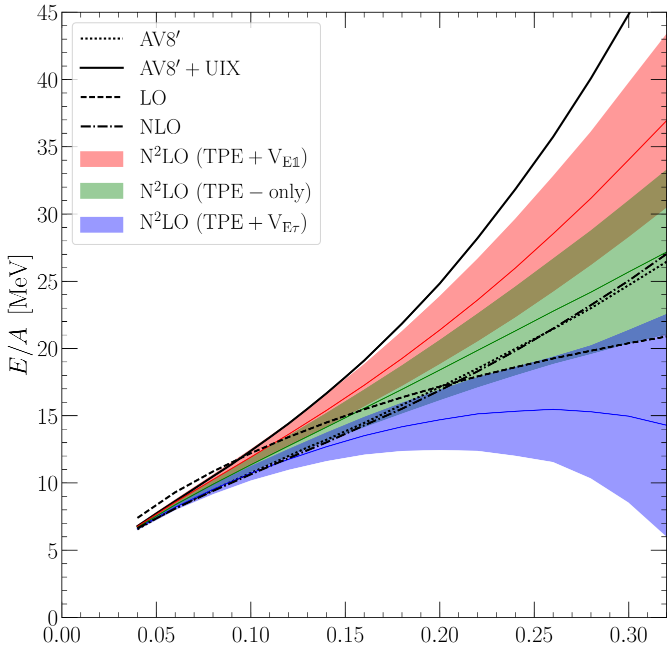


- $\mu_B < m_N - E_0$: vacuum with $P = 0$ and $n_B = 0$
- $\mu_B = m_N - E_0$: first-order phase transition to nuclear matter with $P = 0$ and $n_B = n_0$
- $\mu_B > m_N - E_0$: nuclear matter with $P > 0$ and $n_B > n_0$

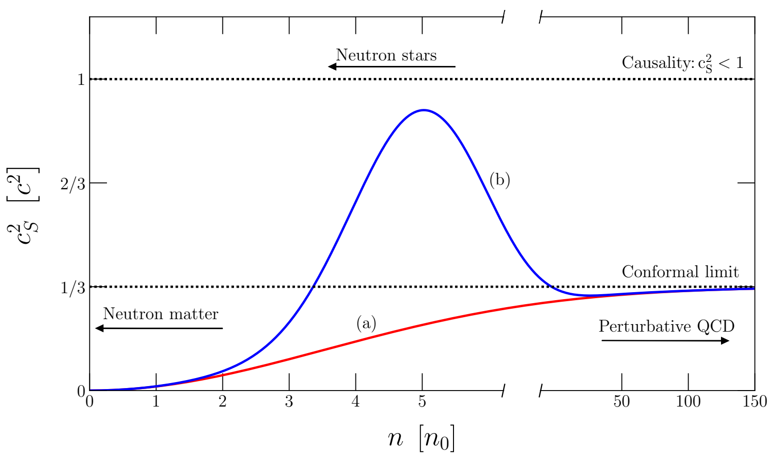
Two examples from current research

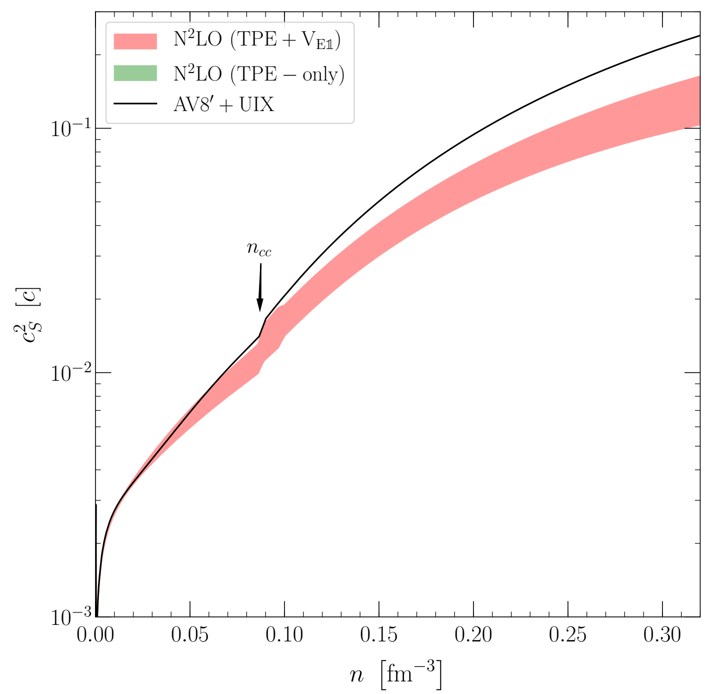
Chiral effective theory and speed of sound

I. Tews, J. Carlson, S. Gandolfi and S. Reddy, Astrophys.J. 860, 149 (2018)



Chiral effective theory:
systematic expansion in momentum scale for low-density nuclear matter based on QCD symmetries

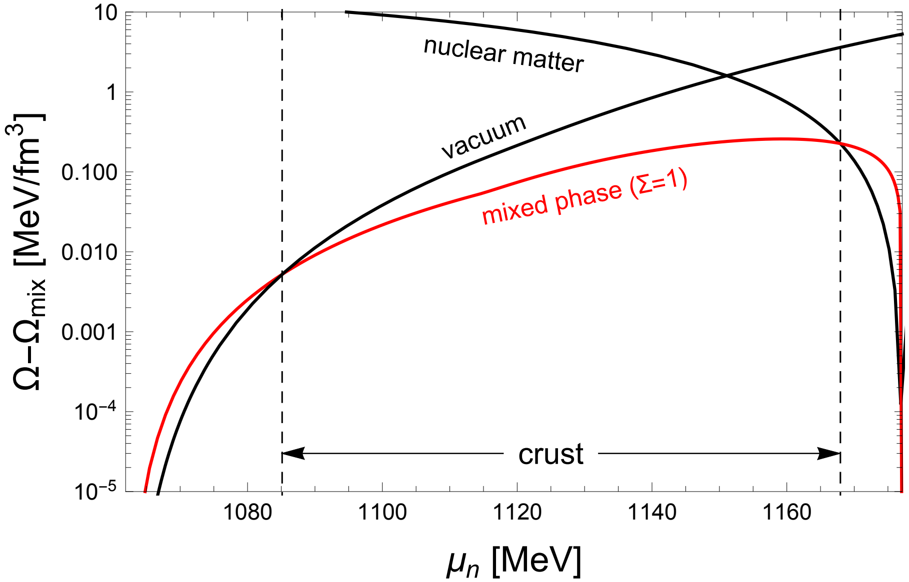




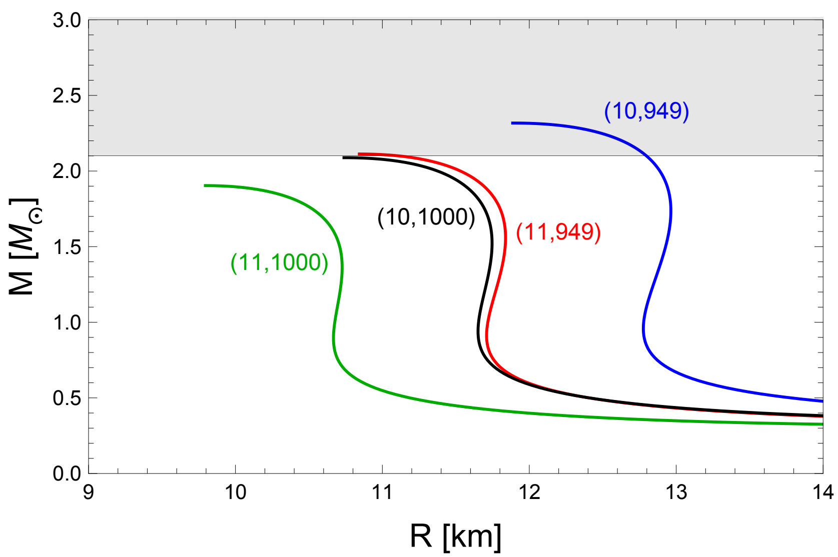
Nuclear matter and neutron stars from holography

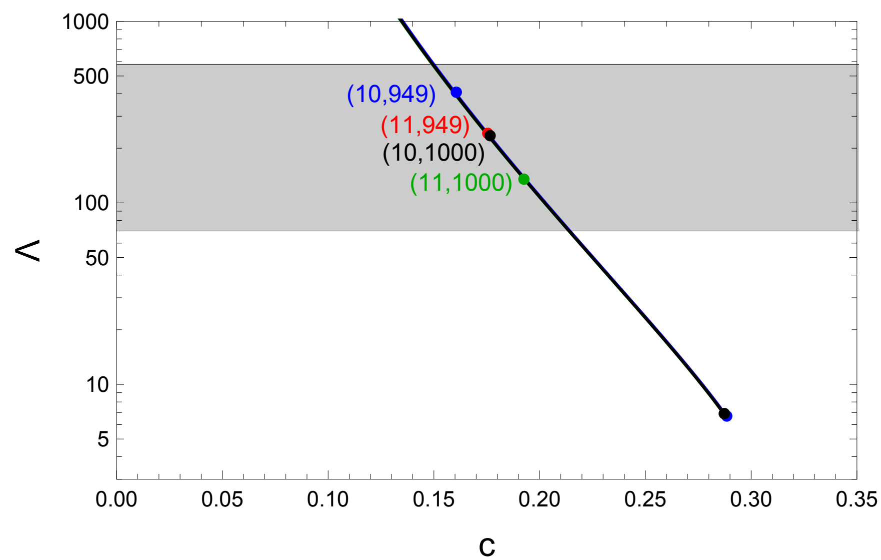
N. Kovensky and A. Schmitt, SciPost Phys. 11, 029 (2021)

N. Kovensky, A. Poole and A. Schmitt, PRD 105, 034022 (2022)



- Nuclear matter from Witten-Sakai-Sugimoto model with two parameters $(\lambda, M_{\rm KK})$
- Construct entire star from holography (including crust)





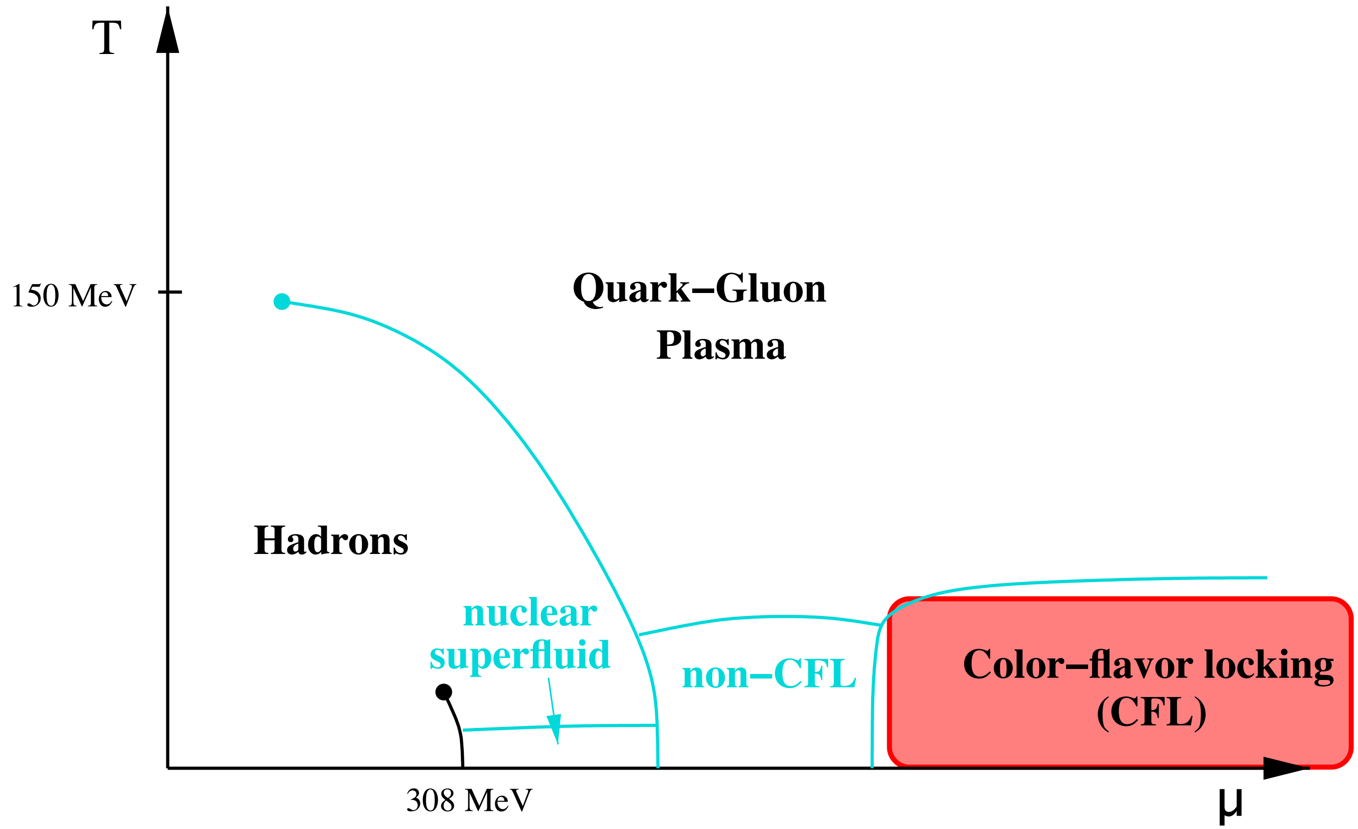
Summary: nuclear matter

- neutral nuclear matter in β-equilibrium is neutron-rich
  $\rightarrow$ "neutron star"
- symmetric nuclear matter has a "saturation density" n_0 and a "binding energy" E_0
- as a consequence, there is a first-order baryon onset (liquid-gas transition) in the QCD phase diagram
- first-principle studies of nuclear matter are possible at low densities (chiral effective theory)
- models, extra(intra)polations are needed for larger densities (e.g. holographic methods)

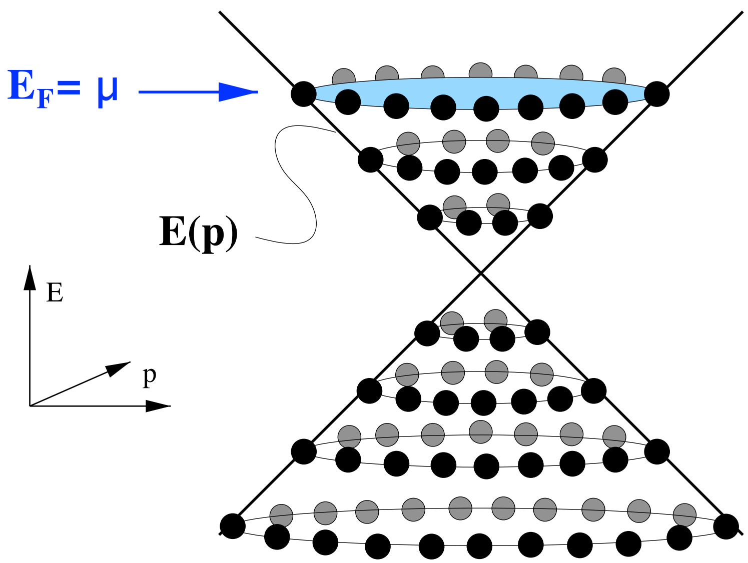
Outline

- Connecting QCD to astrophysical observables
  - Basics of QCD and phase diagram
  - Neutron stars as laboratories for dense (and hot) QCD
- Equation of state
  - Unpaired quark matter at asymptotically large densities
  - Nuclear matter in a simple approximation (intermezzo: thermal field theory)
- Color superconductivity
  - QCD gap equation
  - Color-flavor locking and other color superconductors
- Transport in dense QCD
  - Brief overview of transport in neutron stars
  - Bulk viscosity of (color-superconducting) quark matter

Color superconductivity and color-flavor locking



Cooper pairing of fermions



- free energy $\Omega = E - \mu N$
- no interactions: add fermion at $E = \mu$ without cost
- attractive interaction: add pair with gain
- pairs condense $\rightarrow$ "Cooper pairing"

This Bardeen-Cooper-Schrieffer (BCS) argument holds for electrons in a metal, ^{3}He atoms, nucleons, quarks, ...