Thermal field theory, WS 13/14

Andreas Schmitt

Contents

I. INTRODUCTION AND OUTLINE

We shall discuss systems in equilibrium at finite temperatures and chemical potentials. In most parts we will follow the book by Kapusta [1], see also the book by Le Bellac [2] or the online lecture notes by Laine [3] for additional reference. We will focus on the functional integral approach, for a different approach using second quantization see the book by Fetter and Walecka [4]. We shall learn the tools of functional integration, Matsubara summation, perturbation techniques, and discuss important theoretical concepts such as spontaneous symmetry breaking and restoration thereof at large temperatures. Applications, to be discussed after learning these techniques, are

- early universe, cosmology
  - inflation, $T \sim 10^{15}\,\mathrm{GeV}$
  - electroweak phase transition $T \sim 10^{2}\,\mathrm{GeV}$
  - QCD phase transition, $T \sim 10^{2}\,\mathrm{MeV}$ (for comparison, this is $\sim 10^{12}\,\mathrm{K}$)
  - baryogenesis
- QCD phase transitions
  - heavy-ion collisions ("little bang" vs. "big bang")
  - chiral symmetry (spontaneous breaking thereof)
  - lattice QCD
- compact stars, $T \lesssim 10\,\mathrm{MeV}$, $\mu_q \sim 400\,\mathrm{MeV}$
  - dense nuclear matter
  - neutrino emissivity
  - quark matter, color superconductivity

The physics of compact stars are discussed in a separate lecture, for the lecture notes see Ref. [5] (some methods of thermal field theory are also used in my lectures about superfluids, see Ref. [6]). For a review of the basic features of thermal field theory with emphasis on heavy-ion collisions, see Ref. [7]; for a review of more advanced techniques, in particular in the context of QCD, see Ref. [8].

[End of 1st lecture, Oct 7th, 2013.]

II. BASICS OF STATISTICAL QUANTUM MECHANICS

This chapter serves as a quick reminder of the main ingredients of statistical quantum mechanics and its relation to thermodynamic quantities. A good textbook about statistical physics is the book by Nolting [9]. The goal of this reminder is to explain the meaning and form of the partition function which, in later chapters in the functional integral representation, plays a central role in thermal field theory.

A. Statistical operator

We start by recalling that statistical quantum mechanics involves probabilities on two levels. First, on the fundamental level, quantum mechanics itself involves some kind of statistics, i.e., we can only give probabilities to measure a certain value of an observable. Let $\hat{A}$ be an observable with a set of complete orthogonal eigenstates $|n\rangle$ and eigenvalues a_n,

$$\hat{A}|n\rangle = a_n|n\rangle\,, \tag{1}$$

with $\langle m|n\rangle = \delta_{mn}$, $\sum_n |n\rangle\langle n| = \mathbf{1}$. We can expand any state $|\psi\rangle$ in terms of these eigenstates,

$$|\psi\rangle = \sum_n c_n|n\rangle\,. \tag{2}$$

The probability to measure the value a_n in the state $|\psi\rangle$ is $|c_n|^2$ with $c_n = \langle n|\psi\rangle$. Then, the expectation value of $\hat{A}$ in the state $|\psi\rangle$ is

$$\begin{aligned}\langle\hat{A}\rangle &= \sum_n |c_n|^2 a_n \\ &= \sum_n |\langle n|\psi\rangle|^2 a_n \\ &= \sum_n \langle n|\psi\rangle\langle\psi|\hat{A}|n\rangle \\ &= \langle\psi|\hat{A}|\psi\rangle\,. \end{aligned} \tag{3}$$

Second, and independent of the quantum probability interpretation, there is a level of uncertainty for macroscopic systems of which we don't know (and are not interested in) the microscopic details. We now consider many possible (orthogonal) quantum mechanical states $|\psi_m\rangle$ each of which we find with a probability p_m, $0 < p_m < 1$. Then, the expectation value for $\hat{A}$ is not only given by the quantum mechanical averaging but also by averaging over the possible states $|\psi_m\rangle$, i.e.,

$$\langle\hat{A}\rangle = \sum_m p_m \langle\psi_m|\hat{A}|\psi_m\rangle\,. \tag{4}$$

If we define the statistical operator

$$\hat{\rho} \equiv \sum_m p_m |\psi_m\rangle\langle\psi_m|\,, \tag{5}$$

we can write the expectation value as

$$\langle\hat{A}\rangle = \mathrm{Tr}(\hat{\rho}\,\hat{A})\,. \tag{6}$$

Proof:

$$\begin{aligned}\langle\hat{A}\rangle &= \sum_m p_m \langle\psi_m|\hat{A}|\psi_m\rangle = \sum_{i,j,m} p_m \langle\psi_m|\phi_i\rangle\langle\phi_i|\hat{A}|\phi_j\rangle\langle\phi_j|\psi_m\rangle \\ &= \sum_{i,j} \hat{\rho}_{ji}\hat{A}_{ij} = \mathrm{Tr}(\hat{\rho}\,\hat{A})\,. \end{aligned} \tag{7}$$

We obviously have $\hat{\rho}^\dagger = \hat{\rho}$ and

$$\mathrm{Tr}\,\hat{\rho} = 1 \tag{8}$$

(which is clear from setting $\hat{A} = \mathbf{1}$ above). The meaning of $\hat{\rho}$ can also be understood from the analogy to classical statistical mechanics. In this case, there is a probability distribution $\rho(p,q)$ in the $6N$-dimensional phase space (N be the number of particles which each moves on a trajectory in phase space); then, $d^{3N}p\,d^{3N}q\,\rho(p,q)$ is the probability to find the system in the small region $d^{3N}p\,d^{3N}q$ in phase space. An observable A has a value $A(p,q)$ if the system sits on the point (p,q) in phase space, and its expectation value is given by

$$\langle A\rangle = \frac{1}{(2\pi\hbar)^{3N}N!}\int d^{3N}p\,d^{3N}q\,\rho(p,q)\,A(p,q)\,, \tag{9}$$

where the factor $N!$ in the normalization refers to the exchange of particles and the factor $(2\pi\hbar)^{3N}$ is included to do the transition to a quantum mechanical system. Comparing Eq. (6) with Eq. (9) shows the formal similarity between the quantum and classical expectation values (the trace is replaced by the phase space integral). This formal correspondence goes further, e.g., the Liouville equation for the classical probability distribution

$$\frac{\partial\rho}{\partial t} = -\{H,\rho\}\,, \tag{10}$$

where H is the Hamilton function of the system and $\{-,-\}$ the Poisson bracket, is very similar to the Heisenberg equation for the statistical operator

$$\frac{\partial\hat{\rho}}{\partial t} = -\frac{i}{\hbar}[\hat{H},\hat{\rho}]\,, \tag{11}$$

with the Hamilton operator $\hat{H}$. Here we shall be mostly concerned with equilibrium situations and thus $[\hat{H},\hat{\rho}] = 0$.

B. Grand canonical ensemble

Let us now recall the different ensembles in statistical physics (both classical and quantum mechanical). One should in the following always think of an ensemble as a collection of many different systems with the same fixed macroscopic properties but different microscopic configurations. Which macroscopic properties are fixed depends on the ensemble:

- microcanonical ensemble (E, N, V)
- canonical ensemble (T, N, V)
- grand canonical ensemble (T, μ, V)

with the volume V, the energy E, the particle number (charge) N, the temperature T, and the chemical potential μ (associated with the charge N); in general, there can be more than one conserved charge and chemical potential. We shall mostly be concerned with the grand canonical ensemble. Therefore, let us give a brief derivation of the statistical operator in the grand canonical ensemble. Consider a system Σ with fixed energy E, charge N, and volume V. We are interested in a small subsystem $\Sigma^{(1)}$ such that $\Sigma = \Sigma^{(1)} \cup \Sigma^{(2)}$ and such that $\Sigma^{(1)}$ and $\Sigma^{(2)}$ are separated by walls through which charge and energy can be exchanged. Let us denote the energy, charge, and volume of the subsystems by $E^{(i)}$, $N^{(i)}$, $V^{(i)}$. We then have $E = E^{(1)} + E^{(2)}$, $N = N^{(1)} + N^{(2)}$. We assume that the Hamilton and charge operators commute, $[\hat{H}^{(i)}, \hat{N}^{(i)}] = 0$, and thus that there is a set of simultaneous eigenstates $|E_m, n\rangle$ for the subsystem $\Sigma^{(1)}$,

$$\hat{H}^{(1)}|E_m, n\rangle \;=\; E_m|E_m, n\rangle\,, \tag{12a}$$

$$\hat{N}^{(1)}|E_m, n\rangle \;=\; n|E_m, n\rangle\,. \tag{12b}$$

We can write the statistical operator as

$$\hat{\rho} = \sum_{m,n} p_{m,n}|E_m, n\rangle\langle E_m, n|\,. \tag{13}$$

We are interested in the probability $p_{m,n}$ to find the system $\Sigma^{(1)}$ in the state $|E_m, n\rangle$ with energy E_m and charge $N^{(1)} = n$. This probability is proportional to the number of states Γ available in the complementary system $\Sigma^{(2)}$,

$$p_{m,n} \propto \Gamma^{(2)}(E - E_m, N - n, V^{(2)})\,. \tag{14}$$

The system $\Sigma^{(2)}$ acts as a heat and particle bath for $\Sigma^{(1)}$ and thus $E_m \ll E$, $n \ll N$. We can then expand the logarithm of Γ,

$$\ln\Gamma^{(2)}(E - E_m, N - n, V^{(2)}) \simeq \frac{1}{k_B}S^{(2)}(E, N, V^{(2)}) - \frac{E_m}{k_B}\,\frac{\partial S^{(2)}}{\partial E}\bigg|_{E_m=n=0} - \frac{n}{k_B}\,\frac{\partial S^{(2)}}{\partial N}\bigg|_{E_m=n=0}\,, \tag{15}$$

where we have used the definition for the entropy $S = k_B \ln\Gamma$ where k_B is the Boltzmann constant. Now we can use the thermodynamic relations $\partial S/\partial E = 1/T$, $\partial S/\partial N = -\mu/T$ (in fact, one can view the derivatives in Eq. (15) as the definitions of temperature and chemical potential) to obtain

$$p_{m,n} \propto e^{-\beta(E_m - \mu n)}\,, \tag{16}$$

with

$$\beta \equiv \frac{1}{k_B T}\,. \tag{17}$$

(Later we shall always use units where $\hbar = c = k_B = 1$ such that β is simply the inverse temperature, $\beta = 1/T$.) Here we have dropped the contribution $S^{(2)}/k_B$ which only depends on the system $\Sigma^{(2)}$ and thus can be absorbed in the normalization, to be determined below. Inserting this into the statistical operator (13) yields

$$\hat{\rho} \propto \sum_{m,n} e^{-\beta(E_m - \mu n)} |E_m, n\rangle\langle E_m, n| = e^{-\beta(\hat{H}^{(1)} - \mu \hat{N}^{(1)})} \sum_{m,n} |E_m, n\rangle\langle E_m, n| = e^{-\beta(\hat{H}^{(1)} - \mu \hat{N}^{(1)})} \,. \tag{18}$$

Here we have used that the exponential of an operator is defined via the expansion in powers of the operator, $e^{\hat{A}} = \sum_\ell \frac{\hat{A}^\ell}{\ell!}$. Therefore, one has for instance $e^{\hat{A}}|n\rangle = \sum_\ell \frac{\hat{A}^\ell}{\ell!}|n\rangle = \sum_\ell \frac{a_n^\ell}{\ell!}|n\rangle = e^{a_n}|n\rangle$. We have also used that $\hat{H}^{(1)}$ and $\hat{N}^{(1)}$ commute, such that we can factorize the exponential, $e^{-\beta(\hat{H}^{(1)} - \mu \hat{N}^{(1)})} = e^{-\beta \hat{H}^{(1)}} e^{\beta \mu \hat{N}^{(1)}}$.

To fulfill the normalization (8) we find (dropping the superscript "(1)", from now on we are only talking about the subsystem $\Sigma^{(1)}$)

$$\hat{\rho} = \frac{e^{-\beta(\hat{H} - \mu \hat{N})}}{Z} \,, \tag{19}$$

where

$$Z \equiv \mathrm{Tr}\, e^{-\beta(\hat{H} - \mu \hat{N})} \tag{20}$$

is the *partition function* of the system. This means that, with Eq. (6), the expectation value of an observable $\hat{A}$ is given by

$$\langle \hat{A} \rangle = \frac{\mathrm{Tr}[\hat{A} e^{-\beta(\hat{H} - \mu \hat{N})}]}{Z} \,. \tag{21}$$

We can derive all thermodynamic quantities from the partition function, which thus is the central quantity in statistical physics. For instance we have the grand canonical potential (sometimes called the thermodynamic potential)

$$\Omega(T, \mu, V) \equiv -\frac{1}{\beta} \ln Z \,. \tag{22}$$

This function is related to the other thermodynamic quantities via

$$\Omega = -PV = E - \mu N - TS \,, \tag{23}$$

where P is the pressure. From the partition function we immediately get, in accordance with (23),

$$\begin{aligned} \frac{\partial \Omega}{\partial \mu} &= -\frac{1}{\beta} \frac{\partial \ln Z}{\partial \mu} \\ &= -\frac{1}{\beta Z} \frac{\partial Z}{\partial \mu} \\ &= -\frac{\mathrm{Tr} \hat{N} e^{-\beta(\hat{H} - \mu \hat{N})}}{Z} \\ &= -\langle \hat{N} \rangle \equiv -N \,, \end{aligned} \tag{24}$$

and

$$\frac{\partial \Omega}{\partial T} = -\ln Z - \beta \frac{\mathrm{Tr}\left[(\hat{H} - \mu \hat{N}) e^{-\beta(\hat{H} - \mu \hat{N})}\right]}{Z} = \langle \ln \hat{\rho} \rangle = -\langle S \rangle \,, \tag{25}$$

since $\ln \hat{\rho} = \ln\, e^{-\beta(\hat{H} - \mu \hat{N})} - \ln Z = -\beta(\hat{H} - \mu \hat{N}) - \ln Z$.

Let us compute the partition function for the simplest cases before we turn to the field theoretical description. Let us first consider a single energy state with energy ω which we can fill with bosons. Remember that this simple many-particle system resembles the one-particle harmonic oscillator because the Hamiltonian is $\hat{H} = \omega(\hat{N} + 1/2)$,

where we drop the zero-point energy $\omega/2$. Since arbitrarily many bosons can populate the state, the partition function is

$$Z = \sum_{n=0}^{\infty} \langle n|e^{-\beta(\omega-\mu)\hat{N}}|n\rangle = \sum_{n=0}^{\infty} e^{-\beta(\omega-\mu)n}\langle n|n\rangle = \frac{1}{1-e^{-\beta(\omega-\mu)}}\,, \tag{26}$$

where, for the last step, we have assumed $\mu < \omega$, such that we can apply the formula for the geometric series $\sum_{n=0}^{\infty} q^n = \frac{1}{1-q}$ for $0 < q < 1$. The thermodynamic potential is thus

$$\Omega = -T\ln Z = T\ln\left[1-e^{-\beta(\omega-\mu)}\right] \tag{27}$$

and the particle number

$$N = -\frac{\partial\Omega}{\partial\mu} = \frac{1}{e^{\beta(\omega-\mu)}-1}\,. \tag{28}$$

One can already see from this simple example that something interesting happens if the chemical potential approaches the energy ω, since in this case the particle number seems to diverge. This is a first hint of Bose-Einstein condensation which we shall discuss later. We also see that the chemical potential cannot assume values larger than ω in order to avoid negative N. This is a restriction for non-interacting systems.

For fermions we get a similar expression, with the difference that we can only put one fermion at most into the energy state. Consequently, the sum only runs over $n = 0, 1$, and we get

$$Z = \sum_{n=0}^{1} \langle n|e^{-\beta(\omega-\mu)\hat{N}}|n\rangle = 1 + e^{-\beta(\omega-\mu)}\,, \tag{29}$$

where the $n = 0$ term yields the 1 from the expansion of the exponential (which is obvious from writing $e^{-\beta(\omega-\mu)\hat{N}} = \mathbf{1} - \beta(\omega-\mu)\hat{N} + \ldots$). In this case, the thermodynamic potential is

$$\Omega = -T\ln Z = -T\ln\left[1+e^{-\beta(\omega-\mu)}\right] \tag{30}$$

and the particle number

$$N = -\frac{\partial\Omega}{\partial\mu} = \frac{1}{e^{\beta(\omega-\mu)}+1}\,. \tag{31}$$

In this simple case of just one single energy ω, the particle number at vanishing temperature is $N = 0$ for $\mu < \omega$ and $N = 1$ for $\mu > \omega$. At nonzero temperature, N assumes values in between 0 and 1.

Next we allow for a dispersion of the particles, i.e., the energy may depend on the (modulus of the) momentum, $\omega_p \equiv \omega(p)$. We start with a box with size L in all three dimensions. The box size must be an integer times half of the wavelength λ, $L = n\lambda/2$. With the de Broglie relation for the wavelength $p = 2\pi/\lambda$ (we set $\hbar = 1$) we have $p = n\pi/L$. The log of the full partition function is the sum over all partition functions of the single modes,

$$\ln Z = \sum_{\mathbf{n}=(n_x,n_y,n_z)} \ln Z_{\mathbf{n}} = V\int\frac{d^3p}{(2\pi)^3}\ln Z_p = \mp V\sum_{e=\pm}\int\frac{d^3p}{(2\pi)^3}\ln[1\mp e^{-\beta(\omega_p-e\mu)}]\,. \tag{32}$$

Here we have employed the infinite volume limit $L\to\infty$ where we can replace the sums over n_i by integrals $\int_1^\infty dn_i$. Then, we have used $dn_i \to L/\pi\, dp_i$ from the above de Broglie relation and have doubled the range of the three momentum integrals, hence the the factor 2^3 in the denominator. Also, we have defined the volume $V = L^3$. In the second step, we have inserted the above expressions for the log of the partition function for a single mode for bosons (upper sign) and fermions (lower sign). Finally, we have added a sum over $e = \pm$ accounting for particles and antiparticles which differ in the sign of their chemical potential. The conserved charge is thus

$$N = V\sum_{e=\pm} e\int\frac{d^3p}{(2\pi)^3}\frac{1}{e^{\beta(\omega_p-e\mu)}\mp 1}\,. \tag{33}$$

We shall often consider the charge density $n \equiv N/V$ instead. In the case of bosons (upper sign) we see that we have to require $-\min\omega_p < \mu < \min\omega_p$ in order to have positive occupation numbers.

[End of 2nd lecture, Oct 14th, 2013.]

III. PARTITION FUNCTION IN THE PATH INTEGRAL FORMALISM

Here we derive the expression for the partition function in quantum field theory as opposed to usual quantum mechanics. Remember that in usual quantum theory, the projection of an eigenstate $\langle \mathbf{x}|$ of the position operator $\hat{\mathbf{x}}$ onto the eigenstate $|\mathbf{p}\rangle$ of the momentum operator $\hat{\mathbf{p}}$ is given by a plane wave,

$$\langle \mathbf{x}|\mathbf{p}\rangle = e^{i\mathbf{p}\cdot\mathbf{x}}\,. \tag{34}$$

In quantum field theory, the discrete sum $\mathbf{p}\cdot\mathbf{x} = \sum_i p_i x_i$ becomes an integral,

$$\langle \phi|\pi\rangle = e^{i\int d^3x\,\pi(\mathbf{x})\phi(\mathbf{x})}\,. \tag{35}$$

Here, $\phi(\mathbf{x})$ and $\pi(\mathbf{x})$ are eigenvalues (better: eigenfunctions) of the field operator (at $t=0$) $\hat{\phi}(\mathbf{x},0)$ and its conjugate momentum operator $\hat{\pi}(\mathbf{x},0)$. We have the following completeness and orthogonality conditions,

$$\int d\phi(\mathbf{x})\,|\phi\rangle\langle\phi| = 1\,, \qquad \langle\phi_a|\phi_b\rangle = \delta[\phi_a(\mathbf{x}) - \phi_b(\mathbf{x})]\,, \tag{36a}$$

$$\int \frac{d\pi(\mathbf{x})}{2\pi}\,|\pi\rangle\langle\pi| = 1\,, \qquad \langle\pi_a|\pi_b\rangle = \delta[\pi_a(\mathbf{x}) - \pi_b(\mathbf{x})]\,. \tag{36b}$$

The Hamiltonian $\hat{H}$ of the system is given by the Hamilton density $\mathcal{H}$ which can be expressed in terms of the field operators,

$$\hat{H} = \int d^3x\,\mathcal{H}(\hat{\pi}(\mathbf{x},t),\hat{\phi}(\mathbf{x},t))\,. \tag{37}$$

In the following we shall write the partition function in terms of the fields ϕ and π and get rid of all operators. To this end, we first compute a transition amplitude with identical initial and final state, say ϕ_a, at times $t=0$ and $t=t_f$. The initial state evolves in time upon applying the unitary operator $e^{-i\hat{H}t}$, assuming that $\hat{H}$ does not depend explicitly on t. (We are not interested in the general case with time-dependent $\hat{H}$ since eventually we want to compute the partition function; the statistical operator has the form $e^{-\beta\hat{H}}$ for all $\hat{H}$.) We divide the time interval $[0,t_f]$ into N pieces with length Δt. Then we can write the transition amplitude as

$$\begin{aligned}
\langle\phi_a|e^{-i\hat{H}t_f}|\phi_a\rangle &= \lim_{N\to\infty}\langle\phi_a|e^{-i\hat{H}\Delta t}e^{-i\hat{H}\Delta t}\ldots e^{-i\hat{H}\Delta t}|\phi_a\rangle \\
&= \lim_{N\to\infty}\int\prod_{i=1}^{N}\frac{d\pi_i(\mathbf{x})}{2\pi}d\phi_i(\mathbf{x})\langle\phi_a|\pi_N\rangle\langle\pi_N|e^{-i\hat{H}\Delta t}|\phi_N\rangle\langle\phi_N|\pi_{N-1}\rangle\langle\pi_{N-1}|e^{-i\hat{H}\Delta t}|\phi_{N-1}\rangle \\
&\quad\times\ldots\times\langle\phi_2|\pi_1\rangle\langle\pi_1|e^{-i\hat{H}\Delta t}|\phi_1\rangle\langle\phi_1|\phi_a\rangle\,,
\end{aligned} \tag{38}$$

where we inserted each of the completeness relations in Eqs. (36) N times alternatingly. Now, for the scalar products of the form $\langle\phi_{i+1}|\pi_i\rangle$ we use Eq. (35). For the factors involving the Hamiltonian we use

$$\begin{aligned}
\langle\pi_i|e^{-i\hat{H}\Delta t}|\phi_i\rangle &\simeq e^{-i\Delta t\int d^3x\,\mathcal{H}(\phi_i,\pi_i)}\langle\pi_i|\phi_i\rangle \\
&= e^{-i\Delta t\int d^3x\,\mathcal{H}(\phi_i,\pi_i)}e^{-i\int d^3x\pi_i\phi_i}\,,
\end{aligned} \tag{39}$$

where we have approximated $e^{-i\hat{H}\Delta t}\simeq 1 - i\hat{H}\Delta t$ and used that the Hamiltonian at a given time labelled by i is a sum of powers of the fields ϕ_i and π_i. This is needed to replace the operators in the exponential by their eigenfunctions, by applying all field operators $\hat{\phi}$ to $|\phi_i\rangle$ and all momentum operators $\hat{\pi}$ to $\langle\pi_i|$. Moreover, we have used that $\langle\pi|\phi\rangle = \langle\phi|\pi\rangle^*$, hence the minus in the exponent compared to the scalar product in Eq. (35). From the last

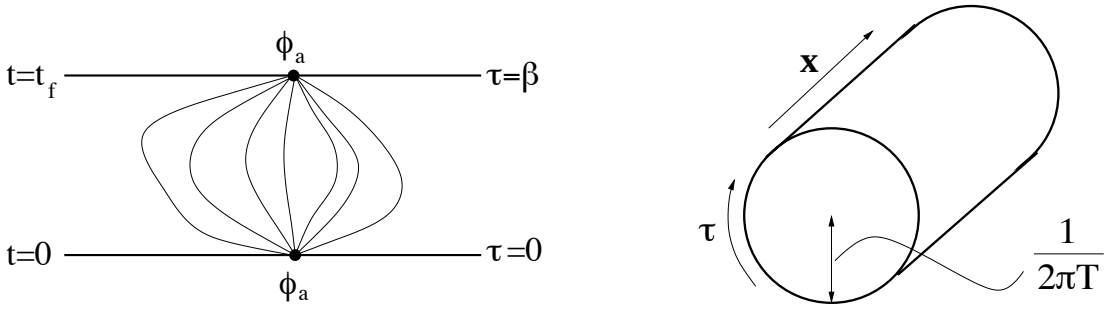

FIG. 1: Left: illustration of the functional integration over fields with identical inital and final states at times $t = 0$ and $t = t_f$. In thermal field theory, we work with imaginary time where the field is periodic in the interval with boundaries $\tau = 0$ and $\tau = \beta$. The integral is performed over all ϕ_a, each of them with the shown periodicity. Right: due to this periodicity, space-time in thermal field theory can effectively be viewed as a cylinder whose radius is proportional to the inverse temperature. For zero temperature, the radius goes to infinity and the flat topology is recovered.

factor in the integrand in Eq. (38) we obtain $\langle\phi_1|\phi_a\rangle = \delta(\phi_a - \phi_1)$. Consequently,

$$\begin{aligned}\langle\phi_a|e^{-i\hat{H}t_f}|\phi_a\rangle &= \lim_{N\to\infty}\int\prod_{i=1}^{N}\frac{d\pi_i(\mathbf{x})}{2\pi}d\phi_i(\mathbf{x})\delta[\phi_a(\mathbf{x})-\phi_1(\mathbf{x})]e^{i\int d^3x\,[\pi_N(\phi-\phi_N)+\pi_{N-1}(\phi_N-\phi_{N-1})+\ldots+\pi_1(\phi_2-\phi_1)]} \\ &\quad\times e^{-i\Delta t\int d^3x\,[\mathcal{H}(\phi_N,\pi_N)+\ldots+\mathcal{H}(\phi_1,\pi_1)]} \\ &= \lim_{N\to\infty}\int\prod_{i=1}^{N}\frac{d\pi_i(\mathbf{x})}{2\pi}d\phi_i(\mathbf{x})\delta[\phi_a(\mathbf{x})-\phi_1(\mathbf{x})]\exp\left\{i\sum_{j=1}^{N}\Delta t\int d^3x\,\left[\pi_j\frac{\phi_{j+1}-\phi_j}{\Delta t}-\mathcal{H}(\phi_j,\pi_j)\right]\right\} \quad (40)\end{aligned}$$

where we have denoted $\phi_{N+1} \equiv \phi_a$. We can now take the limit $N \to \infty$ to obtain

$$\langle\phi_a|e^{-i\hat{H}t_f}|\phi_a\rangle = \int\mathcal{D}\pi\int_{\phi(\mathbf{x},0)=\phi_a(\mathbf{x})}^{\phi(\mathbf{x},t_f)=\phi_a(\mathbf{x})}\mathcal{D}\phi\exp\left\{i\int_0^{t_f}dt\int d^3x\,\left[\pi(\mathbf{x},t)\partial_t\phi(\mathbf{x},t)-\mathcal{H}(\phi(\mathbf{x},t),\pi(\mathbf{x},t))\right]\right\}. \quad (41)$$

We have denoted the continuum limit of the functional integration as

$$\int\prod_{i=1}^{N}\frac{d\pi_i(\mathbf{x})}{2\pi}\to\int\mathcal{D}\pi\,,\qquad\int\prod_{i=1}^{N}d\phi_i(\mathbf{x})\to\int\mathcal{D}\phi\,. \quad (42)$$

We can now use the result (41) to compute the partition function. To this end we compare Eq. (41) with Eq. (20). We see that the trace looks like a transition amplitude with identical initial and final states,

$$\begin{aligned}Z &= \mathrm{Tr}\,e^{-\beta(\hat{H}-\mu\hat{N})} \\ &= \int d\phi\,\langle\phi|e^{-\beta(\hat{H}-\mu\hat{N})}|\phi\rangle \\ &= \int\mathcal{D}\pi\int_{\mathrm{periodic}}\mathcal{D}\phi\,\exp\left[\int_0^\beta d\tau\int d^3x\,(i\pi\partial_\tau\phi-\mathcal{H}+\mu\mathcal{N})\right]. \quad (43)\end{aligned}$$

Here, we have, upon comparing with Eq. (41), identified the inverse temperature with "imaginary time"

$$\tau = it\,, \quad (44)$$

such that the integration over τ goes from 0 to the inverse temperature $\beta = 1/T$. The term "periodic" for the ϕ integral means that all functions ϕ have to be periodic in the imaginary time direction, $\phi(\mathbf{x}, 0) = \phi(\mathbf{x}, \beta)$. The integral over $d\phi$ integrates over all boundary values which are fixed in Eq. (41). We are left with a partition function which is given entirely in terms of the fields, all operators are gone.

IV. REAL NON-INTERACTING SCALAR FIELD

We now compute the partition function (43) for the simplest case, a real non-interacting scalar field which is described by the Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 = \frac{1}{2}\left[(\partial_0\phi)^2 - (\nabla\phi)^2 - m^2\phi^2\right] . \tag{45}$$

Here and throughout the lecture our convention for the Minkowski metric is $g^{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$. In the case of a real scalar field there is no continuous symmetry of the Lagrangian, hence there is no conserved charge and thus no chemical potential. We shall introduce the chemical potential for a charged complex field in the subsequent section.

For the partition function (43) we need the combination $\pi\partial_0\phi - \mathcal{H}$. First we compute $\mathcal{H}$. Remember that $\mathcal{L}$ and $\mathcal{H}$ are connected via a Legendre transformation which changes the independent variable $\partial_0\phi$ (velocity $\dot{q}$ in classical mechanics) to the conjugate momentum π (momentum p in classical mechanics), i.e., $\mathcal{L} = \mathcal{L}(\partial_0\phi, \phi, \nabla\phi)$, while $\mathcal{H} = \mathcal{H}(\pi, \phi, \nabla\phi)$. In order to perform the Legendre transform we need the conjugate momentum

$$\pi = \frac{\partial\mathcal{L}}{\partial(\partial_0\phi)} = \partial_0\phi\,. \tag{46}$$

Therefore, the Hamiltonian is

$$\mathcal{H}(\pi,\phi,\nabla\phi) = \left[\pi\partial_0\phi - \mathcal{L}(\partial_0\phi,\phi,\nabla\phi)\right]_{\partial_0\phi=\pi} = \frac{1}{2}\left[\pi^2 + (\nabla\phi)^2 + m^2\phi^2\right] , \tag{47}$$

and thus

$$\begin{aligned}
\pi\partial_0\phi - \mathcal{H} &= \pi\partial_0\phi - \frac{1}{2}\left[\pi^2 + (\nabla\phi)^2 + m^2\phi^2\right] \\
&= \frac{1}{2}\left[(\partial_0\phi)^2 - (\nabla\phi)^2 - m^2\phi^2\right] - \frac{1}{2}(\pi - \partial_0\phi)^2 \\
&= \mathcal{L} - \frac{1}{2}\tilde{\pi}^2\,,
\end{aligned} \tag{48}$$

with the shifted momentum $\tilde{\pi} \equiv \pi - \partial_0\phi$. If we use this shifted momentum as our new integration variable, the integration over the field ϕ separates from the integration over the momentum $\tilde{\pi}$, and we obtain

$$\begin{aligned}
Z &= \int \mathcal{D}\tilde{\pi}\, \exp\left(-\frac{1}{2}\int_X \tilde{\pi}^2(\tau,\mathbf{x})\right) \int \mathcal{D}\phi\, \exp\int_X \mathcal{L} \\
&= N \int \mathcal{D}\phi\, \exp\int_X \mathcal{L}\,,
\end{aligned} \tag{49}$$

where we have absorbed the result of the Gaussian momentum integral into an irrelevant constant[1], and where we have abbreviated

$$\int_X \equiv \int_0^\beta d\tau \int d^3x\,. \tag{52}$$

---

[1] This constant is infinite, but indeed independent of temperature, as one can see by introducing the Fourier components for the conjugate momenta,

$$\tilde{\pi}(X) = \sqrt{\frac{T}{V}}\sum_K e^{-iK\cdot X}\tilde{\pi}(K)\,. \tag{50}$$

Then, with Eq. (56),

$$\int \mathcal{D}\tilde{\pi}\, \exp\left(-\frac{1}{2}\int_X \tilde{\pi}^2\right) = \int \mathcal{D}\tilde{\pi}\, \exp\left[-\frac{1}{2}\sum_K \tilde{\pi}(-K)\tilde{\pi}(K)\right] . \tag{51}$$

This integral can formally be computed by using Eq. (61).

It remains to perform the integral over the Lagrangian which can be done exactly for a non-interacting field. We denote four-momenta by capital letters,

$$X \equiv (t, \mathbf{x}) = (-i\tau, \mathbf{x}) \,, \qquad K \equiv (k_0, \mathbf{k}) = (-i\omega_n, \mathbf{k}) \,, \tag{53}$$

where ω_n are the "Matsubara frequencies" which we explain now. The Fourier transform of the field is

$$\phi(X) = \frac{1}{\sqrt{TV}} \sum_K e^{-iK\cdot X} \phi(K) = \frac{1}{\sqrt{TV}} \sum_K e^{i(\omega_n \tau + \mathbf{k}\cdot\mathbf{x})} \phi(K) \,, \tag{54}$$

with the Minkowski scalar product $K \cdot X = k_0 x_0 - \mathbf{k}\cdot\mathbf{x}$. Note that in terms of ω_n, τ, the scalar product is Euclidean. The sum is over discrete values k_0, $\mathbf{k}$ (the summation over $\mathbf{k}$ will become an integral over continuous $\mathbf{k}$ when we take the thermodynamic limit below). The normalization is chosen such that the Fourier-transformed fields $\phi(K)$ are dimensionless. We know from the previous section that the field has to be periodic, $\phi(0, \mathbf{x}) = \phi(\beta, \mathbf{x})$. To fulfill this periodicity requirement we need $e^{i\omega_n \beta} = 1$, i.e., $\omega_n \beta$ has to be an integer multiple of 2π, or

$$\omega_n = 2\pi n T \,, \qquad n \in \mathbb{Z} \,. \tag{55}$$

[End of 3rd lecture, Oct 21st, 2013.]

With the Fourier transform (54), and the relation

$$\int_X e^{iK\cdot X} = \frac{V}{T} \delta_{K,0} \,, \tag{56}$$

we have

$$\begin{aligned} \int_X \mathcal{L} &= -\frac{1}{2} \int_X \left[(\partial_\tau \phi)^2 + (\nabla\phi)^2 + m^2 \phi^2 \right] \\ &= -\frac{1}{2} \sum_K \phi(-K) \frac{D_0^{-1}(K)}{T^2} \phi(K) \,, \end{aligned} \tag{57}$$

with the free (hence the subscript "0") inverse propagator in momentum space

$$D_0^{-1}(K) = \omega_n^2 + k^2 + m^2 = -K^2 + m^2 \,. \tag{58}$$

Explicitly, we have for example for the first term,

$$\begin{aligned} \int_X (\partial_\tau \phi)^2 &= \frac{1}{TV} \int_X \sum_{K,Q} [\partial_\tau e^{i(\omega_n \tau + \mathbf{k}\cdot\mathbf{x})} \phi(K)][\partial_\tau e^{i(\omega_m \tau + \mathbf{q}\cdot\mathbf{x})} \phi(Q)] \\ &= -\frac{1}{TV} \int_X \sum_{K,Q} \omega_n \omega_m e^{-i(K+Q)\cdot X} \phi(K)\phi(Q) \\ &= \frac{1}{T^2} \sum_K \omega_n^2 \phi(-K)\phi(K) \,. \end{aligned} \tag{59}$$

Since $\phi(X)$ is real we have $\phi(K) = \phi^*(-K)$ and thus

$$Z = N \int \mathcal{D}\phi \exp\left[-\frac{1}{2} \sum_K \phi^*(K) \frac{D_0^{-1}(K)}{T^2} \phi(K) \right] . \tag{60}$$

We can evaluate this integral by using the general formula

$$\int d^D x \, e^{-\frac{1}{2}\mathbf{x}\cdot\hat{A}\mathbf{x}} = (2\pi)^{D/2} (\det \hat{A})^{-1/2} \,, \tag{61}$$

for a hermitian, positive definite matrix $\hat{A}$. This identity is a generalization of the one-dimensional gaussian integral

$$\int_{-\infty}^{\infty} dx \, e^{-\frac{1}{2}\alpha x^2} = \sqrt{\frac{2\pi}{\alpha}} \,, \tag{62}$$

and can easily be shown by writing the bilinear $\mathbf{x}\cdot\hat{A}\mathbf{x}$ in terms of the eigenvalues of $\hat{A}$ and then using Eq. (62). Consequently,

$$Z = N'\left(\det\frac{D_0^{-1}(K)}{T^2}\right)^{-1/2}, \tag{63}$$

where we have absorbed the constant factor into the new constant N', and where the determinant is taken over momentum space (in which the inverse propagator is diagonal). Hence the log of the partition function is, up to a constant,

$$\begin{aligned}\ln Z &= -\frac{1}{2}\ln\det\frac{D_0^{-1}(K)}{T^2}\left(=-\frac{1}{2}\mathrm{Tr}\ln\frac{D_0^{-1}(K)}{T^2}\right)\\ &= -\frac{1}{2}\ln\prod_K\frac{D_0^{-1}(K)}{T^2}\\ &= -\frac{1}{2}\sum_K\ln\frac{D_0^{-1}(K)}{T^2}\,.\end{aligned} \tag{64}$$

Next we perform the summation over Matsubara frequencies (recall that the sum over K is a sum over $k_0=-i\omega_n$ and over $\mathbf{k}$; the latter will become an integral in the thermodynamic limit).

A. Summation over bosonic Matsubara frequencies

Here we prove the identity

$$\sum_n\ln\frac{\omega_n^2+\epsilon_k^2}{T^2} = \frac{\epsilon_k}{T} + 2\ln\left(1-e^{-\epsilon_k/T}\right) + \text{const}\,, \tag{65}$$

where, in our case, $\epsilon_k^2 = k^2+m^2$ (however, for the following calculation we only need that ϵ_k is a real number), and where const is an (infinite) number independent of temperature and momentum. First, in order to get rid of the log, we write

$$\sum_n\ln\frac{\omega_n^2+\epsilon_k^2}{T^2} = \int_1^{(\epsilon_k/T)^2}dx^2\sum_n\frac{1}{(2n\pi)^2+x^2} + \sum_n\ln[1+(2n\pi)^2]\,. \tag{66}$$

We now perform the sum in the integrand which, denoting $\epsilon_k\equiv Tx$, we write as a contour integral,

$$\frac{1}{T}\sum_n\frac{1}{(2n\pi)^2+x^2} = T\sum_n\frac{1}{\omega_n^2+\epsilon_k^2} = -\frac{1}{2\pi i}\oint_C d\omega\,\frac{1}{\omega^2-\epsilon_k^2}\frac{1}{2}\coth\frac{\omega}{2T}\,. \tag{67}$$

The second identity follows from the residue theorem,

$$\frac{1}{2\pi i}\oint_C dz\,f(z) = \sum_n\mathrm{Res}\,f(z)|_{z=z_n}\,, \tag{68}$$

where z_n are the poles of $f(z)$ in the area enclosed by the contour C. If we can write the function f as $f(z)=\varphi(z)/\psi(z)$, with analytic functions $\varphi(z)$, $\psi(z)$, the residues are

$$\mathrm{Res}\,f(z)|_{z=z_n} = \frac{\varphi(z_n)}{\psi'(z_n)}\,. \tag{69}$$

The contour C in Eq. (67) encloses all poles of $\coth[\omega/(2T)]$ (and none of $1/(\omega^2-\epsilon_k^2)$), as shown in Fig. 2. The denominator of $\coth[\omega/(2T)]$ is $e^{\omega/2T}-e^{-\omega/2T}$ which vanishes when $\omega/2T$ is an integer multiple of $i\pi$, i.e., when $\omega=i\omega_n$ with the Matsubara frequencies ω_n. Hence, in the above notation,

$$\begin{aligned}\varphi(\omega) &= \frac{1}{2}\frac{e^{\omega/(2T)}+e^{-\omega/(2T)}}{\omega^2-\epsilon_k^2}\,, \qquad \psi(\omega)=e^{\omega/(2T)}-e^{-\omega/(2T)}\,,\\ \Rightarrow\ &\frac{\varphi(i\omega_n)}{\psi'(i\omega_n)} = -T\frac{1}{\omega_n^2+\epsilon_k^2}\,,\end{aligned} \tag{70}$$

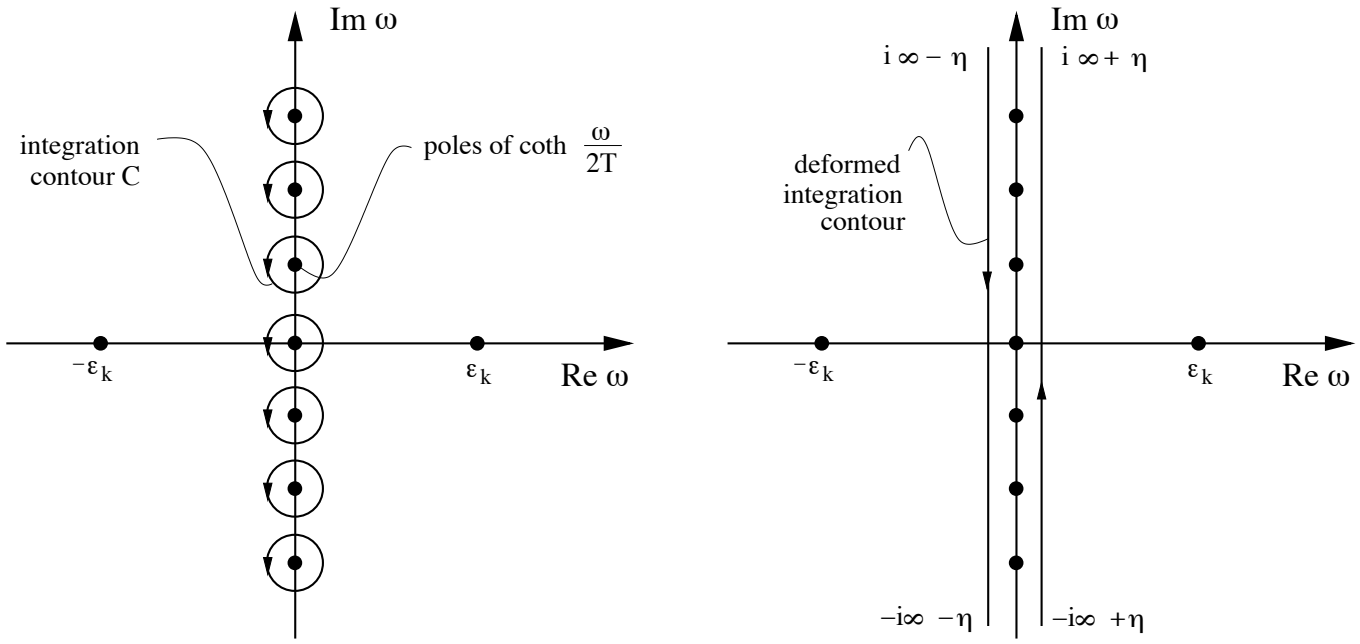

FIG. 2: Left: integration contour in the complex ω plane used in Eq. (67). Right: deformed integration contour from Eq. (71).

from which Eq. (67) follows immediately. Next, we deform the contour (which consists of infinitely many circles surrounding the poles) and obtain

$$\begin{aligned} T\sum_n \frac{1}{\omega_n^2+\epsilon_k^2} &= -\frac{1}{2\pi i}\int_{-i\infty+\eta}^{i\infty+\eta} d\omega\, \frac{1}{\omega^2-\epsilon_k^2}\frac{1}{2}\coth\frac{\omega}{2T} - \frac{1}{2\pi i}\int_{i\infty-\eta}^{-i\infty-\eta} d\omega\, \frac{1}{\omega^2-\epsilon_k^2}\frac{1}{2}\coth\frac{\omega}{2T} \\ &= -\frac{1}{2\pi i}\int_{-i\infty+\eta}^{i\infty+\eta} d\omega\, \frac{1}{\omega^2-\epsilon_k^2}\coth\frac{\omega}{2T}\,, \end{aligned} \tag{71}$$

where we have changed the integration variable $\omega \to -\omega$ in the second integral of the first line. We now use the residue theorem a second time: we can close the contour in the positive half plane and pick up the poles $\omega = \pm\epsilon_k$. (in our simple case $\epsilon_k > 0$, but we can keep the result general in order to use it later for the case of a nonvanishing chemical potential),

$$\begin{aligned} T\sum_n \frac{1}{\omega_n^2+\epsilon_k^2} &= \Theta(\epsilon_k)\frac{1}{2\epsilon_k}\coth\frac{\epsilon_k}{2T} - \Theta(-\epsilon_k)\frac{1}{2\epsilon_k}\coth\frac{-\epsilon_k}{2T} \\ &= \frac{1}{2\epsilon_k}\coth\frac{\epsilon_k}{2T} \\ &= \frac{1}{2\epsilon_k}[1+2f_B(\epsilon_k)]\,, \end{aligned} \tag{72}$$

(note minus sign from clockwise contour integration) with the Bose distribution function

$$f_B(\epsilon) \equiv \frac{1}{e^{\epsilon/T}-1}\,. \tag{73}$$

We thus have found

$$\frac{1}{T}\sum_n \frac{1}{(2n\pi)^2+x^2} = \frac{1}{Tx}\left(\frac{1}{2}+\frac{1}{e^x-1}\right)\,. \tag{74}$$

Now we insert the result into the original expression (66) and integrate over x^2 to obtain (with const denoting T-independent constants)

$$\sum_n \ln \frac{\omega_n^2+\epsilon_k^2}{T^2} = \int_1^{(\epsilon_k/T)^2} dx^2 \frac{1}{x}\left(\frac{1}{2}+\frac{1}{e^x-1}\right)+\text{const}$$
$$= \frac{\epsilon_k}{T}+2\ln\left(1-e^{-\epsilon_k/T}\right)+\text{const}\,, \tag{75}$$

which is the result we wanted to prove.

Exercise 1: *Show via contour integration that*

$$T\sum_{k_0}\frac{1}{[(p_0-k_0)^2-\omega_q^2](k_0^2-\omega_k^2)} = -\sum_{e_1,e_2=\pm}\frac{e_1e_2}{4\omega_k\omega_q}\frac{1}{p_0-e_1\omega_k-e_2\omega_q}[1+f_B(e_1\omega_k)+f_B(e_2\omega_q)]\,, \tag{76}$$

with $k_0=-i\omega_n$, $p_0=-i\omega_m$ bosonic Matsubara frequencies and $\omega_k,\omega_q>0$.

B. Pressure of a scalar field

Inserting the result from the Matsubara sum into Eq. (64) and taking the thermodynamic limit yields the (log of) the bosonic partition function,

$$\ln Z = -V\int\frac{d^3k}{(2\pi)^3}\left[\frac{\epsilon_k}{2T}+\ln\left(1-e^{-\epsilon_k/T}\right)\right]. \tag{77}$$

Consequently, the thermodynamic potential (density) is

$$\frac{\Omega}{V} = -\frac{T}{V}\ln Z = \int\frac{d^3k}{(2\pi)^3}\left[\frac{\epsilon_k}{2}+T\ln\left(1-e^{-\epsilon_k/T}\right)\right]. \tag{78}$$

The first term on the right-hand side is infinite. We have to renormalize the potential by subtracting the zero-temperature result,

$$\frac{\Omega_{\rm ren}}{V}\equiv\frac{\Omega-\Omega_{T=0}}{V} = T\int\frac{d^3k}{(2\pi)^3}\ln\left(1-e^{-\epsilon_k/T}\right), \tag{79}$$

where we have used $\lim_{T\to0}T\ln\left(1-e^{-\epsilon_k/T}\right)=-\epsilon_k\Theta(-\epsilon_k)=0$. We have thus recovered the result from Eq. (32). We can compute the potential analytically for $T\gg m$, in which case we can approximate $\epsilon_k/T\simeq k/T$,

$$\frac{\Omega_{\rm ren}}{V}\simeq\frac{T}{2\pi^2}\int_0^\infty dk\,k^2\ln\left(1-e^{-k/T}\right) = \frac{T^4}{2\pi^2}\underbrace{\int_0^\infty dx\,x^2\ln\left(1-e^{-x}\right)}_{-\frac{\pi^4}{45}} = -\frac{\pi^2T^4}{90}. \tag{80}$$

This result gives the pressure of a noninteracting scalar field for large temperatures $T\gg m$ (for all temperatures if the field is massless, $m=0$),

$$P=-\frac{\Omega}{V}=\frac{\pi^2T^4}{90}. \tag{81}$$

[End of 4th lecture, Oct 28th, 2013.]

V. COMPLEX NON-INTERACTING SCALAR FIELD

Next we discuss a complex bosonic field. Although we still neglect interactions, this will already lead to new physics compared to the real field, namely Bose-Einstein condensation. We start from the Lagrangian

$$\mathcal{L}=\partial_\mu\phi^*\partial^\mu\phi-m^2|\phi|^2-\lambda|\phi|^4. \tag{82}$$

We set the coupling to zero, $\lambda=0$.

A. Conserved charge and chemical potential

We see that $\mathcal{L}$ is invariant under $U(1)$ rotations of the field,

$$\phi \to e^{-i\alpha}\phi\,. \tag{83}$$

Since this rotation leaves $\mathcal{L}$ only invariant if α is constant, the symmetry is called global (= the same rotation is applied at every point of space-time). We know from Noether's theorem that a system with a continuous symmetry has a conserved current. This is in contrast to the previous case of a real scalar field, where there was only a discrete $\mathbb{Z}_2$ symmetry $\phi \to -\phi$. The conserved current will allow us to introduce a chemical potential associated with the corresponding charge.

To identify the conserved current we formally extend the symmetry to a local symmetry $\alpha(x)$ and transform the Lagrangian,

$$\mathcal{L} \to \mathcal{L} + |\phi|^2 \partial_\mu\alpha\partial^\mu\alpha + i\partial_\mu\alpha(\phi^*\partial^\mu\phi - \phi\,\partial^\mu\phi^*)\,. \tag{84}$$

Now we write down the equation of motion for α. We see that the transformed Lagrangian does not depend on α, but only on its derivative. Consequently, the quantity

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\alpha)} = 2|\phi|^2\partial^\mu\alpha + i(\phi^*\partial^\mu\phi - \phi\,\partial^\mu\phi^*) \tag{85}$$

is conserved. If we now go back to constant α we see that we have the conserved current

$$j^\mu \equiv i(\phi^*\partial^\mu\phi - \phi\,\partial^\mu\phi^*)\,, \qquad \partial_\mu j^\mu = 0\,. \tag{86}$$

The conserved charge (density) is thus

$$j^0 = i(\phi^*\partial^0\phi - \phi\,\partial^0\phi^*)\,. \tag{87}$$

This is needed to introduce a chemical potential μ. In the following we want to see how the chemical potential enters the Lagrangian. One might think that we simply have to add a term μj^0 to $\mathcal{L}$, because $j^0 = \mathcal{N}$ is the charge density and the Lagrangian has the form $\mathcal{H} - \mu\mathcal{N}$. However, we need to be more careful. We know that the partition function is (in a straightforward generalization from the real scalar field)

$$Z \;=\; \int \mathcal{D}\pi\mathcal{D}\pi^* \int_{\text{periodic}} \mathcal{D}\phi\mathcal{D}\phi^* \,\exp\left[\int_0^\beta d\tau \int d^3x\,(\pi^*\partial_0\phi + \pi\partial_0\phi^* - \mathcal{H} + \mu\mathcal{N})\right], \tag{88}$$

It is convenient to introduce real and imaginary parts of ϕ and the conjugate momentum π,

$$\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)\,, \qquad \pi = \frac{1}{\sqrt{2}}(\pi_1 + i\pi_2)\,. \tag{89}$$

Then, the Lagrangian becomes

$$\mathcal{L} = \frac{1}{2}\left[\partial_\mu\phi_1\partial^\mu\phi_1 + \partial_\mu\phi_2\partial^\mu\phi_2 - m^2(\phi_1^2 + \phi_2^2)\right], \tag{90}$$

and the conjugate momenta are

$$\pi_i = \frac{\partial\mathcal{L}}{\partial(\partial_0\phi_i)} = \partial^0\phi_i\,. \tag{91}$$

Now, with Eqs. (87), (89), and (91) we find $j^0 = \phi_2\pi_1 - \phi_1\pi_2$. This yields the Hamiltonian

$$\begin{aligned}\mathcal{H} - \mu\mathcal{N} \;&=\; \pi_1\partial_0\phi_1 + \pi_2\partial_0\phi_2 - \mathcal{L} - \mu\mathcal{N} \\ &=\; \frac{1}{2}\left[\pi_1^2 + \pi_2^2 + (\nabla\phi_1)^2 + (\nabla\phi_2)^2 + m^2(\phi_1^2 + \phi_2^2)\right] - \mu(\phi_2\pi_1 - \phi_1\pi_2)\,.\end{aligned} \tag{92}$$

For the partition function (88), using $\pi^*\partial_0\phi + \pi\partial_0\phi^* = \pi_1\partial_0\phi_1 + \pi_2\partial_0\phi_2$, we need

$$\begin{aligned}\pi_1\partial_0\phi_1 + \pi_2\partial_0\phi_2 - \mathcal{H} + \mu\mathcal{N} &= \pi_1\partial_0\phi_1 + \pi_2\partial_0\phi_2 - \frac{1}{2}\left[\pi_1^2 + \pi_2^2 + (\nabla\phi_1)^2 + (\nabla\phi_2)^2 + m^2(\phi_1^2+\phi_2^2)\right] + \mu(\phi_2\pi_1 - \phi_1\pi_2) \\ &= \frac{1}{2}\left[(\partial_0\phi_1)^2 + (\partial_0\phi_2)^2 - (\nabla\phi_1)^2 - (\nabla\phi_2)^2 + (\mu^2-m^2)(\phi_1^2+\phi_2^2) + 2\mu(\phi_2\partial_0\phi_1 - \phi_1\partial_0\phi_2)\right] \\ &\quad -\frac{1}{2}\left[(\pi_1 - \partial_0\phi_1 - \mu\phi_2)^2 + (\pi_2 - \partial_0\phi_2 + \mu\phi_1)^2\right] \\ &= \mathcal{L}' - \frac{1}{2}(\tilde{\pi}_1^2 + \tilde{\pi}_2^2)\,, \end{aligned} \tag{93}$$

with the shifted momenta $\tilde{\pi}_1 \equiv \pi_1 - \partial_0\phi_1 - \mu\phi_2$, $\tilde{\pi}_2 \equiv \pi_2 - \partial_0\phi_2 + \mu\phi_1$, and the new Lagrangian that now includes the chemical potential,

$$\mathcal{L}' = \frac{1}{2}\left[(\partial_0\phi_1)^2 + (\partial_0\phi_2)^2 - (\nabla\phi_1)^2 - (\nabla\phi_2)^2 + (\mu^2-m^2)(\phi_1^2+\phi_2^2) + 2\mu(\phi_2\partial_0\phi_1 - \phi_1\partial_0\phi_2)\right]\,. \tag{94}$$

In terms of the complex field ϕ, the Lagrangian reads

$$\mathcal{L}' = |(\partial_0 - i\mu)\phi|^2 - |\nabla\phi|^2 - m^2|\phi|^2\,. \tag{95}$$

Thus we see that the effect of the chemical potential is to add, besides the expected term μj^0, the additional term $\mu^2(\phi_1^2+\phi_2^2)/2$. As a result, the chemical potential enters the Lagrangian in the same way as the temporal component of a gauge field.

In order to compute the partition function, we Fourier transform the fields ϕ_1, ϕ_2 as discussed for the scalar field. However, anticipating Bose-Einstein condensation, we separate the zero-momentum mode $\zeta_i = \phi_i(K=0)$,

$$\phi_i(X) = \zeta_i + \frac{1}{\sqrt{TV}}\sum_{K\neq 0} e^{-iK\cdot X}\phi_i(K)\,. \tag{96}$$

The condensate ζ_i plays the role of a vacuum expectation value of the field. It breaks the $U(1)$ symmetry spontaneously. We can choose any of the degenerate directions in the complex plane, for instance $\zeta_2 = 0$ and will denote $\zeta \equiv \zeta_1$. Moreover, we assume ζ to be constant in space-time. With the Lagrangian (94) the action then becomes

$$\int_X \mathcal{L}' = \frac{V}{T}\frac{\mu^2-m^2}{2}\zeta^2 - \frac{1}{2}\sum_K (\phi_1(-K), \phi_2(-K))\frac{D_0^{-1}(K)}{T^2}\begin{pmatrix}\phi_1(K)\\ \phi_2(K)\end{pmatrix}\,, \tag{97}$$

with the 2×2 inverse propagator

$$D_0^{-1}(K) = \begin{pmatrix} -K^2 + m^2 - \mu^2 & -2i\mu k_0 \\ 2i\mu k_0 & -K^2 + m^2 - \mu^2 \end{pmatrix}\,. \tag{98}$$

In deriving the action (97) we have used that the integrals over mixed terms, i.e., over a product of the condensate ζ and the momentum sum (excluding the mode $K=0$), vanish. We see that the chemical potential induces off-diagonal terms in the propagator.

Now from the partition function

$$Z = N\int \mathcal{D}\phi_1\mathcal{D}\phi_2 \exp\int_X \mathcal{L}\,, \tag{99}$$

we obtain, dropping the constant terms,

$$\begin{aligned}
\ln Z &= \frac{V}{T}\frac{\mu^2-m^2}{2}\zeta^2 - \frac{1}{2}\ln\left(\det\frac{D_0^{-1}(K)}{T^2}\right) \\
&= \frac{V}{T}\frac{\mu^2-m^2}{2}\zeta^2 - \frac{1}{2}\ln\prod_K \frac{1}{T^4}[(-K^2+m^2-\mu^2)^2 - 4\mu^2 k_0^2] \\
&= \frac{V}{T}\frac{\mu^2-m^2}{2}\zeta^2 - \frac{1}{2}\ln\prod_K \frac{1}{T^4}[(\epsilon_k-\mu)^2 - k_0^2][(\epsilon_k+\mu)^2 - k_0^2] \\
&= \frac{V}{T}\frac{\mu^2-m^2}{2}\zeta^2 - \frac{1}{2}\sum_K \left[\ln\frac{(\epsilon_k-\mu)^2-k_0^2}{T^2} + \ln\frac{(\epsilon_k+\mu)^2-k_0^2}{T^2}\right], \qquad (100)
\end{aligned}$$

where we defined

$$\epsilon_k \equiv \sqrt{k^2+m^2}\,. \qquad (101)$$

We can now use the result of the Matsubara summation from above, Eq. (65), to obtain

$$\ln Z = \frac{V}{T}\frac{\mu^2-m^2}{2}\zeta^2 - V\int\frac{d^3k}{(2\pi)^3}\left[\frac{\epsilon_k}{T} + \ln\left(1-e^{-(\epsilon_k-\mu)/T}\right) + \ln\left(1-e^{-(\epsilon_k+\mu)/T}\right)\right]. \qquad (102)$$

This gives the thermodynamic potential

$$\frac{\Omega}{V} = \frac{m^2-\mu^2}{2}\zeta^2 + T\int\frac{d^3k}{(2\pi)^3}\left[\frac{\epsilon_k}{T} + \ln\left(1-e^{-(\epsilon_k-\mu)/T}\right) + \ln\left(1-e^{-(\epsilon_k+\mu)/T}\right)\right]. \qquad (103)$$

We see that in order to avoid complex values of the potential we need to require

$$-m < \mu < m\,. \qquad (104)$$

This restriction for μ is a consequence of neglecting any interaction. Had we included an interaction term, the condensate would have had an effect on the dispersion relations ϵ_k. In our non-interacting system, they are not affected.

As discussed for the case of the scalar field, we need to renormalize the potential by subtracting the "vacuum contribution", in this case

$$P_{\rm vac} = -\frac{\Omega_{T=\mu=0}}{V} = -\int\frac{d^3k}{(2\pi)^3}\,\epsilon_k\,, \qquad (105)$$

where we used that $\zeta(\mu=0)=0$ (which we shall show below), and $\lim_{T\to 0} T\ln(1-e^{-E/T}) = -E\Theta(-E)$. Consequently,

$$\frac{\Omega_{\rm ren}}{V} = \frac{\Omega-\Omega_{T=\mu=0}}{V} = \frac{m^2-\mu^2}{2}\zeta^2 + T\int\frac{d^3k}{(2\pi)^3}\left[\ln\left(1-e^{-(\epsilon_k-\mu)/T}\right) + \ln\left(1-e^{-(\epsilon_k+\mu)/T}\right)\right]. \qquad (106)$$

In the following we shall drop the subscript "ren" again since for all physical purposes the renormalized potential is used and thus no confusion is possible. As for the scalar field, we may compute the pressure $P=-\Omega/V$ at sufficiently large temperatures $T \gg m, \mu$ (where $\zeta = 0$),

$$P \simeq -2T\int\frac{d^3k}{(2\pi)^3}\ln\left(1-e^{-k/T}\right) = -2\frac{T^4}{2\pi^2}\int_0^\infty dx\,x^2\ln\left(1-e^{-x}\right) = 2\,\frac{\pi^2T^4}{90}\,. \qquad (107)$$

The additional factor 2 compared to Eq. (81) is due to the two degrees of freedom of the complex field.

The charge density is

$$Q = -\frac{1}{V}\frac{\partial\Omega}{\partial\mu} = \mu\zeta^2 + \sum_{e=\pm} e\int\frac{d^3k}{(2\pi)^3}\frac{1}{e^{(\epsilon_k-e\mu)/T}-1}\,. \qquad (108)$$

We may approximate the thermal part for small and large temperatures. We first introduce the new integration variable $x = k/T$ to obtain

$$Q = \frac{T^3}{2\pi^2}\int_0^\infty dx\, x^2 \left[\frac{1}{e^{\sqrt{x^2+(m/T)^2}-\mu/T}-1} - \frac{1}{e^{\sqrt{x^2+(m/T)^2}+\mu/T}-1}\right] . \tag{109}$$

Now we expand for small T, $\sqrt{x^2+(m/T)^2} \simeq m/T + Tx^2/(2m)$. Then, with the new integration variable $y = \sqrt{T/(2m)}\,x$, we have

$$\begin{aligned} Q &\simeq \frac{T^3}{2\pi^2}\int_0^\infty dx\, x^2 \left[\frac{e^{-(m-\mu)/T}}{e^{Tx^2/(2m)}} - \frac{e^{-(m+\mu)/T}}{e^{Tx^2/(2m)}}\right] \\ &= \left(\frac{2m}{T}\right)^{3/2}\frac{T^3}{2\pi^2}\left[e^{-(m-\mu)/T} - e^{-(m+\mu)/T}\right]\int_0^\infty dy\, y^2 e^{-y^2} \\ &= \frac{m^{3/2}T^{3/2}}{2\sqrt{2}\pi^{3/2}}\left[e^{-(m-\mu)/T} - e^{-(m+\mu)/T}\right] , \end{aligned} \tag{110}$$

where we have used that the remaining y-integral evaluates to $\sqrt{\pi}/4$. We see that the density is exponentially suppressed for small temperatures. This exponential suppression for massive particles is also typical for other quantities such as the specific heat.

For large temperatures we can use Eq. (109) and neglect the terms m/T and μ/T in the integrand to obtain

$$Q_+ = Q_- = \frac{T^3}{2\pi^2}\underbrace{\int_0^\infty dx\,\frac{x^2}{e^x-1}}_{2\zeta(3)} = \frac{\zeta(3)T^3}{\pi^2} , \tag{111}$$

where Q_+ and Q_- are the particle and antiparticle contributions, respectively. We see that they become identical for large T and thus the total charge $Q = Q_+ - Q_-$ vanishes. This is easy to understand: the difference in energies between particles and antiparticles is 2μ for all momenta. If T is sufficiently large, i.e., $T \gg \mu$, then this difference is not "resolved" and particle and antiparticle states become practically equally populated. For T of the order of μ or smaller, the chemical potential induces an asymmetry between particles and antiparticles, favoring particles for $\mu > 0$ and antiparticles for $\mu < 0$.

Exercise 2: *Compute the specific heat (at constant chemical potential) $c_V = T\partial S/\partial T$, where $S = -\partial\Omega/\partial T$ is the entropy, and find analytic approximations for the limits of small and large temperatures. Compare these approximations with the full result in a numerical plot.*

[End of 5th lecture, Nov 4th, 2013.]

B. Bose-Einstein condensation

Let us now discuss the condensate. The condensate ζ has to be determined from minimizing the potential,

$$0 = \frac{\partial\Omega}{\partial\zeta} = (m^2-\mu^2)\zeta . \tag{112}$$

We see that $\zeta = 0$ for $|\mu| < m$. In this case, there is no Bose condensation and all particles sit in the thermal states. For $|\mu| = m$, ζ remains undetermined. This is due to our neglecting the interactions. From usual ϕ^4 theory at zero temperature we know that the interactions may lead to a nonvanishing vacuum expectation value ("mexican hat potential"). But for now we have dropped the ϕ^4 term for simplicity. In this case, we can determine ζ by fixing the density. This may or may not correspond to the physical situation one is interested in.

For the charge density Q, there is a zero-temperature contribution $\mu\zeta^2$ coming from the bosons in the zero-momentum state. For a given density, the system populates as many thermal states as possible until there is no more "space". Note that the contribution of the thermal integral is bounded with its maximum at $\mu^2 = m^2$. This maximum value defines a critical density for a given temperature T. For densities larger than this critical density,

the condensate gets populated. The population is "macroscopic", i.e., proportional to the volume. The value of the condensate is given by

$$\zeta^2 = \frac{1}{m}\left(Q - \sum_{e=\pm} e\int \frac{d^3k}{(2\pi)^3}\frac{1}{e^{(\epsilon_k - em)/T}-1}\right). \tag{113}$$

The critical temperature T_c for a given charge density Q is then given by the implicit equation

$$Q = \sum_{e=\pm} e\int \frac{d^3k}{(2\pi)^3}\frac{1}{e^{(\epsilon_k - em)/T_c}-1}\,. \tag{114}$$

In the nonrelativistic limit $\sqrt{k^2+m^2}-\mu$ is replaced by $\frac{k^2}{2m}-\mu$ (note that this defines a "nonrelativistic μ" which includes the rest energy m), and condensation occurs for $\mu = 0$. In this case, T_c can be computed as

$$\begin{aligned} Q &= \int \frac{d^3k}{(2\pi)^3}\frac{1}{e^{k^2/(2mT_c)}-1} \\ &= \frac{1}{2\pi^2}\int_0^\infty dk\, k^2 \frac{1}{e^{k^2/(2mT_c)}-1} \\ &= \frac{(2mT_c)^{3/2}}{2\pi^2}\int_0^\infty dx\,\frac{x^2}{e^{x^2}-1} \\ &= \frac{(2mT_c)^{3/2}}{2\pi^2}\frac{\sqrt{\pi}}{4}\zeta(3/2)\,, \end{aligned} \tag{115}$$

which implies

$$T_c = \frac{2\pi}{m}\left(\frac{Q}{\zeta(3/2)}\right)^{2/3}. \tag{116}$$

In the ultrarelativistic limit it is instructive to compute particle and antiparticle contributions separately. With $\mu = m$ and $\epsilon_k \mp m \simeq k \mp m + \mathcal{O}(m^2)$ we have

$$Q_\pm \simeq \frac{T_c^3}{2\pi^2}\int_0^\infty dx\,\frac{x^2}{e^{x\mp m/T_c}-1}\,. \tag{117}$$

Up to first order in m/T_c we have

$$\frac{1}{e^{x\mp m/T_c}-1} = \frac{e^{\pm m/T_c}}{e^x - e^{\pm m/T_c}} \simeq \frac{1}{e^x-1}\left(1\pm\frac{m}{T_c}\right) \pm \frac{m/T_c}{(e^x-1)^2}\,. \tag{118}$$

Consequently,

$$\begin{aligned} Q_\pm &\simeq \frac{T_c^3}{2\pi^2}\int_0^\infty dx\frac{x^2}{e^x-1} \pm \frac{T_c^2 m}{2\pi^2}\Big[\int_0^\infty dx\frac{x^2}{e^x-1} + \underbrace{\int_0^\infty dx\frac{x^2}{(e^x-1)^2}}_{\frac{\pi^2}{3}-2\zeta(3)}\Big] \\ &= \frac{T_c^3}{2\pi^2}\left[2\zeta(3)\pm\frac{m\pi^2}{3T_c}\right]. \end{aligned} \tag{119}$$

We see that the antiparticles have an interesting effect: had we neglected antiparticles, i.e., $Q = Q_+$, the critical temperature would have been $T_c \propto Q_+^{1/3}$. In the full result $Q = Q_+ - Q_-$, however, the leading term cancels and we get the very different result

$$T_c = \sqrt{\frac{3Q}{m}}\,. \tag{120}$$

Bose-Einstein condensation is a phenomenon occurring in a huge variety of systems. It was first directly observed with bosonic atoms in 1995, awarded with the Nobel prize 2001. It often has spectacular phenomenological consequences, such as in superfluid He-4. It can also occur for excitons in semiconductors, and for mesons such as pions and kaons in neutron stars. One can even think of superconductivity in fermionic systems as a Bose-Einstein condensate, since Cooper pairs of fermions can be viewed as bosons. Very recent experiments have shown that this picture indeed is valid, i.e., there is a crossover from a superfluid (at weak coupling) to a Bose-Einstein condensate (at strong coupling), not a phase transition.

We will come back to Bose-Einstein condensation in Sec. IX, where we include interactions and discuss the Goldstone mode that appears due to the spontaneous breaking of a global symmetry.

Exercise 2a: *Repeat exercise 2, but now at a fixed density (as opposed to a fixed chemical potential). This allows you to include Bose-Einstein condensation. Plot the full result for c_V for all temperatures and show that c_V is continuous, but not differentiable*[2] *at the critical temperature T_c. (Hint: find the derivative $\partial\mu/\partial T$ with the help of the implicit function theorem.)*

VI. NON-INTERACTING FERMIONS

We shall now turn to fermions and compute their partition function. We shall see that there are two important differences to the bosonic case. Firstly, the fields over which we integrate in the functional integral are anticommuting, which yields a different result for the functional integration. Secondly, we shall have antiperiodicity instead of periodicity in the fields, which yields different Matsubara frequencies. Both differences are related to the Pauli principle.

A. Grassmann Algebra and antiperiodicity in β for fermion fields

We start by defining the so-called Grassmann Algebra: on an r-dimensional vector space with basis vectors $\eta_1, \ldots, \eta_r$ we define an anticommuting product

$$\eta_i \eta_j = -\eta_j \eta_i \,, \tag{121}$$

to obtain the Grassmann Algebra $\mathcal{A}$. The algebra has 2^r basis elements $1, \eta_i, \eta_i\eta_j, \ldots, \eta_1\eta_2\ldots\eta_r$. Note that Eq. (121) implies $\eta_i^2 = 0$. One needs a sign convention to define the derivatives on this space. For example, for $j \neq k$,

$$\frac{\partial}{\partial\eta_j}\eta_j\eta_k = \eta_k \,, \qquad \frac{\partial}{\partial\eta_k}\eta_j\eta_k = -\eta_j \,. \tag{122}$$

This is a convenient convention since one can think of the derivative operator as anticommuting with the variable itself. (In other words, we have defined the derivative to act from the left, not from the right.) Second derivatives of any product of η's vanish (they vanish if there is at most one factor of the variable with respect to which the derivative is taken; if there are two factors the product itself vanishes). This already shows that integration on the Grassmann space is a bit different than one is used to: since the differential operator squared vanishes, there is no operation inverse to differentiation. ("Usually", that would be integration.) We require the integral to be translationally invariant and linear. Restricting ourselves for the moment to a one-dimensional vector space (i.e., a two-dimensional Grassmann algrabra) this means

$$\int d\eta\, f(\eta) = \int d\eta\, f(\eta+\zeta) \,, \qquad \int d\eta\,(a\eta + b) = a\int d\eta\,\eta + b\int d\eta \,, \tag{123}$$

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[2] According to the traditional classification of Ehrenfest, a phase transition is called *n-th order phase transition* if the n-th derivative of the thermodynamical potential is discontinuous. Since the specific heat is given by the second derivative of Ω, the result of this exercise shows that Bose-Einstein condensation in a free Bose gas is a third-order phase transition. In a more modern terminology one distinguishes only between phase transitions where the order parameter is discontinuous at the critical point ("first-order phase transition") and where it is continuous (then somewhat confusingly called "second-order phase transition", including all higher-order transitions according to Ehrenfest). This terminology is more closely related to symmetries of the system: discontinuous transitions can occur even though no symmetry is spontaneously broken; continuous transitions (where the order parameter must be zero in one of the phases) imply spontaneous symmetry breaking; see Sec. IX for a detailed discussion of spontaneous symemtry breaking. Here, in the case of Bose-Einstein transformation, the order parameter is the condensate. It behaves continuously at the critical point.

with a, b complex numbers and $\zeta \in \mathcal{A}$. Here, the integration range is always the whole space, i.e., we can only talk about "definite" integrals; "indefinite" integrals, as one is used to from c-numbers do not exist (since this would be an operation inverse to differentiation). Because of $\eta^2 = 0$ the most general form of a function of η in our one-dimensional example is $f(\eta) = a\eta + b$. Then, because of linearity,

$$\int d\eta\, f(\eta + c) = \int d\eta\, (a\eta + b) + ac \int d\eta = \int d\eta\, f(\eta) + ac \int d\eta\,, \tag{124}$$

and translational invariance yields

$$\int d\eta = 0\,. \tag{125}$$

We also normalize

$$\int d\eta\, \eta = 1\,. \tag{126}$$

Equipped with these properties we can turn to fermions.

First we consider a simple system with two states $|0\rangle$ and $|1\rangle$, and creation and annihilation operators a and $a^\dagger$ which obey the anticommutation relations

$$\{a, a^\dagger\} = 1\,, \qquad a^2 = (a^\dagger)^2 = 0\,. \tag{127}$$

Moreover we consider the Grassmann Algebra generated by the two variables η and η^* (these shall correspond to the fermion fields later), and the states

$$|\eta\rangle \;\equiv\; e^{-\eta a^\dagger}|0\rangle = (1 - \eta a^\dagger)|0\rangle = |0\rangle - \eta|1\rangle\,, \tag{128a}$$

$$\langle\eta| \;\equiv\; \langle 0|e^{-a\eta^*} = \langle 0|(1 - a\eta^*) = \langle 0| - \langle 1|\eta^*\,. \tag{128b}$$

We also need

$$\langle\eta|0\rangle = \langle 0|\eta\rangle = 1\,, \qquad \langle 1|\eta\rangle = \langle\eta|1\rangle^* = -\eta\,, \tag{129}$$

which is obvious from Eqs. (128), and

$$\langle\eta|\eta\rangle = e^{\eta^*\eta}\,, \tag{130}$$

which follows from inserting $1 = |0\rangle\langle 0| + |1\rangle\langle 1|$ and using Eqs. (129). Also, with Eqs. (128) and the rules for integration (125) (generalized to two dimensions) we find

$$\begin{aligned}\int d\eta^* d\eta\, e^{-\eta^*\eta}|\eta\rangle\langle\eta| \;&=\; \int d\eta^* d\eta\, (1 - \eta^*\eta)\, (|0\rangle\langle 0| - \eta|1\rangle\langle 0| - |0\rangle\langle 1|\eta^* + |1\rangle\langle 1|\eta\eta^*) \\ &=\; |0\rangle\langle 0| + |1\rangle\langle 1| = 1\,.\end{aligned} \tag{131}$$

And, finally, upon inserting unity twice and using Eqs. (129)

$$\begin{aligned}\int d\eta^* d\eta\, e^{-\eta^*\eta}\langle -\eta|A|\eta\rangle \;&=\; \int d\eta^* d\eta\, (1 - \eta^*\eta)\, (\langle 0|A|0\rangle + \eta^*\langle 1|A|0\rangle - \eta\langle 0|A|1\rangle - \eta^*\eta\langle 1|A|1\rangle) \\ &=\; \langle 0|A|0\rangle + \langle 1|A|1\rangle = \mathrm{Tr}\, A\,.\end{aligned} \tag{132}$$

Eqs. (131) and (132) are the ingredients we need to compute the fermionic partition function in the path integral formalism in analogy to the bosonic case. First, from Eq. (132) we compute the partition function for the Hamiltonian $\hat{H} = \omega a^\dagger a$,

$$Z = \mathrm{Tr} e^{-\beta\hat{H}} = \int d\eta^* d\eta\, e^{-\eta^*\eta}\langle -\eta|e^{-\beta\hat{H}}|\eta\rangle\,. \tag{133}$$

The important difference to the bosonic case can already be seen here, namely the $-\eta$ as the final state of the transition amplitude. Compare this to Eq. (43) which is the bosonic analogue. We can now proceed analogously to the bosonic

case by dividing the "time" interval into N pieces of width Δt and inserting unity from Eq. (131) $N-1$ times. We obtain

$$Z = \int_{\eta^*(\beta)=-\eta^*(0)} \mathcal{D}\eta^* \int_{\eta(\beta)=-\eta(0)} \mathcal{D}\eta \exp\left(-\int_0^\beta d\tau \left[\eta^* \partial_\tau \eta + H(\eta^*, \eta)\right]\right) . \tag{134}$$

Before we generalize this to the case of Dirac fields let us discuss the fermionic Lagrangian.

[End of 6th lecture, Nov 11th, 2013.]

B. Fermionic Lagrangian and conserved charge

We start with the non-interacting Lagrangian

$$\mathcal{L} = \bar{\psi}\left(i\gamma^\mu \partial_\mu - m\right)\psi , \tag{135}$$

where $\bar{\psi} = \psi^\dagger \gamma^0$, and where the Dirac matrices are given in the Dirac representation by

$$\gamma^0 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} , \qquad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} , \tag{136}$$

with the Pauli matrices σ_i. The general properties of the Dirac matrices are

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} , \qquad (\gamma^0)^2 = 1 , \qquad (\gamma^i)^2 = -1 , \qquad (\gamma^0)^\dagger = \gamma^0 , \qquad (\gamma^i)^\dagger = -\gamma^i , \tag{137}$$

where $g^{\mu\nu}$ is the Minkowski metric.

As for the bosons we are interested in the theory with a chemical potential. To this end, we determine the conserved current with the same method as above. The Lagrangian is invariant under the transformation $\psi \to e^{-i\alpha}\psi$. Considering a local transformation $\alpha(x)$, we have

$$\mathcal{L} \to \mathcal{L} + \bar{\psi}\gamma^\mu(\partial_\mu \alpha)\psi . \tag{138}$$

From the equation of motion for α we then conclude that the current

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \alpha)} = \bar{\psi}\gamma^\mu \psi \tag{139}$$

is conserved, i.e.,

$$\partial_\mu j^\mu = 0 , \tag{140}$$

and the conserved charge (density) is given by

$$Q = \psi^\dagger \psi . \tag{141}$$

The conjugate momentum is

$$\pi = \frac{\partial \mathcal{L}}{\partial(\partial_0 \psi)} = i\psi^\dagger . \tag{142}$$

We see that we have to treat ψ and $\psi^\dagger$ as independent variables, in accordance to what we have discussed before in terms of η and η^*. Consequently, the Hamiltonian becomes

$$\mathcal{H} = \pi \partial_0 \psi - \mathcal{L} = \bar{\psi}(i\boldsymbol{\gamma}\cdot\nabla + m)\psi . \tag{143}$$

Here and in the following we mean by the scalar product $\boldsymbol{\gamma}\cdot\nabla$ the product where the Dirac matrices appear with a lower index γ_i, i.e., the negative of the γ^i given in Eq. (136).

C. Partition function for fermions

Now we recall that for the partition function we need $i\pi\partial_\tau\psi - \mathcal{H} + \mu\mathcal{N}$ (see for instance Eq. (43)). With the Hamiltonian (143) and the generalization of the fermionic partition function (134) to fields ψ, $\psi^\dagger$ we obtain

$$Z = \int_{\text{antiperiodic}} \mathcal{D}\psi^\dagger \mathcal{D}\psi \, \exp\left[\int_X \bar{\psi}\left(-\gamma^0\partial_\tau - i\boldsymbol{\gamma}\cdot\nabla + \gamma^0\mu - m\right)\psi\right] . \tag{144}$$

In this case we cannot separate the $\pi \sim \psi^\dagger$ integration from the ψ integration. Remember that, in the bosonic case, this led to a new Lagrangian which contains the chemical potential not just in the term $j^0\mu$. Here, the Lagrangian with chemical potential simply is

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu\partial_\mu + \gamma^0\mu - m)\psi \, . \tag{145}$$

Note that again the chemical potential enters just like the temporal component of a gauge field that couples to the fermions. Analogously to the bosonic case, we introduce the Fourier transform (note the different dimensionality of fields compared to bosons; here the field has mass dimension 3/2)

$$\psi(X) = \frac{1}{\sqrt{V}}\sum_K e^{-iK\cdot X}\psi(K) \, , \qquad \bar{\psi}(X) = \frac{1}{\sqrt{V}}\sum_K e^{iK\cdot X}\bar{\psi}(K) \, , \qquad \int_X e^{iK\cdot X} = \frac{V}{T}\delta_{K,0} \, , \tag{146}$$

again with $k_0 = -i\omega_n$ such that $K\cdot X = -(\omega_n\tau + \mathbf{k}\cdot\mathbf{x})$. Now antiperiodicity requires $\psi(0,\mathbf{x}) = -\psi(\beta,\mathbf{x})$, which implies $e^{i\omega_n\beta} = -1$ and thus the fermionic Matsubara frequencies are

$$\omega_n = (2n+1)\pi T \, . \tag{147}$$

With the Fourier decomposition we find

$$\int_X \bar{\psi}\left(i\gamma^\mu\partial_\mu + \gamma^0\mu - m\right)\psi = -\sum_K \bar{\psi}(K)\frac{G_0^{-1}(K)}{T}\psi(K) \, , \tag{148}$$

with the free inverse fermion propagator in momentum space[3]

$$G_0^{-1}(K) = -\gamma^\mu K_\mu - \gamma^0\mu + m \, . \tag{154}$$

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[3] The inverse propagator (149) can also be written in terms of energy projectors. This form will not be needed here but is very helpful for more difficult calculations. In particular it allows inversion in a simple way. We can write

$$G_0^{-1}(K) = -\sum_{e=\pm}(k_0 + \mu - e\epsilon_k)\gamma^0\Lambda^e_{\mathbf{k}} \, , \tag{149}$$

where $\epsilon_k \equiv \sqrt{k^2+m^2}$, and where the projectors onto positive and negative energy states are given by

$$\Lambda^e_{\mathbf{k}} \equiv \frac{1}{2}\left(1 + e\gamma^0\frac{\boldsymbol{\gamma}\cdot\mathbf{k} + m}{\epsilon_k}\right) . \tag{150}$$

These (hermitian) projectors are complete and orthogonal,

$$\Lambda^+_{\mathbf{k}} + \Lambda^-_{\mathbf{k}} = 1 \, , \qquad \Lambda^e_{\mathbf{k}}\Lambda^{e'}_{\mathbf{k}} = \delta_{e,e'}\Lambda^e_{\mathbf{k}} \, . \tag{151}$$

The first property is trivial to see, the second follows with the anticommutation property $\{\gamma^0,\gamma^i\} = 0$ which follows from the general anticommutation property in Eq. (137) and with $(\boldsymbol{\gamma}\cdot\mathbf{k})^2 = -k^2$. From the form of the inverse propagator (149) we can immediately read off the propagator itself,

$$G_0(K) = -\sum_{e=\pm}\frac{\Lambda^e_{\mathbf{k}}\gamma^0}{k_0 + \mu - e\epsilon_k} \, . \tag{152}$$

With the properties (151) one easily checks that $G_0^{-1}G_0 = 1$. One can also rewrite (152) as

$$G_0(K) = \frac{-\gamma^\mu K_\mu - \gamma^0\mu - m}{(k_0+\mu)^2 - \epsilon_k^2} \, . \tag{153}$$

For the functional integration we use

$$\int \prod_k^N d\eta_k^\dagger d\eta_k \exp\left(-\sum_{i,j}^N \eta_i^\dagger D_{ij}\eta_j\right) = \det D\,. \tag{155}$$

Exercise 3*: Prove this relation by using the above properties of the Grassmann variables.*

Note the difference of this Grassmann integration for fermions with the corresponding formula for bosons (61). We obtain for the partition function

$$\begin{aligned} Z &= \int_{\text{antiperiodic}} \mathcal{D}\psi^\dagger \mathcal{D}\psi \, \exp\left[-\sum_K \psi^\dagger(K)\gamma^0 \frac{G_0^{-1}(K)}{T}\psi(K)\right] \\ &= \det \frac{G_0^{-1}(K)}{T} \\ &= \det \frac{1}{T}\begin{pmatrix} -(k_0+\mu)+m & -\boldsymbol{\sigma}\cdot\mathbf{k} \\ \boldsymbol{\sigma}\cdot\mathbf{k} & (k_0+\mu)+m \end{pmatrix}, \end{aligned} \tag{156}$$

where we have used $\det\gamma^0 = 1$, and where the determinant is taken over Dirac space and momentum space.

We can use the general formula

$$\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(AD - BD^{-1}CD)\,, \tag{157}$$

for matrices A, B, C, D with D invertible, to get

$$\det \frac{G_0^{-1}(K)}{T} = \prod_K \left(\frac{k^2+m^2-(k_0+\mu)^2}{T^2}\right)^2 . \tag{158}$$

Here we have used $(\boldsymbol{\sigma}\cdot\mathbf{k})^2 = k^2$. Consequently,

$$\ln Z = \sum_K \ln\left(\frac{\epsilon_k^2-(k_0+\mu)^2}{T^2}\right)^2, \qquad \epsilon_k \equiv \sqrt{k^2+m^2}\,. \tag{159}$$

With $k_0 = -i\omega_n$ we can write this as

$$\begin{aligned} \ln Z &= \sum_K \ln\left(\frac{\epsilon_k^2+(\omega_n+i\mu)^2}{T^2}\right)^2 \\ &= \sum_K \left(\ln\frac{\epsilon_k^2+(\omega_n+i\mu)^2}{T^2} + \ln\frac{\epsilon_k^2+(-\omega_n+i\mu)^2}{T^2}\right) \\ &= \sum_K \left(\ln\frac{\omega_n^2+(\epsilon_k-\mu)^2}{T^2} + \ln\frac{\omega_n^2+(\epsilon_k+\mu)^2}{T^2}\right), \end{aligned} \tag{160}$$

where, in the second step, we have replaced ω_n by $-\omega_n$ which does not change the result since we sum over all $n \in \mathbb{Z}$. Then, the third step can be easily checked by multiplying out all terms,

$$[\epsilon_k^2+(\omega_n+i\mu)^2][\epsilon_k^2+(-\omega_n+i\mu)^2] = [\omega_n^2+(\epsilon_k-\mu)^2][\omega_n^2+(\epsilon_k+\mu)^2]\,. \tag{161}$$

D. Summation over fermionic Matsubara frequencies

We have written the log of the fermionic partition function in a form which is identical to the bosonic one, compare Eq. (160) with Eq. (100). The only difference is the form of the Matsubara frequencies. We can thus compute the sum over fermionic Matsubara frequencies analogous to the sum over bosonic ones, explained in Sec. IV A. As above, we write

$$\sum_n \ln \frac{\omega_n^2+\epsilon_k^2}{T^2} = \int_1^{(\epsilon_k/T)^2} dx^2 \sum_n \frac{1}{(2n+1)^2\pi^2+x^2} + \sum_n \ln[1+(2n+1)^2\pi^2]\,. \tag{162}$$

And as above, we write the sum as a contour integral, this time with the tanh instead of the coth,

$$\frac{1}{T}\sum_n \frac{1}{(2n+1)^2\pi^2+x^2} = T\sum_n \frac{1}{\omega_n^2+\epsilon_k^2} = -\frac{1}{2\pi i}\oint_C d\omega\, \frac{1}{\omega^2-\epsilon_k^2}\frac{1}{2}\tanh\frac{\omega}{2T}\,. \tag{163}$$

(We have denoted $\epsilon_k \equiv xT$.) The contour C encloses all poles of the tanh (and none of $\frac{1}{\omega^2-\epsilon_k^2}$). The poles of the tanh are given by the zeros of $e^{\omega/(2T)}+e^{-\omega/(2T)}$, i.e., $\omega/(2T)$ must be an odd integer multiple of $i\pi/2$. Therefore, the poles are located at i times the fermionic Matsubara frequencies, $\omega = i\omega_n$. Then, with the residue theorem and with

$$\left(e^{\omega/(2T)}-e^{-\omega/(2T)}\right)\Big|_{\omega=i\omega_n} = 2i(-1)^n\,, \qquad \frac{d}{d\omega}\left(e^{\omega/(2T)}+e^{-\omega/(2T)}\right)\Big|_{\omega=i\omega_n} = \frac{i(-1)^n}{T}\,, \tag{164}$$

one sees Eq. (163). We can then proceed as for bosons, i.e., we close the contour in the positive half-plane to obtain with the residue theorem

$$\begin{aligned} T\sum_n \frac{1}{\omega_n^2+\epsilon_k^2} &= -\frac{1}{2\pi i}\int_{-i\infty+\eta}^{i\infty+\eta} d\omega\, \frac{1}{\omega^2-\epsilon_k^2}\tanh\frac{\omega}{2T} \\ &= \frac{1}{2\epsilon_k}\tanh\frac{\epsilon_k}{2T} \\ &= \frac{1}{2\epsilon_k}[1-2f_F(\epsilon_k)]\,, \end{aligned} \tag{165}$$

where

$$f_F(\epsilon) \equiv \frac{1}{e^{\epsilon/T}+1} \tag{166}$$

is the Fermi distribution function. Inserting this result into the original expression (162) yields

$$\begin{aligned} \sum_n \ln\frac{\omega_n^2+\epsilon_k^2}{T^2} &= \int_1^{(\epsilon_k/T)^2} dx^2 \frac{1}{x}\left(\frac{1}{2}-\frac{1}{e^x+1}\right) + \text{const} \\ &= \frac{\epsilon_k}{T} + 2\ln\left(1+e^{-\epsilon_k/T}\right) + \text{const}\,. \end{aligned} \tag{167}$$

Exercise 4*: Prove via contour integration the following result for the summation over fermionic Matsubara frequencies,*

$$T\sum_{k_0} \frac{(k_0+\xi_1)(k_0+q_0+\xi_2)}{(k_0^2-\epsilon_1^2)[(k_0+q_0)^2-\epsilon_2^2]} = -\frac{1}{4\epsilon_1\epsilon_2}\sum_{e_1,e_2=\pm} \frac{(\epsilon_1-e_1\xi_1)(\epsilon_2-e_2\xi_2)}{q_0-e_1\epsilon_1+e_2\epsilon_2}\frac{f_F(-e_1\epsilon_1)f_F(e_2\epsilon_2)}{f_B(-e_1\epsilon_1+e_2\epsilon_2)}\,, \tag{168}$$

where $k_0 = -i\omega_n$ with fermionic Matsubara frequencies ω_n, and $q_0 = -i\omega_m$ with bosonic (!) Matsubara frequencies ω_m, and where $\xi_1, \xi_2, \epsilon_1, \epsilon_2 > 0$ are real numbers.

E. Thermodynamic potential for fermions

The result for the Matsubara sum (167) can now be inserted into the partition function (160) to obtain

$$\ln Z = 2V \int \frac{d^3k}{(2\pi)^3} \left[\frac{\epsilon_k}{T} + \ln\left(1 + e^{-(\epsilon_k - \mu)/T}\right) + \ln\left(1 + e^{-(\epsilon_k + \mu)/T}\right)\right] . \tag{169}$$

Consequently, the thermodynamic potential $\Omega = -T \ln Z$ becomes

$$\frac{\Omega}{V} = -2 \int \frac{d^3k}{(2\pi)^3} \left[\epsilon_k + T \ln\left(1 + e^{-(\epsilon_k - \mu)/T}\right) + T \ln\left(1 + e^{-(\epsilon_k + \mu)/T}\right)\right] . \tag{170}$$

Note the overall factor 2 which accounts for the two spin states of the spin-1/2 fermion. Together with the particle/antiparticle degree of freedom (from $e = \pm 1$) we thus see all four degrees of freedom of the Dirac spinor.

[End of 7th lecture, Nov 18th, 2013.]

VII. GAUGE FIELDS

A. Lagrangians for QCD and QED

In this section we shall compute the partition function for gauge fields. Many applications of thermal field theory in modern research can be found in Quantum Chromodynamics (QCD), for instance heavy-ion collisions and neutron star (quark star) physics. We shall, for the calculation of the partition function, focus on the simpler case of Quantum Electrodynamics (QED). But first we write down the QCD Lagrangian from which we obtain the QED Lagrangian as a limit. We have

$$\mathcal{L}_{\rm QCD} = -\frac{1}{2}\mathrm{Tr}[G_{\mu\nu}G^{\mu\nu}] + \bar{\psi}(i\gamma^\mu D_\mu + \gamma^0\mu - m)\psi \, . \tag{171}$$

Let us explain the meaning of the various quantities and their structure. The field strengths are

$$G_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu] \, , \tag{172}$$

where g is the QCD coupling constant, and where A_μ are matrices in the Lie Algebra of the gauge group $SU(N_c)$ where $N_c = 3$ is the number of colors. Here, $SU(N_c)$ is the group of unitary $N_c \times N_c$ matrices with determinant 1. The dimension of $SU(N_c)$ is $N_c^2 - 1$, thus in this case there are eight generators T_a which fulfil

$$[T_a, T_b] = i f_{abc} T_c \, , \qquad T_a^\dagger = T_a \, , \qquad \mathrm{Tr}[T_a T_b] = \frac{\delta_{ab}}{2} \, , \tag{173}$$

with the so-called structure constants f_{abc}. The generators (more precisely, twice the generators $\lambda_a = 2T_a$) are called Gell-Mann matrices. The gauge fields, which are called gluons, and field strengths can thus be written as

$$A_\mu = A_\mu^a T_a \, , \qquad G_{\mu\nu} = G_{\mu\nu}^a T_a \, , \qquad G_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c \, . \tag{174}$$

The Dirac spinors ψ describe quarks and are spinors in a $4N_f N_c$-dimensional space with the number of flavors N_f; the covariant derivative is

$$D_\mu = \partial_\mu - igA_\mu \, . \tag{175}$$

With fundamental color indices $\alpha, \beta \leq 3$, the adjoint color index $a \leq 8$, and flavor indices $i, j \leq N_f$ we can thus write the Lagrangian as

$$\mathcal{L}_{\rm QCD} = -\frac{1}{4} G_{\mu\nu}^a G_a^{\mu\nu} + \bar{\psi}_i^\alpha \delta_{ij} [i\gamma^\mu (\delta^{\alpha\beta}\partial_\mu - igA_\mu^a T_a^{\alpha\beta}) + \delta^{\alpha\beta}(\gamma^0 \mu_i - m_i)]\psi_j^\beta \, . \tag{176}$$

Here m and μ are matrices in flavor space, with different masses and chemical potentials for different flavors.

The Lagrangian is invariant under gauge transformations $U = e^{ig\theta_a(X)T^a} \in SU(N_c)$. The fermion fields and the gauge fields transform as

$$\psi \to U\psi \, , \qquad A_\mu \to U A_\mu U^{-1} + \frac{i}{g} U \partial_\mu U^{-1} \, , \tag{177}$$

where $U = U(\mathbf{x}, t)$ may depend on space-time, i.e., the symmetry is local. We can easily check that the Lagrangian is invariant under gauge transformations: one uses $0 = \partial_\mu(UU^{-1}) = (\partial_\mu U)U^{-1} + U(\partial_\mu U^{-1})$ to find

$$G_{\mu\nu} \to U G_{\mu\nu} U^{-1} \,. \tag{178}$$

Therefore, $\mathrm{Tr}[G_{\mu\nu}G^{\mu\nu}]$ is obviously invariant under gauge transformations. For the quark part we find

$$D_\mu \psi \to U D_\mu \psi \,, \tag{179}$$

from which we conclude that $\bar{\psi} D_\mu \psi$ is invariant and thus we see that $\mathcal{L}_{\mathrm{QCD}}$ is invariant.

For simplicity, we shall consider QED in the following calculation. In this case the gauge group is $U(1)$ which is an abelian symmetry. For many physical applications and many calculations this makes the theory tremendously simpler than QCD. For the latter, controled rigorous calculations from first principles are only valid for very few systems such as systems at very large densities or temperatures. This is due to asymptotic freedom which makes the theory weakly coupled for large momentum transfers. In many other cases, however, the theory is strongly coupled and the theoretical treatment becomes very complicated.

In QED there is no commutator term in the field strengths,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \,, \tag{180}$$

and a gauge transformation is simply given by

$$U(X) = e^{-ie\alpha(X)} \,, \qquad A_\mu \to A_\mu + \frac{i}{e} U \partial_\mu U^{-1} = A_\mu - \partial_\mu \alpha \,. \tag{181}$$

Since $U(1)$ is a one-dimensional Lie group, there is only one gauge boson, the photon (compared to eight gluons in QCD). Due to the missing commutator term, the photon has no self-coupling (whereas gluons interact with each other). The fermions are leptons instead of quarks, and the coupling is denoted by e instead of g. The Lagrangian, invariant under $U(1)$, is

$$\mathcal{L}_{\mathrm{QED}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}(i\gamma^\mu D_\mu + \gamma^0 \mu - m)\psi \,, \tag{182}$$

with the covariant derivative

$$D_\mu = \partial_\mu - ieA_\mu \,. \tag{183}$$

B. Partition function in QED

We now focus on the gauge part of the QED Lagrangian (182), i.e., we are interested in

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = \frac{1}{2} F_{0i} F_{0i} - \frac{1}{4} F_{ij} F_{ij} \,. \tag{184}$$

The electric and magnetic fields are given by

$$E_i = -F_{0i} = F_{i0} \,, \qquad \mathbf{B} = \nabla \times \mathbf{A} \;\Rightarrow\; B_i = \frac{1}{2} \epsilon_{ijk} F_{jk} \,. \tag{185}$$

We thus have

$$B^2 = \frac{1}{2} F_{jk} F_{jk} \,, \tag{186}$$

and the Lagrangian becomes

$$\mathcal{L} = \frac{1}{2} E^2 - \frac{1}{2} B^2 \,. \tag{187}$$

In the following we shall work in the so-called axial gauge

$$A_3 = 0 \,. \tag{188}$$

This does not completely fix the gauge and we will see how the residual gauge freedom appears. With

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = -\frac{1}{2}\left(\delta^\mu_\rho\delta^\nu_\sigma - \delta^\mu_\sigma\delta^\nu_\rho\right)F^{\rho\sigma} = -\frac{1}{2}(F^{\mu\nu} - F^{\nu\mu}) = -F^{\mu\nu}\,, \tag{189}$$

we find the conjugate momenta

$$\pi^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_0 A_\mu)} = -F^{0\mu}\,. \tag{190}$$

We see that there is no momentum conjugate to A_0. Consequently, A_0 is not a dynamical field. The spatial components of the momentum are

$$\pi_i = F_{0i} = -E_i\,. \tag{191}$$

Therefore, formally there is a conjugate momentum

$$\pi_3 = -E_3\,, \tag{192}$$

even though $A_3 = 0$ in the chosen gauge, i.e., π_3 is not an independent variable. It can be determined from Gauss' law, which, in the absence of charges, is

$$\nabla\cdot\mathbf{E} = 0\,. \tag{193}$$

Consequently, we have $\partial_3 E_3 = \partial_1\pi_1 + \partial_2\pi_2$ and thus

$$E_3 = \int_{x_{30}}^{x_3} dx_3'(\partial_1\pi_1 + \partial_2\pi_2) + P(x_1, x_2, t)\,, \tag{194}$$

and

$$A_0 = \int_{x_{30}}^{x_3} dx_3' E_3 + Q(x_1, x_2, t)\,. \tag{195}$$

The integration constants P and Q correspond to the residual gauge freedom. Next we determine the Hamiltonian in terms of the independent variables π_1, π_2, A_1, A_2,

$$\mathcal{H} \;=\; \pi_1\partial_0 A_1 + \pi_2\partial_0 A_2 - \mathcal{L} \tag{196}$$

We use $\partial_0 A_i = \pi_i + \partial_i A_0$ (from Eq. (191)) and $(\pi_1^2+\pi_2^2)/2 = (E_1^2+E_2^2)/2$ to obtain

$$\begin{aligned}\mathcal{H} \;&=\; \frac{1}{2}(\pi_1^2+\pi_2^2) - \frac{1}{2}E_3^2 + \frac{1}{2}B^2 + \pi_1\partial_1 A_0 + \pi_2\partial_2 A_0 \\ &=\; \frac{1}{2}(\pi_1^2+\pi_2^2) + \frac{1}{2}E_3^2 + \frac{1}{2}B^2\,,\end{aligned} \tag{197}$$

where we used partial integration and dropped the surface terms (i.e., this identity only holds under the integral d^3x): $\pi_1\partial_1 A_0 + \pi_2\partial_2 A_0 \to -A_0(\partial_1\pi_1 + \partial_2\pi_2) = -A_0\partial_3 E_3 \to E_3\partial_3 A_0 = E_3^2$. The Hamiltonian now has the familiar form $\mathcal{H} = E^2/2 + B^2/2$. The partition function for the bosonic fields A_1, A_2 and their conjugate momenta is

$$Z = \int \mathcal{D}\pi_1\mathcal{D}\pi_2 \int_{\text{periodic}} \mathcal{D}A_1\mathcal{D}A_2 \exp\int_X \left(i\pi_1\partial_\tau A_1 + i\pi_2\partial_\tau A_2 - \mathcal{H}\right)\,. \tag{198}$$

We rewrite the partition function in the following way. First we insert

$$1 = \int \mathcal{D}\pi_3\,\delta(\pi_3 + E_3(\pi_1,\pi_2))\,. \tag{199}$$

This can be rewritten upon using

$$\delta(\nabla\cdot\boldsymbol{\pi}) = \left(\det\frac{\partial(\nabla\cdot\boldsymbol{\pi})}{\partial\pi_3}\right)^{-1}\delta(\pi_3 + E_3(\pi_1,\pi_2))\,. \tag{200}$$

Here one should remember the more familiar form of this identity

$$\delta(f(x)) = \frac{1}{|f'(x_0)|}\delta(x - x_0)\,, \tag{201}$$

where x_0 is the zero of the function f. Moreover we use

$$\det\frac{\partial(\nabla\cdot\boldsymbol{\pi})}{\partial\pi_3} = \det(\partial_3)\,, \tag{202}$$

and we write the δ-function in its integral representation,

$$\delta(\nabla\cdot\boldsymbol{\pi}) = \int \mathcal{D}A_0 \exp\left(i\int_X A_0\nabla\cdot\boldsymbol{\pi}\right)\,. \tag{203}$$

Here, in the exponential, we have replaced $A_0 \to iA_0$ since this yields the replacement $i\int d^4x A_0\nabla\cdot\boldsymbol{\pi} \to i\int_X A_0\nabla\cdot\boldsymbol{\pi}$ (note that we also have to replace dx_0 by $-id\tau$) [4]. Inserting all this into Eq. (199) yields

$$1 = \int \mathcal{D}\pi_3 \int \mathcal{D}A_0 \det(\partial_3)\exp\left(i\int_X A_0\nabla\cdot\boldsymbol{\pi}\right)\,, \tag{204}$$

and the partition function becomes (after a partial integration $A_0\nabla\cdot\boldsymbol{\pi} \to -(\nabla A_0)\cdot\boldsymbol{\pi}$)

$$Z = \int \mathcal{D}\pi_1\mathcal{D}\pi_2\mathcal{D}\pi_3 \int_{\text{periodic}} \mathcal{D}A_0\mathcal{D}A_1\mathcal{D}A_2 \det(\partial_3)\,\exp\int_X\left[i\pi_1\partial_\tau A_1 + i\pi_2\partial_\tau A_2 - i(\nabla A_0)\cdot\boldsymbol{\pi} - \frac{1}{2}\pi^2 - \frac{1}{2}B^2\right]\,. \tag{205}$$

The momentum integral now becomes trivial as we have seen in the case of scalar bosons. To this end, we rewrite the exponential with the help of

$$i\pi_1\partial_\tau A_1 + i\pi_2\partial_\tau A_2 - i(\nabla A_0)\cdot\boldsymbol{\pi} - \frac{1}{2}\pi^2 = -\frac{1}{2}(\boldsymbol{\pi} - i\partial_\tau\mathbf{A} + i\nabla A_0)^2 - \frac{1}{2}(\partial_\tau\mathbf{A} - \nabla A_0)^2\,, \tag{206}$$

where $\mathbf{A} = (A_1, A_2, 0)$ in the axial gauge we use. Now the integration over the shifted momentum $\boldsymbol{\pi} - i\partial_\tau\mathbf{A} + i\nabla A_0$ can be performed and yields an irrelevant constant factor which we omit in the following. Consequently,

$$Z = \int_{\text{periodic}} \mathcal{D}A_0\mathcal{D}A_1\mathcal{D}A_2 \det(\partial_3)\,\exp\int_X \mathcal{L}\,. \tag{207}$$

We have recovered the Lagrangian in the exponential since

$$-(\partial_\tau\mathbf{A} - \nabla A_0)^2 = E^2\,. \tag{208}$$

(To see this, one simply "undoes" the finite-temperature replacements $\partial_0 \to i\partial_\tau$, $A_0 \to iA_0$.) Hence we get the Lagrangian in the form (187).

Before we proceed with Eq. (207) we notice that the general form of the partition function, without specifying a gauge, is

$$Z = \int_{\text{periodic}} \mathcal{D}A_\mu\,\delta(F)\det\frac{\partial F}{\partial\alpha}\,\exp\int_X \mathcal{L}\,, \tag{209}$$

where $\mathcal{D}A_\mu \equiv \mathcal{D}A_0\mathcal{D}A_1\mathcal{D}A_2\mathcal{D}A_3$, where F is a function of the gauge fields and the condition $F = 0$ fixes the gauge. In our case, $F = A_3' = A_3 - \partial_3\alpha$. Then, with $\partial F/\partial\alpha = \partial_3$ we recover Eq. (207). The more general form shows that we integrate over the space of gauge fields "modulo gauge transformations". In other words, for each point in the space of gauge fields, we choose a fixed gauge given by the function F and fixed by the factor $\delta(F)$. Then $\det(\partial F/\partial\alpha)$ is the determinant of the Jacobian of the transformation $A_\mu' = F(A_\mu) = A_\mu - \partial_\mu\alpha$, i.e., it accounts for the change of integration variables according to the gauge transformation. The partition function in the form (209) is manifestly gauge invariant.

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[4] Another way of saying this is that in the field strength $F_{i0} = \partial_i A_0 - \partial_0 A_i$ for finite temperature we have to replace ∂_0 by $i\partial_\tau$. To get the same factor i from the first term we need to replace A_0 by iA_0.

Let us now come back to our expression (207) in the axial gauge and compute the functional integral. With Eq. (186) we find

$$\begin{aligned} E^2 - B^2 &= -(\partial_\tau \mathbf{A} - \nabla A_0)^2 - B^2 \\ &= -(\partial_\tau \mathbf{A})^2 - (\nabla A_0)^2 + 2\partial_\tau \mathbf{A} \cdot \nabla A_0 \\ &\quad - (\partial_1 A_2)^2 - (\partial_2 A_1)^2 - (\partial_3 A_1)^2 - (\partial_3 A_2)^2 + 2(\partial_1 A_2)(\partial_2 A_1)\,. \end{aligned} \tag{210}$$

As above, we introduce the Fourier transform of the gauge fields,

$$A_\mu(X) = \frac{1}{\sqrt{TV}} \sum_K e^{-iK\cdot X} A_\mu(K)\,. \tag{211}$$

This yields

$$\int_X (\partial_\tau \mathbf{A})^2 = -\frac{1}{T^2} \sum_K k_0^2 \mathbf{A}(-K)\cdot\mathbf{A}(K)\,, \tag{212a}$$

$$\int_X (\nabla A_0)^2 = \frac{1}{T^2} \sum_K k^2 A_0(-K) A_0(K)\,, \tag{212b}$$

$$\int_X \partial_\tau \mathbf{A}\cdot\nabla A_0 = \frac{1}{T^2} \sum_K ik_0 \mathbf{k}\cdot\mathbf{A}(-K) A_0(K) = \frac{1}{T^2} \sum_K ik_0 \mathbf{k}\cdot\mathbf{A}(K) A_0(-K)\,, \tag{212c}$$

$$\int_X (\partial_1 A_2)^2 = \frac{1}{T^2} \sum_K k_1^2 A_2(-K) A_2(K)\,, \tag{212d}$$

$$\int_X (\partial_1 A_2)(\partial_2 A_1) = \frac{1}{T^2} \sum_K k_1 k_2 A_1(-K) A_2(K) = \frac{1}{T^2} \sum_K k_1 k_2 A_1(K) A_2(-K)\,. \tag{212e}$$

The other terms $(\partial_2 A_1)^2$, $(\partial_3 A_1)^2$, $(\partial_3 A_2)^2$ are obtained analogously to Eq. (212d). We thus find[5]

$$\int_X \mathcal{L} = -\frac{1}{2T^2} \sum_K (A_0(-K), A_1(-K), A_2(-K)) \begin{pmatrix} k^2 & -ik_0k_1 & -ik_0k_2 \\ -ik_0k_1 & -k_0^2+k_2^2+k_3^2 & -k_1k_2 \\ -ik_0k_2 & -k_1k_2 & -k_0^2+k_1^2+k_3^2 \end{pmatrix} \begin{pmatrix} A_0(K) \\ A_1(K) \\ A_2(K) \end{pmatrix}. \tag{214}$$

The 3×3 matrix is the inverse gauge field propagator in momentum space which we denote by $\mathcal{D}_0^{-1}(K)$. Here we have symmetrized the appearing matrix in the exponential. This is important since $\mathbf{A}(K)$ and $\mathbf{A}(-K)$ are not independent variables. So suppose we had used some asymmetric "propagator" $\tilde{\mathcal{D}}_0$. Then we have to write

$$\begin{aligned} \sum_K A_a(-K)[\tilde{\mathcal{D}}_0^{-1}(K)]_{ab} A_b(K) &= \sum_{K>0} A_a(-K)[\tilde{\mathcal{D}}_0^{-1}(K)]_{ab} A_b(K) + \sum_{K<0} A_a(-K)[\tilde{\mathcal{D}}_0^{-1}(K)]_{ab} A_b(K) \\ &= \sum_{K>0} A_a(-K)\left\{[\tilde{\mathcal{D}}_0^{-1}(K)]_{ab} + [\tilde{\mathcal{D}}_0^{-1}(K)]_{ba}\right\} A_b(K)\,, \end{aligned} \tag{215}$$

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[5] If we start from

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} = -\frac{1}{2}(\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\mu A_\nu \partial^\nu A^\mu)\,,$$

and insert the Fourier transform (211), we obtain

$$\int_X \mathcal{L} = -\frac{1}{2T^2} \sum_K A_\mu(-K)(K^2 g^{\mu\nu} - K^\mu K^\nu) A_\nu(K)\,. \tag{213}$$

Dropping the 3-component and replacing $A_0 \to iA_0$ yields Eq. (214).

and arrive at the symmetrized propagator.

We can now use Eq. (61) for the integration to obtain

$$\begin{aligned} Z &= \det(\partial_3) \left(\det \frac{\mathcal{D}_0^{-1}}{T^2} \right)^{-1/2} \\ &= \det(\partial_3) \left(\prod_K \frac{K^4 k_3^2}{T^6} \right)^{-1/2} \\ &= \det(\partial_3) \left(\prod_K \frac{K^2}{T^2} \right)^{-2/2} \left(\prod_K \frac{k_3^2}{T^2} \right)^{-1/2} , \end{aligned} \tag{216}$$

and thus

$$\ln Z = \ln \det(\partial_3) - 2\,\frac{1}{2} \sum_K \ln \frac{k_0^2 - k^2}{T^2} - \frac{1}{2} \sum_K \ln \frac{k_3^2}{T^2} \,. \tag{217}$$

It remains to evaluate the so-called Fadeev-Popov determinant $\det(\partial_3)$. With Eq. (155) we can write this determinant as a functional integral over Grassmann variables $\bar{C}$, C,

$$\det(\partial_3) = \int \mathcal{D}\bar{C}\mathcal{D}C \, \exp\left(- \int_X \bar{C} \partial_3 C \right) . \tag{218}$$

Here C is a complex, scalar field, i.e., it seems to describe a spin-0 boson. On the other hand, the integration goes over Grassmann variables, indicating fermionic properties. This unphysical field is called a Fadeev-Popov ghost field. It plays a more important role in non-abelian gauge theories but we see that it is needed also here. With the Fourier transform

$$C(X) = \frac{1}{\sqrt{V}} \sum_K e^{-iK\cdot X} C(K) \,, \tag{219}$$

(bosonic Matsubara frequencies!) we have

$$- \int_X \bar{C} \partial_3 C = - \sum_K \bar{C}(K) \frac{ik_3}{T} C(K) \,. \tag{220}$$

Consequently, the ghost contribution is

$$\det(\partial_3) = \det \frac{ik_3}{T} = \prod_K \frac{ik_3}{T} \propto \prod_K \frac{k_3}{T} \,. \tag{221}$$

We see that this term exactly cancels the third term on the right-hand side of Eq. (217) and we are left with

$$\ln Z = -2\,\frac{1}{2} \sum_K \ln \frac{k_0^2 - k^2}{T^2} \,. \tag{222}$$

This result shows the two degrees of freedom of the gauge field. The third degree of freedom, unphysical due to gauge symmetry, is cancelled by the ghosts.

[End of 8th lecture, Dec 2nd, 2013.]

VIII. INTERACTIONS

A. Perturbative expansion in $\lambda\phi^4$ theory

We add an interaction term with coupling constant λ to the Lagrangian for a real scalar field (45) to obtain the Lagrangian

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 - \lambda \phi^4 \,. \tag{223}$$

We use the index 0 for the contribution we have already computed above. The partition function then is

$$Z = \int \mathcal{D}\phi\, e^{S}\,, \tag{224}$$

with the action

$$S = S_0 + S_I = \int_X \mathcal{L}_0 + \int_X \mathcal{L}_I\,, \qquad S_I = -\lambda \int_X \phi^4\,. \tag{225}$$

Without interaction, $S_I = 0$, we could compute $\ln Z$ exactly. In the presence of interactions this is not possible. Therefore, we need to apply an approximation. The simplest approximation is to consider the coupling constant λ as a small expansion parameter and then truncate the expansion at a given order in λ. We shall discuss this procedure in the following. Denoting the noninteracting part by

$$Z_0 \equiv \int \mathcal{D}\phi\, e^{S_0} \tag{226}$$

we can write the expansion as

$$\begin{aligned} \ln Z &= \ln \int \mathcal{D}\phi\, e^{S_0+S_I} \\ &= \ln \int \mathcal{D}\phi\, e^{S_0} \sum_{n=0}^{\infty} \frac{S_I^n}{n!}\,. \end{aligned} \tag{227}$$

Now if we add and subtract $\ln Z_0$ we can write this as

$$\begin{aligned} \ln Z &= \ln Z_0 + \ln \frac{\int \mathcal{D}\phi\, e^{S_0} \sum_{n=0}^{\infty} \frac{S_I^n}{n!}}{\int \mathcal{D}\phi\, e^{S_0}} \\ &= \ln Z_0 + \ln Z_I\,, \end{aligned} \tag{228}$$

with

$$\ln Z_I \equiv \ln\left(1 + \sum_{n=1}^{\infty} \frac{1}{n!} \frac{\int \mathcal{D}\phi\, e^{S_0} S_I^n}{\int \mathcal{D}\phi\, e^{S_0}}\right) = \ln\left(1 + \sum_{n=1}^{\infty} \frac{\langle S_I^n\rangle_0}{n!}\right)\,. \tag{229}$$

Here $\langle - \rangle_0$ denotes the ensemble average over the noninteracting ensemble. From the definition of S_I we know that each factor of S_I comes with one power of λ. If we expand $\ln Z_I$ to, say, third order in the coupling, we thus obtain, using $\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} x^n/n$,

$$\begin{aligned} \ln Z_I &\simeq \ln\left(1 + \langle S_I\rangle_0 + \frac{\langle S_I^2\rangle_0}{2} + \frac{\langle S_I^3\rangle_0}{6}\right) \\ &\simeq \langle S_I\rangle_0 + \frac{1}{2}\left(\langle S_I^2\rangle_0 - \langle S_I\rangle_0^2\right) + \frac{1}{6}\left(\langle S_I^3\rangle_0 - 3\langle S_I\rangle_0\langle S_I^2\rangle_0 + 2\langle S_I\rangle_0^3\right)\,, \end{aligned} \tag{230}$$

where we have ordered the contributions according to the powers λ, λ^2, λ^3. Denoting the n-th order correction to $\ln Z$ by $\ln Z_I^{(n)}$, we thus have

$$\ln Z_I^{(1)} = \langle S_I\rangle_0\,, \tag{231a}$$

$$\ln Z_I^{(2)} = \frac{1}{2}\left(\langle S_I^2\rangle_0 - \langle S_I\rangle_0^2\right)\,, \tag{231b}$$

$$\ln Z_I^{(3)} = \frac{1}{6}\left(\langle S_I^3\rangle_0 - 3\langle S_I\rangle_0\langle S_I^2\rangle_0 + 2\langle S_I\rangle_0^3\right)\,. \tag{231c}$$

Let us compute the first correction $\ln Z_I^{(1)} \propto \lambda$ explicitly. We have

$$\langle S_I\rangle_0 = -\lambda \frac{\int \mathcal{D}\phi\, e^{S_0} \int_X \phi^4(X)}{\int \mathcal{D}\phi\, e^{S_0}}\,. \tag{232}$$

From Sec. IV we know that

$$e^{S_0} = \exp\left[-\frac{1}{2}\sum_K \phi(-K)\frac{D_0^{-1}(K)}{T^2}\phi(K)\right] = \prod_K \exp\left[-\frac{1}{2}\phi(-K)\frac{D_0^{-1}(K)}{T^2}\phi(K)\right], \tag{233}$$

with the inverse propagator

$$D_0^{-1}(K) = \omega_n^2 + k^2 + m^2\,. \tag{234}$$

In momentum space, the ϕ^4 term becomes

$$\begin{aligned}\int_X \phi^4(X) &= \frac{1}{T^2V^2}\sum_{K_1,\ldots,K_4}\int_X e^{i(K_1+\ldots+K_4)\cdot X}\,\phi(K_1)\ldots\phi(K_4)\\ &= \frac{1}{T^3V}\sum_{K_1,\ldots,K_4}\delta(K_1+\ldots+K_4)\,\phi(K_1)\ldots\phi(K_4)\,.\end{aligned} \tag{235}$$

Inserting Eqs. (233) and (235) into Eq. (232) yields

$$\langle S_I\rangle_0 = -\frac{\lambda}{T^3V}\frac{\sum_{K_1,\ldots,K_4}\delta(K_1+\ldots+K_4)\prod_K\int d\phi(K)\,e^{-\frac{1}{2}\phi(-K)\frac{D_0^{-1}(K)}{T^2}\phi(K)}\phi(K_1)\ldots\phi(K_4)}{\prod_K\int d\phi(K)\,e^{-\frac{1}{2}\phi(-K)\frac{D_0^{-1}(K)}{T^2}\phi(K)}}\,. \tag{236}$$

The integral in the numerator is only nonvanishing if the four momenta K_1, K_2, K_3, K_4 cancel each other pairwise. Otherwise, if there is a single power of $\phi(K)$, the integral over $\phi(K)$ is zero by symmetry. (Remember that $\phi(-K) = \phi^*(K)$ and thus $\phi(K)\phi(-K) = |\phi(K)|^2$.) Hence we have for instance $K_1 = -K_2 \equiv Q$ and $K_3 = -K_4 \equiv P$, and the Kronecker-delta is automatically fulfilled. There are 3 possibilities for the momenta to be pairwise identical and thus we obtain

$$\langle S_I\rangle_0 = -\frac{3\lambda}{T^3V}\sum_{Q,P}\frac{\prod_K\int d\phi(K)\,e^{-\frac{1}{2}\phi(-K)\frac{D_0^{-1}(K)}{T^2}\phi(K)}\phi(-Q)\phi(Q)\phi(-P)\phi(P)}{\prod_K\int d\phi(K)\,e^{-\frac{1}{2}\phi(-K)\frac{D_0^{-1}(K)}{T^2}\phi(K)}}\,. \tag{237}$$

Now we notice that all integrals over $K \neq P, Q$ appear identically in numerator and denominator and thus cancel. The $\phi(P)$ and $\phi(Q)$ integrals factorize and we obtain

$$\langle S_I\rangle_0 = -\frac{3\lambda}{T^3V}\left[\sum_Q\frac{\int d\phi(Q)\,e^{-\frac{1}{2}\phi(-Q)\frac{D_0^{-1}(Q)}{T^2}\phi(Q)}\phi(Q)\phi(-Q)}{\int d\phi(Q)\,e^{-\frac{1}{2}\phi(-Q)\frac{D_0^{-1}(Q)}{T^2}\phi(Q)}}\right]^2. \tag{238}$$

Now we use

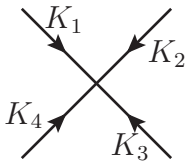
$$\frac{\int_{-\infty}^{\infty} dx\,x^2 e^{-ax^2/2}}{\int_{-\infty}^{\infty} dx\,e^{-ax^2/2}} = \frac{\frac{\sqrt{2\pi}}{a^{3/2}}}{\frac{\sqrt{2\pi}}{a^{1/2}}} = \frac{1}{a}\,, \tag{239}$$

to obtain

$$\ln Z_I^{(1)} = \langle S_I\rangle_0 = -3\lambda\frac{T}{V}\left[\sum_Q D_0(Q)\right]^2. \tag{240}$$

We shall evaluate $\ln Z_I^{(1)}$ further in Sec. VIII C. Here we proceed by introducing Feynman diagrams: it is convenient to translate the complicated algebraic perturbative expansion into a diagrammatic form. One starts by representing

each field by a line with a direction, the direction indicating whether the field is ingoing or outgoing (i.e., inverting the direction corresponds to $\phi(K) \to \phi(-K)$). Then the interaction term $-\lambda\phi^4$ is represented in momentum space by



(241)

By convention, we have chosen the signs of the momenta such that all lines are ingoing. Since the momenta K_1, K_2, K_3, K_4 have to cancel pairwise, we connect the lines pairwise and interpret each resulting line as a propagator. (From the explicit integration above we know how the propagators come about.) There are three possibilities to connect the four lines pairwise and thus the algebraic result is translated into a Feynman diagram as follows,

$$\langle S_I\rangle_0 = -3\lambda\frac{T}{V}\left[\sum_Q D_0(Q)\right]^2 = 3$$

(242)

In summary, the vertex gives a factor $-\lambda$, the factor 3 is a combinatorical factor, the closed line is a propagator $T/V\sum_Q D_0(Q)$, and momentum conservation gives $V/T\,\delta(K_{\rm in}-K_{\rm out})$ which here is automatically fulfilled and thus simply gives a factor V/T.

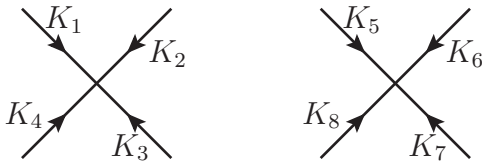
In general, the rules to find all contributions to the logarithm of the partition function for a given order λ^n are

1. Draw all connected diagrams with combinatorical prefactors.
2. Each (closed) line gives a propagator $\frac{T}{V}\sum_K D_0(K)$.
3. Each vertex gives a factor $-\lambda$ and a momentum-conserving Kronecker-delta $\frac{V}{T}\delta(K_{\rm in}-K_{\rm out})$.

In the first-order contibution it is clear that there is only a connected diagram. We shall explain now, for the second-order corrections to $\ln Z$, why the disconnected diagrams cancel. The second-order terms of the partition function are

$$\ln Z_I^{(2)} = \frac{1}{2}\left(\langle S_I^2\rangle_0 - \langle S_I\rangle_0^2\right) . \tag{243}$$

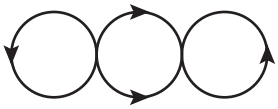
From Eq. (242) we know the diagrammatic representation of $\langle S_I\rangle_0^2$. For $\langle S_I^2\rangle_0$ we need to start from $(-\lambda\phi^4)^2$ which, in analogy to Eq. (241) is represented as



(244)

Again we have to construct all possible diagrams by connecting the eight lines pairwise. One of the diagrams we obtain is the product of two disconnected "double-bubbles" (each with a combinatorical factor 3) which exactly cancels the term $\langle S_I\rangle_0^2$. We are left with

$$\ln Z_I^{(2)} = 36$$



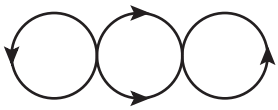
$$+\,12$$

(245)

The combinatorical factors arise as follows. First diagram: pick two fixed lines from the first vertex, say 2 and 3 such that 2 is the upper line, 3 the lower line. This fixed pair of lines now gets connected with a pair of the second vertex. There are 6 such pairs, and each pair can connect in 2 ways (if we choose the pair 5, 8 we can connect (2,5), (3,8) or (2,8), (3,5)). We are now at 12 diagrams. This has to be multiplied by 6 since there are 6 pairs from the first diagram which we could have started with. Then we are done since once the two pairs in the middle are connected

there is no more choice. Consequently, we obtain 72 possibilities which has to be divided by 2 because of Eq. (243). Second diagram: simply fix the lines from the left diagram in the order 1,2,3,4 from top to bottom. Then there are 4! possibilities to attach the lines from the second diagram to them. This makes 24 which also has to be divided by 2.

Translating the second-order diagrams into momentum sums yields



$$= (-\lambda)^2 \left(\frac{T}{V}\right)^4 \left(\frac{V}{T}\right)^2 \sum_{K_1,\ldots,K_4} \delta(K_2+K_3) D_0(K_1)\ldots D_0(K_4)$$

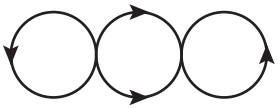
$$= \lambda^2 \frac{T^2}{V^2} \left[\sum_K D_0(K)\right]^2 \sum_{K_2,K_3} \delta(K_2+K_3) D_0(K_2) D_0(K_3)\,, \tag{246a}$$

$$= (-\lambda)^2 \left(\frac{T}{V}\right)^4 \left(\frac{V}{T}\right)^2 \sum_{K_1,\ldots,K_4} \delta(K_1+\ldots+K_4) D_0(K_1)\ldots D_0(K_4)\,. \tag{246b}$$

The fact that the disconnected diagrams cancel out is general, i.e., Eq. (229) simplifies to

$$\ln Z_I = \sum_{n=1}^{\infty} \frac{\langle S_I^n\rangle_{0,\text{connected}}}{n!}\,. \tag{247}$$

As a summary, we have computed the partition function diagrammatically up to second order in the coupling constant,

$\ln Z_I = 3$  $+\,36$  $+\,12$  $+\ldots$ (248)

We shall see below, however, that this result is incomplete. There is in fact a contribution to $\ln Z_I$ of order $\lambda^{3/2}$, see Sec. VIII D.

B. Propagator, self-energy, and one-particle irreducible (1PI) diagrams

We have seen above that the free propagator can be written as the following ensemble average over the non-interacting ensemble (see for instance Eq. (239)),

$$D_0(Q) = \frac{1}{T^2} \frac{\prod_K \int d\phi(K)\, e^{-\frac{1}{2}\phi(-K)\frac{D_0^{-1}(K)}{T^2}\phi(K)} \phi(Q)\phi(-Q)}{\prod_K \int d\phi(K)\, e^{-\frac{1}{2}\phi(-K)\frac{D_0^{-1}(K)}{T^2}\phi(K)}}$$

$$= \frac{1}{T^2}\langle\phi(Q)\phi(-Q)\rangle_0\,, \tag{249}$$

In general, the propagator in position space is defined as

$$D(X_1,X_2) \equiv \langle\phi(X_1)\phi(X_2)\rangle\,. \tag{250}$$

From this definition the above form of $D_0(Q)$ follows for the case of a translationally invariant system: in this case, $D(X_1,X_2)$ only depends on the difference $X_1 - X_2$. Without loss of generality we can thus set $X_2 = 0$ and denote $X = X_1$. Then, the Fourier transform is

$$D(Q) = \int_X e^{iQ\cdot X}\langle\phi(X)\phi(0)\rangle = \frac{1}{TV}\sum_{K_1,K_2}\int_X e^{iQ\cdot X} e^{-iK_1\cdot X}\langle\phi(K_1)\phi(K_2)\rangle$$

$$= \frac{1}{T^2}\sum_{K_2}\langle\phi(Q)\phi(K_2)\rangle = \frac{1}{T^2}\langle\phi(Q)\phi(-Q)\rangle\,. \tag{251}$$

In the last step we have used that the ensemble average is only nonzero for $K_2 = -Q$. The result is the full propagator in momentum space in a translationally invariant system (where the ensemble average is taken over the *interacting* ensemble, in contrast to the free propagator (249)).

Exercise 5*: Compute the free propagator $D_0(X,0)$ in position space and show that*

$$D_0(X,0) \simeq \begin{cases} -\dfrac{1}{4\pi^2 X^2} & \text{for small distances } x,\tau \ll \dfrac{1}{T}, \dfrac{1}{m} \\ T\dfrac{e^{-mx}}{4\pi x} & \text{for large distances } x \gg \dfrac{1}{T} \text{ and } m \ll T \end{cases} .$$

As usual, $X = (-i\tau, \mathbf{x})$, and $x \equiv |\mathbf{x}|$. (Hints: In the Matsubara summation, use the function $1/(e^{\omega/T}-1)$ instead of $1/2 \coth \omega/(2T)$ in the analogue of Eq. (67); this ensures that, when closing the contour, the contribution of the infinite semi-circle in the positive half-plane vanishes. In the Fourier integral, use that the dominant contribution comes from momenta $kx \sim 1$.)

[End of 9th lecture, Dec 9th, 2013.]

For a systematic calculation of the perturbation series it is convenient to divide the full (inverse) propagator into a free part and an interaction part, called self-energy Π. We write

$$D^{-1}(K) = D_0^{-1}(K) + \Pi(K) . \tag{252}$$

The purpose of the following will be to connect the perturbative expansion of the self-energy Π to the expansion of $\ln Z_I$. To this end, we first observe that the propagator of the *interacting* system can be written as a functional derivative of $\ln Z$ with respect to the *free* propagator. Remember from Eq. (227) that

$$\ln Z = \ln \int \mathcal{D}\phi \, e^{S_0} e^{S_I} . \tag{253}$$

Therefore,

$$\begin{aligned} \frac{\delta}{\delta D_0^{-1}(Q)} \ln Z &= \frac{1}{\int \mathcal{D}\phi \, e^{S_0+S_I}} \frac{\delta}{\delta D_0^{-1}} \int \mathcal{D}\phi \, e^{S_0} e^{S_I} \\ &= \frac{1}{\int \mathcal{D}\phi \, e^{S_0+S_I}} \frac{\delta}{\delta D_0^{-1}} \prod_K \int d\phi(K) \, e^{-\frac{1}{2}\phi(-K)\frac{D_0^{-1}}{T^2}\phi(K)} \, e^{S_I} \\ &= -\frac{1}{2T^2} \frac{\int \mathcal{D}\phi \, e^{S_0} e^{S_I} \phi(-Q)\phi(Q)}{\int \mathcal{D}\phi \, e^{S_0+S_I}} \\ &= -\frac{1}{2} D(Q) , \end{aligned} \tag{254}$$

and thus

$$D(Q) = -2\frac{\delta \ln Z}{\delta D_0^{-1}} = 2\frac{\delta \ln Z}{\delta D_0} D_0^2 . \tag{255}$$

Then, from Eq. (252) we have

$$D = (D_0^{-1} + \Pi)^{-1} = (1 + D_0\Pi)^{-1} D_0 . \tag{256}$$

Using Eq. (255) we have

$$(1 + D_0\Pi)^{-1} = D D_0^{-1} = 2\frac{\delta \ln Z}{\delta D_0} D_0 . \tag{257}$$

In the following we shall expand both sides of this equation in λ to obtain the self-energy Π order by order. To this end we first write

$$\Pi = \sum_{n=1}^{\infty} \Pi_n \,, \tag{258}$$

such that Π_n is proportional to λ^n. Then, up to second order in λ, we have

$$(1 + D_0\Pi)^{-1} = 1 - D_0[\Pi_1 + \Pi_2 - \Pi_1 D_0 \Pi_1 + \mathcal{O}(\lambda^3)] \,. \tag{259}$$

This is the left-hand side of Eq. (257). For the right-hand side we use, see Eq. (64),

$$\ln Z_0 = -\frac{1}{2}\sum_K \ln \frac{D_0^{-1}}{T^2} = \frac{1}{2}\sum_K \ln(D_0 T^2) \qquad \Rightarrow \qquad \frac{\delta \ln Z_0}{\delta D_0} = \frac{1}{2} D_0^{-1} \,, \tag{260}$$

and thus

$$\begin{aligned} 2D_0 \frac{\delta \ln Z}{\delta D_0} &= 2D_0 \left(\frac{\delta \ln Z_0}{\delta D_0} + \frac{\delta \ln Z_I}{\delta D_0} \right) \\ &= 1 + 2D_0 \frac{\delta \ln Z_I}{\delta D_0} \\ &= 1 + 2D_0 \left[\frac{\delta \langle S_I \rangle_0}{\delta D_0} + \frac{1}{2} \frac{\delta(\langle S_I^2 \rangle_0 - \langle S_I \rangle_0^2)}{\delta D_0} + \mathcal{O}(\lambda^3) \right] . \end{aligned} \tag{261}$$

Thus, upon comparing Eqs. (259) and (261) we have

$$\Pi_1 + \Pi_2 - \Pi_1 D_0 \Pi_1 + \ldots = -2 \frac{\delta \ln Z_I}{\delta D_0} \,. \tag{262}$$

The first- and second- order contributions are

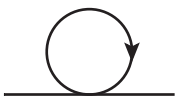
$$\Pi_1 = -2 \frac{\delta \langle S_I \rangle_0}{\delta D_0} \,, \tag{263a}$$

$$\Pi_2 - \Pi_1 D_0 \Pi_1 = -\frac{\delta(\langle S_I^2 \rangle_0 - \langle S_I \rangle_0^2)}{\delta D_0} \,. \tag{263b}$$

The first-order contribution becomes

$$\begin{aligned} \Pi_1 &= 6\lambda \frac{T}{V} \frac{\delta}{\delta D_0} \left[\sum_K D_0(K) \right]^2 \\ &= 12\lambda \frac{T}{V} \sum_K D_0(K) \,. \end{aligned} \tag{264}$$

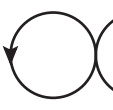
We see that taking the functional derivative with respect to the propagator is equivalent to cutting a line in the Feynman diagram,

$$\begin{aligned} \Pi_1 &= -2 \frac{\delta \langle S_I \rangle_0}{\delta D_0} \\ &= -2 \frac{\delta}{\delta D_0} 3 \quad \text{  } \\ &= -12 \quad \text{  } \end{aligned} \tag{265}$$

The additional factor of 2 appears since each of the two lines can be cut to obtain the same diagram. With this "cutting rule", we can easily determine the second-order contributions with the help of the diagrams. By cutting a line in the respective diagrams we obtain

$$\Pi_2 - \Pi_1 D_0 \Pi_1 = -\frac{\delta(\langle S_I^2\rangle_0 - \langle S_I\rangle_0^2)}{\delta D_0}$$

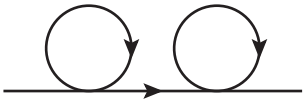
$$= -\frac{\delta}{\delta D_0}\Big[72$$



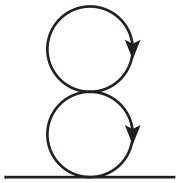
$$+ 24$$

$$\Big]$$

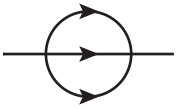
$$= -144$$



$$- 144$$



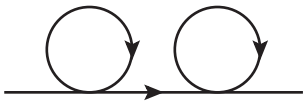
$$- 96$$



(266)

From Eq. (265) we conclude

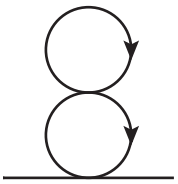
$$\Pi_1 D_0 \Pi_1 = 144$$



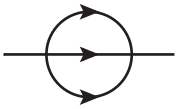
(267)

such that we obtain

$$\Pi_2 = -144$$



$$- 96$$



(268)

We see that the diagram which can be divided into two disconnected diagrams by cutting one line cancels. This is a general fact and the self-energy is given by all diagrams that cannot be divided into two by cutting one line. These diagrams are called "one-particle irreducible (1PI)", and thus Eq. (262) simplifies to

$$\Pi = -2\left(\frac{\delta \ln Z_I}{\delta D_0}\right)_{\text{1PI}}. \tag{269}$$

C. Evaluation of the first-order corrections

We can now compute the first-order contribution to the self-energy (264) and to the pressure from Eq. (240). Making use of Eq. (72) we obtain

$$\Pi_1 = 12\lambda \frac{T}{V}\sum_K \frac{1}{\omega_n^2 + \epsilon_k^2} = \Pi_1^{\text{vac}} + \Pi_1^T\,, \tag{270}$$

with the temperature-independent vacuum part

$$\Pi_1^{\text{vac}} = \Pi_1(T=0) = 6\lambda \int \frac{d^3k}{(2\pi)^3}\frac{1}{\epsilon_k}\,, \tag{271}$$

and the temperature-dependent part

$$\Pi_1^T = 12\lambda \int \frac{d^3k}{(2\pi)^3}\frac{f_B(\epsilon_k)}{\epsilon_k}\,. \tag{272}$$

While Π_1^T is finite, the vacuum part is divergent. This divergence comes from large momenta, $k \to \infty$, and is thus called *ultraviolet* divergence[6].

This divergence requires a renormalization, such that we obtain the renormalized self-energy by subtracting the vacuum part,

$$\Pi_1^{\rm ren} = \Pi_1 - \Pi_1(T=0)\,. \tag{273}$$

With this renormalization condition we could proceed directly to Eq. (279) to compute $\Pi_1^{\rm ren}$ explicitly. Before doing so, let us briefly discuss how the renormalization is usually implemented via counterterms in the Lagrangian.

To this end, we remember that the inverse propagator to first order in λ is [see Eq. (252)]

$$D^{-1}(K) = \omega_n^2 + k^2 + m^2 + \Pi_1\,. \tag{274}$$

This shows that the self-energy plays the role of a mass squared if it is momentum-independent (which is the case in our example of quartic interactions; this would be different for a cubic interaction term). The counterterm we thus add to the Lagrangian is written as a mass term

$$\mathcal{L} \to \mathcal{L} - \frac{1}{2}\delta m^2 \phi^2\,. \tag{275}$$

However, this term is treated as an interaction, i.e., δm^2 has to be thought of as being of order λ. From Eq. (229) we have concluded that the ϕ^4 term produces a "double-bubble" diagram for the first-order correction. Analogously, there is a "single-bubble" contribution (δm^2 replacing a factor ϕ^2, i.e., one closed loop) from $\delta m^2 \phi^2$, denoted as

$$\langle \int_X \delta m^2 \phi^2 \rangle_0 = \delta m^2 \langle \int_X \phi^2 \rangle_0 =$$

(276)

Its contribution to the self-energy is then obtained from cutting one line,

$\delta m^2 =$

(277)

This contribution is now chosen such that the condition (273) is fulfilled, i.e.,

$$\delta m^2 = -\Pi_1^{\rm vac}\,. \tag{278}$$

After this renormalization we can evaluate the first-order self-energy. For the massless case $m = 0$ (or, equivalently, for large temperatures $T \gg m$) we can do so analytically,

$$\Pi_1^{\rm ren} = 12\lambda \int \frac{d^3k}{(2\pi)^3} \frac{f_B(\epsilon_k)}{\epsilon_k} \simeq \lambda T^2\,, \tag{279}$$

where we used

$$\int_0^\infty dx \frac{x}{e^x - 1} = \frac{\pi^2}{6}\,. \tag{280}$$

We see that a massless boson acquires a *thermal mass* λT^2.

We can also compute the corrections to the pressure from interactions. To first order in the coupling, the pressure is given by

$$P = \frac{T}{V} \ln Z_0 + \frac{T}{V} \ln Z_I^{(1)}\,. \tag{281}$$

We already know the pressure of the non-interacting system at large temperatures, see Eq. (81),

$$\frac{T}{V} \ln Z_0 = -T \int \frac{d^3k}{(2\pi)^3} \ln\left(1 - e^{-\epsilon_k/T}\right) \simeq \frac{\pi^2 T^4}{90}\,, \tag{282}$$

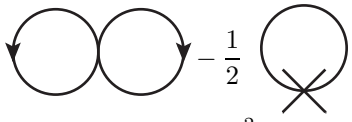
---

[6] In the next section we will encounter an *infrared* divergence, more intimately related to finite temperature effects.

where we have dropped the T-independent contribution which is only an irrelevant shift in the pressure, where we have approximated the expression for $T \gg m$, and where we have used

$$\int_0^\infty dx\, x^2 \ln(1 - e^{-x}) = -\frac{\pi^4}{45}\,. \tag{283}$$

With $\ln Z_I^{(1)}$ from Eq. (240) plus the contribution from the mass counterterm we have

$$\ln Z_I^{(1)} \;=\; 3$$  $$= \; -3\lambda \frac{V}{T}\left[\frac{T}{V}\sum_Q D_0(Q)\right]^2 - \frac{1}{2}\delta m^2 \sum_Q D_0(Q)\,. \tag{284}$$

Using the form of the self-energy (264) and the mass counterterm (278) and dividing both terms into vacuum and temperature-dependent parts according to Eqs. (271) and (272) we have

$$\begin{aligned} \ln Z_I^{(1)} \;&=\; -\frac{V}{T}\frac{1}{48\lambda}\left(\Pi_1^{\rm vac} + \Pi_1^T\right)^2 + \frac{V}{T}\frac{1}{24\lambda}\Pi_1^{\rm vac}\left(\Pi_1^{\rm vac} + \Pi_1^T\right) \\ &=\; \frac{V}{T}\frac{1}{48\lambda}\left[(\Pi_1^{\rm vac})^2 - \left(\Pi_1^T\right)^2\right]\,. \end{aligned} \tag{285}$$

Again, we drop the temperature-independent part to get

$$\frac{T}{V}\ln Z_I^{(1)} \;=\; -3\lambda\left[\int \frac{d^3k}{(2\pi)^3}\frac{f_B(\epsilon_k)}{\epsilon_k}\right]^2 \simeq -\frac{\lambda T^4}{48}\,, \tag{286}$$

again approximating for $T \gg m$ and using Eq. (279). Putting Eqs. (282) and (286) together yields the pressure

$$P = \frac{\pi^2 T^4}{90}\left(1 - \frac{15\lambda}{8\pi^2} + \ldots\right)\,. \tag{287}$$

Exercise 6*: Derive the lowest-order correction $\langle S_I^2\rangle_0$ to $\ln Z$ for a Yukawa interaction $\mathcal{L}_I = g\bar\psi\psi\phi$, with a bosonic scalar field ϕ and a fermionic field ψ. Give $\langle S_I^2\rangle_0$ in terms of diagrams as well as in terms of momentum sums. (Note that odd powers in the interaction term such as $\langle S_I\rangle_0$ vanish. Therefore $\langle S_I^2\rangle_0$ is the lowest-order correction.)*

[End of 10th lecture, Dec 16th, 2013.]

D. Infrared divergence and resummation of ring diagrams

We have seen above that the first-order self-energy Π_1 gives rise to a thermal mass λT^2. In particular, if $m = 0$, the scalar field acquires a finite mass only through a temperature effect. We shall in the following focus on the case $m = 0$. For small energies and momenta, at most of the order of the thermal mass, $\omega_n^2, k^2 \lesssim \lambda T^2$, the free inverse propagator $D_0^{-1} = \omega_n^2 + k^2$ is (at most) of the order of λT^2, and, as we have seen, also the correction through the self energy is of the order of λT^2. This indicates that the naive perturbation series might not be the correct procedure. Indeed, we shall see in the following that one needs to "resum" a certain class of infinitely many diagrams because of an *infrared* divergence, i.e., a divergence coming from small momenta (and energies), as the above simple argument suggests. Another way of saying this is that the temperature introduces a new energy scale. If we work at zero temperature, the only possible correction to the mass (squared) is of the form λm^2. This is always parametrically small compared to m^2. Now, at nonzero temperature, the temperature itself can (and does) give a correction to the mass (squared) of the form λT^2. And, even for arbitrarily small λ, if T is large enough compared to m, this is not a small correction.

Before we discuss the necessary resummation more systematically, let us show a very direct form of a resummation, namely to use the full instead of the free propagator in the one-loop expression of the self-energy,

$$\begin{aligned}\Pi &= 12\lambda\frac{T}{V}\sum_K D(K) \\ &= 12\lambda\frac{T}{V}\sum_K \frac{1}{D_0^{-1}(K)+\Pi}\,. \end{aligned} \tag{288}$$

This is a self-consistent equation for Π. Note that in our simple example of real ϕ^4 theory Π does not depend on K. In other more complicated theories where Π depends on four-momentum, Eq. (288) is a complicated integral equation. But also here the general solution of (288) has to be done numerically. Writing

$$\frac{1}{D_0^{-1}(K)+\Pi} = D_0\sum_{n=0}^{\infty}(-\Pi D_0)^n\,, \tag{289}$$

we see that the self energy now is a loop which itself has any arbitrary number of self-energy insertions, which in turn have self-energy insertions and so on. Such a sum, which formally includes all powers of the coupling constant, is usually termed "resummation". If we replace Π in the denominator of Eq. (288) by the first-order approximation Π_1, the corresponding diagrams consist of a loop with n loops attached to it, sometimes called "daisy". Using the full Π, each of the n loops itself gets additional loops, hence here we sum over "superdaisy" diagrams. (But note that, even if the full self-energy is used, Eq. (288) still has the form of a one-loop self-energy, i.e., even in the "superdaisy" resummation we only sum over a subset of all possible diagrams.)

We evaluate Eq. (288) as follows,

$$\begin{aligned}\Pi &= 12\lambda\int\frac{d^3k}{(2\pi)^3}T\sum_n\frac{1}{\omega_n^2+k^2+\Pi} \\ &= 12\lambda\int\frac{d^3k}{(2\pi)^3}\frac{f_B(\sqrt{k^2+\Pi})}{\sqrt{k^2+\Pi}}\,, \end{aligned} \tag{290}$$

where we used the Matsubara sum (72) and where we dropped the zero-temperature contribution. With the new integration variable $x=\sqrt{k^2/\Pi+1}$ we can write this as

$$\begin{aligned}1 &= \frac{6\lambda}{\pi^2}\int_1^\infty dx\,\sqrt{x^2-1}\,f_B(\Pi^{1/2}x)\,, \\ &= \frac{6\lambda}{\pi^2}\int_1^\infty dx\,\frac{\sqrt{x^2-1}}{e^{x\Pi^{1/2}/T}-1}\,. \end{aligned} \tag{291}$$

If one wants to keep all superdaisy diagrams, one has to proceed numerically now.

Exercise 7*: Solve Eq. (291) numerically and plot* $\Pi^{1/2}/T$ *as a function of* λ*. Compare this curve with the result (279) for* Π_1*.*

We can extract an analytical result by using the expansion

$$\int_1^\infty dx\,\sqrt{x^2-1}\,f_B(ux) = \frac{2\pi^2}{u^2}\left[\frac{1}{12}-\frac{u}{4\pi}+\mathcal{O}(u^2\ln u)\right]. \tag{292}$$

Inserting this expansion into Eq. (291), solving the resulting equation for Π and expanding in powers of λ yields

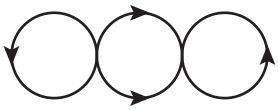
$$\Pi = \lambda T^2 - \frac{3T^2\lambda^{3/2}}{\pi}+\ldots \tag{293}$$

Interestingly, besides the first-order term, we have found a term proportional to $\lambda^{3/2}$. We shall see now that this power also appears in the thermodynamic potential, and we shall see that it is related to the infrared divergence of certain diagrams.

Let us start from Eq. (247) and separate the first-order contribution which we already have computed,

$$\ln Z_I = \langle S_I \rangle_0 + \sum_{N=2}^{\infty} \frac{\langle S_I^N \rangle_{0,\text{connected}}}{N!} \,. \tag{294}$$

The second-order diagrams are given in Eqs. (246). We see that the first of these diagrams, Eq. (246a) can be written as



$$= \frac{\Pi_1^2}{12^2} V \sum_n \int \frac{d^3k}{(2\pi)^3} \frac{1}{(\omega_n^2 + k^2)^2} \,. \tag{295}$$

For $n = 0$ the integrand goes like $1/k^2$ which is infrared divergent. One can check that the second diagram, given in Eq. (246b) does not have this property. The solution to this apparent problem is to resum all diagrams of this "dangerous" kind as we did above for the self energy. More precisely, these are the "daisy" diagrams, where N loops are attached to an interior loop (sometimes also called "ring" diagrams). We have

$$\begin{aligned}
\sum_{N=2}^{\infty} \frac{\langle S_I^N \rangle_{0,\text{connected,daisy}}}{N!} &= \sum_{N=2}^{\infty} \frac{1}{N!} 6^N 2^{N-1} \frac{N!}{N} \quad \text{(} N \text{ loops)} \\
&= \sum_{N=2}^{\infty} \frac{1}{N!} 6^N 2^{N-1} \frac{N!}{N} V \sum_n \int \frac{d^3k}{(2\pi)^3} \frac{(-\Pi_1)^N}{12^N} D_0(K)^N \\
&= \frac{V}{2} \sum_n \int \frac{d^3k}{(2\pi)^3} \sum_{N=2}^{\infty} \frac{1}{N} [-\Pi_1 D_0(K)]^N \\
&= -\frac{V}{2} \sum_n \int \frac{d^3k}{(2\pi)^3} \left\{ \ln\left[1 + \Pi_1 D_0(K)\right] - \Pi_1 D_0(K) \right\} \\
&= -\frac{V}{2} \sum_n \int \frac{d^3k}{(2\pi)^3} \left[\ln\left(1 + \frac{\lambda T^2}{\omega_n^2 + k^2}\right) - \frac{\lambda T^2}{\omega_n^2 + k^2} \right]
\end{aligned} \tag{296}$$

The origin of the combinatorical factors is: 6^N for choosing one pair of lines from each of the N crosses; 2^{N-1} for the number of ways to connect the chosen pairs of lines to obtain a ring diagram; $N!/N$ for the number of ways to order the N loops around the ring. Let us again look at the zero Matsubara mode, $n = 0$. All other Matsubara modes give contributions of higher order in λ (remember that naively, i.e., ignoring any infrared divergence, one would expect every single diagram of the sum to be of order λ^2 or higher). With the noninteracting and first-order results (282) and (286) we thus find the pressure

$$P \simeq \frac{\pi^2 T^4}{90} - \frac{\lambda T^4}{48} - \frac{T}{2} \int \frac{d^3k}{(2\pi)^3} \left[\ln\left(1 + \frac{\lambda T^2}{k^2}\right) - \frac{\lambda T^2}{k^2} \right] . \tag{297}$$

The integral is given by

$$\int dk\, k^2 \left[\ln\left(1 + \frac{a^2}{k^2}\right) - \frac{a^2}{k^2} \right] = -\frac{2a^3}{3} \arctan\frac{k}{a} - \frac{a^2 k}{3} + \frac{k^3}{3} \ln\left(1 + \frac{a^2}{k^2}\right) . \tag{298}$$

Therefore, for the upper boundary we need

$$-\frac{2a^3}{3} \arctan\frac{\Lambda}{a} = -\frac{\pi a^3}{3} + \mathcal{O}\left(\frac{1}{\Lambda}\right) , \qquad \frac{\Lambda^3}{3} \ln\left(1 + \frac{a^2}{\Lambda^2}\right) = \frac{a^2 \Lambda}{3} + \mathcal{O}\left(\frac{1}{\Lambda}\right) , \tag{299}$$

and consequently

$$\int_0^{\infty} dk\, k^2 \left[\ln\left(1 + \frac{a^2}{k^2}\right) - \frac{a^2}{k^2} \right] = -\frac{\pi a^3}{3} \,. \tag{300}$$

Therefore,

$$\int \frac{d^3k}{(2\pi)^3}\left[\ln\left(1+\frac{\lambda T^2}{k^2}\right)-\frac{\lambda T^2}{k^2}\right]=-\frac{\lambda^{3/2}T^3}{6\pi}\,, \tag{301}$$

and thus

$$P=\frac{\pi^2T^4}{90}\left[1-\frac{15}{8}\frac{\lambda}{\pi^2}+\frac{15}{2}\left(\frac{\lambda}{\pi^2}\right)^{3/2}+\dots\right]. \tag{302}$$

Consequently, we have found that the next term in the perturbation series of the pressure is not of order λ^2, but of order $\lambda^{3/2}$, as we have seen above for the self-energy.

IX. BOSE-EINSTEIN CONDENSATION OF AN INTERACTING BOSE GAS

A. Spontaneous symmetry breaking and the Goldstone theorem

In Sec. V we have discussed Bose-Einstein condensation of a non-interacting field. Now, having discussed the basics of an interacting theory, we revisit this effect. The inclusion of interactions is not only more realistic for possible applications, but also will give us a better conceptual understanding of Bose-Einstein condensation. In particular, we shall see that Bose-Einstein condensation is an example of spontaneous symmetry breaking and the Goldstone theorem which are extremely important concepts in various fields of theoretical physics. Another consequence of including interactions is that we can compute the condensate for fixed chemical potential, and not only for fixed charge density, as we have done in Sec. V.

We use the same Lagrangian as in Sec. V, see for instance Eq. (95), i.e., we include a chemical potential, but now also include an interaction term as in the previous section,

$$\mathcal{L}=|(\partial_0-i\mu)\varphi|^2-|\nabla\varphi|^2-m^2|\varphi|^2-\lambda|\varphi|^4\,. \tag{303}$$

This Lagrangian is invariant under $U(1)$ rotations $\varphi\to e^{-i\alpha}\varphi$. As in Sec. V we introduce real fields via

$$\varphi=\frac{1}{\sqrt{2}}(\varphi_1+i\varphi_2)\,, \tag{304}$$

which leads to the Lagrangian

$$\mathcal{L}=\frac{1}{2}\left[(\partial_0\varphi_1)^2+(\partial_0\varphi_2)^2-(\nabla\varphi_1)^2-(\nabla\varphi_2)^2+2\mu(\varphi_2\partial_0\varphi_1-\varphi_1\partial_0\varphi_2)+(\mu^2-m^2)(\varphi_1^2+\varphi_2^2)-\frac{\lambda}{2}(\varphi_1^2+\varphi_2^2)^2\right]. \tag{305}$$

As discussed above for the noninteracting case, we separate the zero-mode ϕ_i, allowing for Bose condensation, $\varphi_i\to\varphi_i+\phi_i$ (remember Eq. (96)). Then the Lagrangian becomes

$$\mathcal{L}=-U+\mathcal{L}^{(2)}+\mathcal{L}^{(3)}+\mathcal{L}^{(4)}, \tag{306}$$

with

$$\begin{aligned}
U &= \frac{m^2-\mu^2}{2}(\phi_1^2+\phi_2^2)+\frac{\lambda}{4}(\phi_1^2+\phi_2^2)^2\,, && \text{(307a)}\\
\mathcal{L}^{(2)} &= -\frac{1}{2}\big[-(\partial_0\varphi_1)^2-(\partial_0\varphi_2)^2+(\nabla\varphi_1)^2+(\nabla\varphi_2)^2-2\mu(\varphi_2\partial_0\varphi_1-\varphi_1\partial_0\varphi_2)\\
&\quad +(m^2-\mu^2)(\varphi_1^2+\varphi_2^2)+\lambda(3\phi_1^2+\phi_2^2)\varphi_1^2+\lambda(\phi_1^2+3\phi_2^2)\varphi_2^2+4\lambda\phi_1\phi_2\varphi_1\varphi_2\big]\,, && \text{(307b)}\\
\mathcal{L}^{(3)} &= -\lambda(\phi_1\varphi_1+\phi_2\varphi_2)(\varphi_1^2+\varphi_2^2)\,, && \text{(307c)}\\
\mathcal{L}^{(4)} &= -\frac{\lambda}{4}(\varphi_1^2+\varphi_2^2)^2\,. && \text{(307d)}
\end{aligned}$$

Here we have assumed that the condensate is constant in space and time, i.e., all derivative terms $\partial^\mu\phi_i$ vanish.

We can ignore the terms linear in the fluctuations because of the following

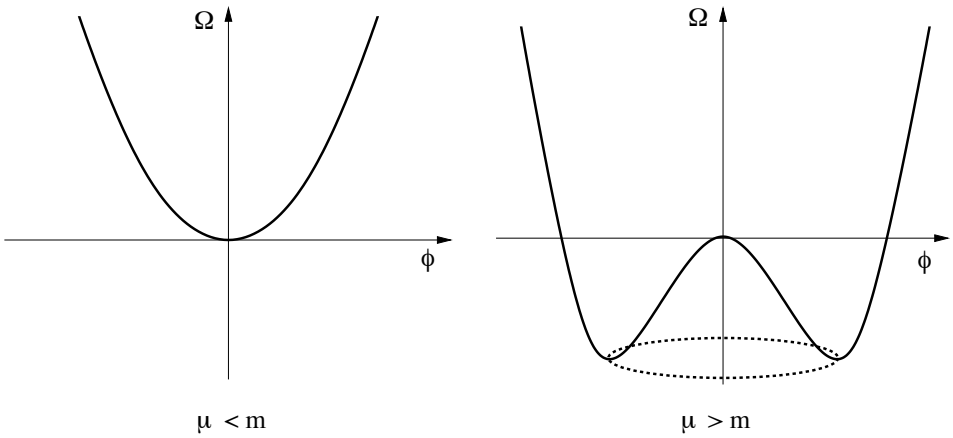


FIG. 3: Illustration of the zero-temperature tree-level potential Ω for $\mu < m$ (left) and $\mu > m$ (right). In the latter case, the order parameter acquires a nonzero value at a fixed, but arbitrary value on the bottom circle of the potential, thus breaking the $U(1)$ symmetry "spontaneously".

Exercise 8*: Show that the contributions linear in the fluctuations vanish after using the classical equations of motion for ϕ_1, ϕ_2. Hint: You need to keep the space-time dependence of ϕ_1, ϕ_2 and drop a boundary term (assuming that the fields vanish at infinity). The classical equations of motion are the equations of motion obtained from U.*

Remarkably, besides the obvious quartic interaction term $\mathcal{L}^{(4)}$, there is an interaction term *cubic* in the fields, induced by the condensate ϕ_i. (Note that here we use the term "condensate" for what is, in the particle physics context, also called "vacuum expectation value" for the field, or shortly "vev".) Both interaction terms will lead to loop corrections as discussed in the previous section. However, let us first discuss the "tree-level" contributions.

Again, for symmetry reasons, we can choose $\phi_2 = 0$ and denote $\phi \equiv \phi_1$. Then, the potential becomes

$$U(\phi^2) = \frac{m^2 - \mu^2}{2}\phi^2 + \frac{\lambda}{4}\phi^4\,, \tag{308}$$

and, following the same steps as in Sec. V, the tree-level propagator is

$$D_0^{-1}(K) = \begin{pmatrix} -K^2 + m^2 + 3\lambda\phi^2 - \mu^2 & -2ik_0\mu \\ 2ik_0\mu & -K^2 + m^2 + \lambda\phi^2 - \mu^2 \end{pmatrix}. \tag{309}$$

Note that D_0 knows about the interaction, although we have only considered the terms quadratic in the fields. This is because of the condensate which appears together with the coupling constant λ. Again following the steps in Sec. V we obtain the tree-level thermodynamic potential,

$$\frac{\Omega}{V} \;=\; U(\phi^2) + \frac{T}{2V}\mathrm{Tr}\ln\frac{D_0^{-1}(K)}{T^2}\,. \tag{310}$$

Firstly, let us discuss the zero-temperature case, $T = 0$. In this case, the thermodynamic potential is simply the potential U,

$$\frac{\Omega(T=0)}{V} = \frac{m^2 - \mu^2}{2}\phi^2 + \frac{\lambda}{4}\phi^4\,. \tag{311}$$

Minimization of Ω with respect to ϕ yields the ground state (corresponding to the state with maximal pressure). For chemical potentials $|\mu| < m$, the minimum is at $\phi = 0$. In accordance with our observation in the noninteracting theory, this means that there is no condensation in this case. One rather needs a negative coefficient in front of the ϕ^2 term for condensation, i.e, $|\mu| > m$ (negative "mass parameter squared" in field-theoretical treatments without chemical potential). In this case, the potential has a "mexican hat" or "bottom of a wine bottle" shape, see Fig. 3 (we consider a repulsive interaction for which $\lambda > 0$; otherwise the potential would be unbounded from below, indicating an unstable system). Remember that we started from a complex field φ, hence the rotationally symmetric wine bottle potential. The minimum is now at $\phi \neq 0$. Such a minimum cannot be invariant under $U(1)$. However, all possible

minima are degenerate, i.e., we are free to choose an arbitrary direction (and have already done this by choosing $\phi = \phi_1$). This mechanism, where the Lagrangian has a symmetry which is not respected by the ground state, is called *spontaneous symmetry breaking*, as already briefly mentioned in Sec. V. The object that breaks the symmetry and which is zero in the symmetric phase (here the condensate ϕ) is called the *order parameter*. Other examples of spontaneous symmetry breaking are:

- ferromagnetism; in this case, the rotational symmetry $SO(3)$ is, by alignment of all microscopic spins and hence a nonzero magnetization, broken down to $U(1)$.
- chiral symmetry breaking; here, the order parameter is a condensate of pairs of a left-handed antiquark and a right-handed quark (or vice versa), and the so-called chiral group $SU(N_f)_L \times SU(N_F)_R$ (with N_f being the number of quark flavors) is broken down to the group of joint left- and right-handed rotations $SU(N_f)_{L+R}$.
- superconductivity; in this case, the order parameter is a condensate of Cooper pairs, breaking the electromagnetic gauge group $U(1)_{\rm em}$
- the Higgs mechanism, where the the Higgs field breaks the electroweak group $SU(2)_I \times U(1)_Y$ of isospin I and hypercharge Y down to the electromagnetic $U(1)_{\rm em}$.

For $\mu^2 > m^2$ we can easily solve the minimization condition

$$\frac{\partial\Omega(T=0)}{\partial\phi} = 0 \quad \Rightarrow \qquad \phi^2 = \frac{\mu^2 - m^2}{\lambda}\,. \tag{312}$$

Additionally, there is of course the trivial solution $\phi = 0$; but, as we easily see, this solution corresponds to a maximum of the free energy and thus to an unstable state.

[End of 11th lecture, Jan 13th, 2014.]

Next, let us discuss the excitations of the system on top of the condensate background. The poles of the tree-level propagator D_0 (i.e., the zeros of the determinant of the inverse propagator D_0^{-1}) correspond to the quasiparticle energies, which we denote by $\epsilon_k^\pm$. They can be used to compute the Tr ln (which is the same as ln det). Defining

$$m_1^2 \equiv m^2 + 3\lambda\phi^2\,, \tag{313a}$$
$$m_2^2 \equiv m^2 + \lambda\phi^2\,, \tag{313b}$$

we obtain (cf. Eq. (100))

$$\begin{aligned}
\ln\det\frac{D_0^{-1}(K)}{T^2} &= \ln\prod_K \frac{1}{T^4}[(-K^2 + m_1^2 - \mu^2)(-K^2 + m_2^2 - \mu^2) - 4\mu^2 k_0^2] \\
&= \ln\prod_K \frac{1}{T^4}[(\epsilon_k^+)^2 - k_0^2][(\epsilon_k^-)^2 - k_0^2] \\
&= \sum_K \left[\ln\frac{(\epsilon_k^+)^2 - k_0^2}{T^2} + \ln\frac{(\epsilon_k^-)^2 - k_0^2}{T^2}\right]\,.
\end{aligned} \tag{314}$$

The quasiparticle energies are

$$\epsilon_k^\pm = \sqrt{E_k^2 + \mu^2 \mp \sqrt{4\mu^2 E_k^2 + \delta M^4}}\,, \tag{315}$$

where we abbreviated

$$E_k \equiv \sqrt{k^2 + M^2}\,, \tag{316}$$

and

$$M^2 \equiv \frac{m_1^2 + m_2^2}{2} = m^2 + 2\lambda\phi^2\,, \tag{317a}$$
$$\delta M^2 \equiv \frac{m_1^2 - m_2^2}{2} = \lambda\phi^2\,. \tag{317b}$$

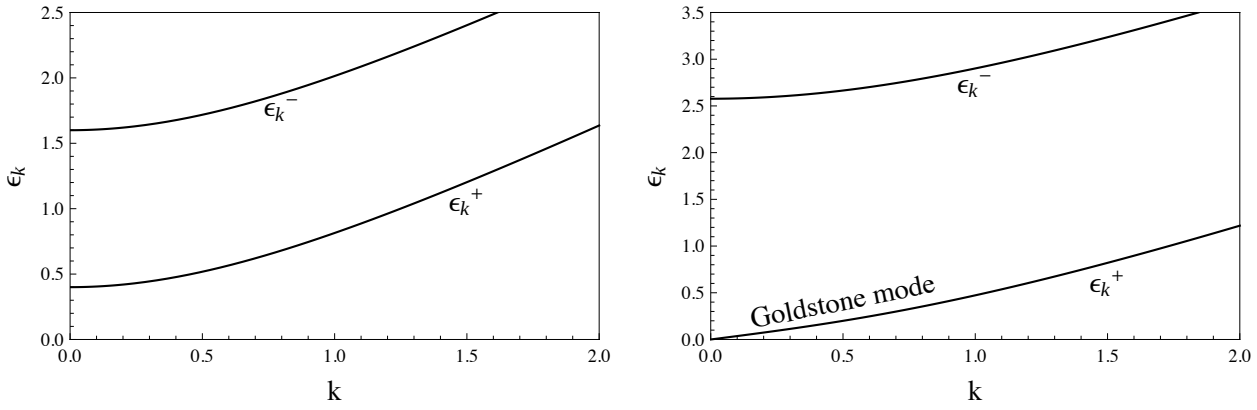


FIG. 4: Dispersions ϵ_k without (left, $\mu < m$) and with (right, $\mu > m$) Bose condensation. In the latter case, one of the modes becomes gapless with a linear behavior for small momenta k, see Eqs. (320) and (321). Such a "Goldstone mode" appears always for a spontaneously broken global symmetry.

We recover the usual particle and antiparticle dispersions when we set the coupling constant to zero, $\lambda = 0$. In this case, we have $M = m$ and $\delta M = 0$, and

$$\epsilon_k^{\pm} = \sqrt{k^2 + m^2} \mp \mu \,. \tag{318}$$

Performing the Matsubara sum in Eq. (314) as usual, inserting the result into the thermodynamic potential (310) and dropping the vacuum contribution yields

$$\frac{\Omega}{V} = \frac{m^2 - \mu^2}{2}\phi^2 + \frac{\lambda}{4}\phi^4 + T\sum_{e=\pm}\int \frac{d^3k}{(2\pi)^3} \ln\left(1 - e^{-\epsilon_k^e/T}\right) . \tag{319}$$

Inserting the solution for the condensate (312) into the definition of M and δM from Eq. (317), we obtain $M^2 = 2\mu^2 - m^2$, $\delta M^2 = \mu^2 - m^2$ and thus

$$\epsilon_k^{\pm} = \sqrt{k^2 + (3\mu^2 - m^2) \mp \sqrt{4\mu^2 k^2 + (3\mu^2 - m^2)^2}} \,. \tag{320}$$

We see that ϵ_k^+ becomes gapless, i.e., $\epsilon_{k=0}^+ = 0$, as shown in Fig. 4. We can expand this mode for small momenta to obtain

$$\epsilon_k^+ \simeq \sqrt{\frac{\mu^2 - m^2}{3\mu^2 - m^2}}\, k = \sqrt{\frac{M^2 - \mu^2}{M^2 + \mu^2}}\, k \,. \tag{321}$$

This gapless mode with linear dispersion relation is called the *Goldstone mode*. Its presence is of great importance since one is very often interested in the low-energy limit of a theory. The Goldstone mode can be excited with arbitrarily small energy. As a consequence, it is populated for arbitrarily small temperature. Moreover, its presence is a very general fact, due to the *Goldstone theorem* which says that in any system with a spontaneously broken global symmetry there is a gapless mode.[7]

The second mode does have an energy gap and behaves quadratically for small k,

$$\epsilon_k^- = \sqrt{2}\sqrt{3\mu^2 - m^2} + \frac{1}{2\sqrt{2}} \frac{5\mu^2 - m^2}{(3\mu^2 - m^2)^{3/2}}\, k^2 \,. \tag{322}$$

[7] Notice the specification *global* symmetry. For a *local* symmetry, for instance in the case of superconductivity, the Goldstone mode is "eaten up" by the gauge fields, giving rise to a magnetic screening mass, the *Meissner mass*. Another example is the Higgs mechanism where the spontaneous breaking of the electroweak group gives rise to the massive $W^{\pm}$ and Z bosons.

B. Symmetry restoration at finite temperature

At a certain temperature, called critical temperature T_c, the symmetry will be restored, i.e., the ground state will be symmetric under the original symmetry group. In our case this means that for $T > T_c$ we have $\phi = 0$ (in the case of a ferromagnet the magnetization vanishes above the critical temperature, there called Curie temperature; in a superconductor, the condensate of Cooper pairs vanishes above T_c etc.). The general treatment of symmetry restoration and the determination of the critical temperature is complicated. Here we shall first discuss the qualitative picture which gives a good physical understanding for the symmetry restoration process. We shall do so with the help of a very simple approximation. We shall also see that this approximation has obvious problems.

For the sake of simplicity let us set $\mu = 0$. This means, in order to have condensation at small temperatures, we need a negative mass parameter squared. Therefore, we introduce the positive square c^2 via

$$m^2 = -c^2 \,. \tag{323}$$

The tree-level propagator is now diagonal,

$$D_0^{-1}(K) = \begin{pmatrix} -K^2 - c^2 + 3\lambda\phi^2 & 0 \\ 0 & -K^2 - c^2 + \lambda\phi^2 \end{pmatrix} , \tag{324}$$

and the excitations energies become

$$\epsilon_k^{\pm} = \sqrt{k^2 + m_{2/1}^2} \,, \qquad m_2^2 = \lambda\phi^2 - c^2 \,, \qquad m_1^2 = 3\lambda\phi^2 - c^2 \,. \tag{325}$$

The potential in Eq. (319) now is

$$\begin{aligned} \frac{\Omega}{V} &= -\frac{c^2}{2}\phi^2 + \frac{\lambda}{4}\phi^4 + T\sum_{e=\pm}\int \frac{d^3k}{(2\pi)^3} \ln\left(1 - e^{-\epsilon_k^e/T}\right) \\ &= -\frac{c^2}{2}\phi^2 + \frac{\lambda}{4}\phi^4 + \frac{T^4}{2\pi^2}\sum_{i=1,2}\int_0^\infty dx\, x^2 \ln\left(1 - e^{-\sqrt{x^2 + (m_i/T)^2}}\right) \\ &\simeq -\frac{c^2}{2}\phi^2 + \frac{\lambda}{4}\phi^4 - \frac{T^4\pi^2}{45} + \frac{T^2(m_1^2 + m_2^2)}{24} \,, \end{aligned} \tag{326}$$

where we have used the high-temperature approximation $T^2 \gg c^2$ and

$$\int_0^\infty dx\, x^2 \ln\left(1 - e^{-\sqrt{x^2+y^2}}\right) = -\frac{\pi^4}{45} + \frac{\pi^2}{12}\, y^2 + \mathcal{O}(y^3) \,, \tag{327}$$

Now, with Eq. (325) this becomes, ordered in powers of ϕ,

$$\frac{\Omega}{V} = \left(-\frac{c^2}{2} + \frac{\lambda T^2}{6}\right)\phi^2 + \frac{\lambda}{4}\phi^4 - \frac{T^4\pi^2}{45} - \frac{c^2T^2}{12} \,. \tag{328}$$

This result shows that the coefficient of the quadratic term becomes larger with increasing temperature until it becomes positive for temperatures larger than the *critical temperature*

$$T_c^2 = \frac{3c^2}{\lambda} \,. \tag{329}$$

This indicates a second-order phase transition to the restored phase. (A first-order phase transition can only occur if there was also a cubic term ϕ^3.) We can compute the condensate as a function of temperature and find

$$\phi^2(T) = \frac{c^2}{\lambda} - \frac{T^2}{3} \quad \Rightarrow \qquad \phi(T) = \frac{T_c}{\sqrt{3}}\sqrt{1 - \frac{T^2}{T_c^2}} \,, \tag{330}$$

for $T < T_c$ and $\phi = 0$ for $T > T_c$.

This approximation has the following severe problem. With Eqs. (325) and (330) we observe that the excitation energies are given by the following masses (for $T < T_c$),

$$m_2^2 = -\frac{\lambda T^2}{3} \,, \tag{331a}$$

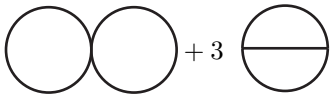
$$m_1^2 = 2c^2 - \lambda T^2 \,. \tag{331b}$$

For $T = 0$ we recover the gapless Goldstone mode and the gapped mode. However, for nonzero temperature, the Goldstone mode acquires an imaginary energy for small momenta which is obviously unphysical. It is no surprise that the applied approximation is incomplete, since we know from the previous section that loops can give corrections to the potential of the order of λT^2. This is exactly the order which was responsible for the symmetry restoration above. Therefore, in the next subsection we improve our result by including loop corrections.

[End of 12th lecture, Jan 20th, 2014.]

C. Including loop corrections

So far we have ignored the contributions from the interaction terms (307c) and (307d) in the Lagrangian. From Sec. VIII A we know that the quartic term gives rise to the "double-bubble" diagram in the thermodynamic potential. The cubic term gives an additional contribution, such that up to two loops we find the contributions (omitting the arrows in the loop diagrams)

$$\ln Z_I^{\text{two loops}} = 3$$  (332)

Note that now each line denotes a 2×2 propagator, given in Eq. (309). Naively, one might say that the second diagram is suppressed compared to the first because of the number of vertices, which each comes with one power of the coupling constant λ. However, there is a condensate sitting on the vertices of the second diagram. We have seen above that the condensate is proportional to the inverse of $\lambda^{1/2}$, and thus the power counting does not work in the usual way. Nevertheless we shall neglect the second diagram for simplicity. The self-energy that arises from this diagram depends on four-momentum (in contrast to the self-energy arising from the first diagram). For an estimate of the self-energy from the second diagram we restrict ourselves to the low-momentum, large-temperature approximation and consider a scalar propagator instead of the full 2×2 structure.

Exercise 9*: Show that the momentum-dependent self-energy*

$$\Sigma(P) \equiv \phi^2 \lambda^2 \frac{T}{V} \sum_K D_0(P-K) D_0(K) \tag{333}$$

with the propagator $D_0(K) = 1/(k_0^2 - \epsilon_k^2)$, $\epsilon_k^2 = k^2 + m^2$, *is given in the* $p_0 = 0$, $\mathbf{p} \to 0$, $T \gg m$ *limit by*

$$\lim_{\mathbf{p}\to 0} \Sigma(0,\mathbf{p}) \simeq \frac{\phi^2 \lambda^2 T}{8\pi m} . \tag{334}$$

(Hint: Use the result from Exercise 1 for the Matsubara sum.)

As we shall see below (and can guess from our results for the real scalar field), the contribution of the first diagram (with quartic interactions) is $\propto \lambda T^2$. We can thus neglect the contribution from the second diagram if the condensate is small, more precisely if $(\mu^2 - m^2)/m \ll T$, i.e., if the chemical potential is only slightly larger than the mass and thus only allows for a small condensate.

To find the result for the first diagram we write the interaction Lagrangian (307d) as

$$\mathcal{L}^{(4)} = -\Lambda_{abcd}\, \varphi_a \varphi_b \varphi_c \varphi_d \,, \qquad a,b,c,d \in \{1,2\} \,, \tag{335}$$

with a symmetrized tensor,

$$\Lambda_{abcd} = \frac{\lambda}{12} (\delta_{ab}\delta_{cd} + \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}) \,. \tag{336}$$

The tensor structure comes from the two components of the complex field φ and replaces the scalar λ for the case of a single scalar field. Consequently,

$$\ln Z_I^{(1)} = -3\Lambda_{abcd} \frac{T}{V} \sum_K D_0^{ab}(K) \sum_Q D_0^{cd}(Q) \,, \tag{337}$$

where D_0^{ab} are the components of the 2×2 propagator

$$D_0(K) = \frac{1}{[(\epsilon_k^+)^2 - k_0^2][(\epsilon_k^-)^2 - k_0^2]} \begin{pmatrix} -K^2 + m_2^2 - \mu^2 & 2ik_0\mu \\ -2ik_0\mu & -K^2 + m_1^2 - \mu^2 \end{pmatrix} . \tag{338}$$

This propagator is obtained from the inverse propagator (309) by using the expression for the inversion of a 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \Rightarrow \qquad A^{-1} = \frac{1}{\det A} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} . \tag{339}$$

Inserting the tensor (336) into Eq. (337) and performing the sum over a, b, c, d yields

$$\ln Z_I^{(1)} = -\frac{\lambda}{4} \frac{T}{V} \left\{ 3 \left[\sum_Q D_0^{11}(Q) \right]^2 + 3 \left[\sum_Q D_0^{22}(Q) \right]^2 + 2 \sum_Q D_0^{11}(Q) \sum_K D_0^{22}(K) \right\} , \tag{340}$$

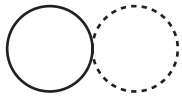
where we have used that the Matsubara sum over the off-diagonal components $D_0^{12} = -D_0^{21}$ vanishes. Therefore we are left with the diagonal components. The self-energy (which, due to the definition (252) now is also a 2×2 matrix) can now be determined from the relation (269). With Eq. (337) the self-energy of first order in λ is

$$\begin{aligned} \Pi^{ab} &= -2 \frac{\delta \ln Z_I^{(1)}}{\delta D_0^{ab}} \\ &= 12 \Lambda_{abcd} \frac{T}{V} \sum_Q D_0^{cd}(Q) \\ &= \lambda(\delta_{ab}\delta_{cd} + \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}) \frac{T}{V} \sum_Q D_0^{cd}(Q) . \end{aligned} \tag{341}$$

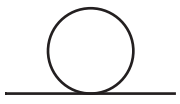
Written as a matrix, this is

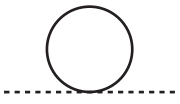
$$\Pi = \lambda \frac{T}{V} \sum_Q \begin{pmatrix} 3D_0^{11}(Q) + D_0^{22}(Q) & 0 \\ 0 & D_0^{11}(Q) + 3D_0^{22}(Q) \end{pmatrix} . \tag{342}$$

This matrix can of course also be obtained from taking the derivatives with respect to D_0^{ab} from the explicit expression (340). Using two different line styles for the two modes 1 and 2 we can bring Eqs. (340) and (342) into a diagrammatic form,

$4 \ln Z_I^{(1)} = 3$  $+ 3$  $+ 2$  , (343)

and

$\Pi^{11} = -3$  $-$  , (344a)

$\Pi^{22} = -3$  $-$  . (344b)

As a check, we see that these self-energy diagrams arise from cutting the respective lines in Eq. (343). If we ignore the second degree of freedom we recover the form of the self-energy (265) for a real scalar field (the different prefactor, 12 vs. 3, is due to the different normalization of the coupling constant in the Lagrangian, λ vs. $\lambda/4$, which originates from the normalization of the real fields φ_1, φ_2 in Eq. (304)).

To evaluate the self-energy from Eq. (342) we again set $\mu = 0$ for simplicity. The result will give us the loop corrections to the masses given in Eq. (331). In the high-temperature approximation, $T \gg m_1, m_2$, the propagators

of the two modes become identical, $D_0^{11} \simeq D_0^{22} \simeq -1/K^2$. Then we can use the results of Sec. VIII C to obtain, after subtracting the vacuum contribution,

$$\lambda \frac{T}{V} \sum_Q D_0^{11}(Q) \simeq \lambda \frac{T}{V} \sum_Q D_0^{22}(Q) \simeq \frac{\lambda T^2}{12} , \tag{345}$$

i.e., $\Pi^{11} \simeq \Pi^{22} \simeq \lambda T^2/3$. Consequently, the corrected masses of the two modes are, for $T < T_c$,

$$m_2^2 + \Pi^{11} = 0 , \tag{346a}$$

$$m_1^2 + \Pi^{22} = 2\left(c^2 - \frac{\lambda T^2}{3}\right) = \frac{2\lambda T_c^2}{3}\left(1 - \frac{T^2}{T_c^2}\right) , \tag{346b}$$

where we have used Eqs. (329) and (331). We see that the Goldstone mode now remains gapless up to $T = T_c$ as it should be. Moreover, the other mode becomes gapless at the phase transition. Then, for $T > T_c$, both masses become identical, $m_1^2 = m_2^2 = -c^2 + \lambda T^2/3 \geq 0$.

The case of a nonvanishing chemical potential is similar. However, in this case, the gapped mode has an energy gap even at $T = T_c$, where the energy is μ (as opposed to 0 in the case without chemical potential). For a self-consistent treatment of this case see for instance Sec. III in Ref. [10].

[End of 13th lecture, Jan 27th, 2014.]

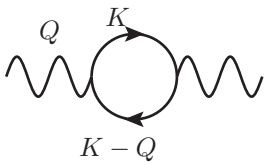
X. THE PHOTON PROPAGATOR IN A QED PLASMA

A. Photon polarization tensor

From Exercise 6 we know the structure of the self-energy diagrams for fermions interacting via a bosonic field through the Yukawa interaction. In QED, discussed in Sec. VII, electrons interact in a similar way with photons, namely through an interaction term

$$\mathcal{L}_I = e\bar{\psi}\gamma^\mu A_\mu \psi , \tag{347}$$

cf. Eq. (182). From this expression we can construct the one-loop photon self-energy

$\Pi_{\mu\nu}(Q) =$ 

$= e^2 \frac{T}{V} \sum_K \mathrm{Tr}[\gamma_\mu G_0(K) \gamma_\nu G_0(P)] ,$ (348)

where the trace is taken over Dirac space, where we have abbreviated

$$P \equiv K - Q , \tag{349}$$

and where the electron propagator is given by Eq. (153). A gauge interaction is of course different from a simple Yukawa interaction. We shall come to the subtleties related to gauge invariance in the next subsection. First we compute the photon self-energy, also called *polarization tensor*. For simplicity, we shall consider the ultrarelativistic limit $m = 0$ and set the chemical potential to zero, $\mu = 0$. Then, the fermion propagator simply becomes

$$G_0(K) = -\frac{\gamma^\mu K_\mu}{K^2} . \tag{350}$$

With the trace

$$\mathrm{Tr}[\gamma_\mu \gamma_\sigma \gamma_\nu \gamma_\rho] = 4(g_{\mu\sigma} g_{\nu\rho} + g_{\mu\rho} g_{\sigma\nu} - g_{\mu\nu} g_{\sigma\rho}) , \tag{351}$$

where $g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$ is the metric tensor, we obtain

$$\mathrm{Tr}[\gamma_\mu \gamma^\sigma K_\sigma \gamma_\nu \gamma^\rho P_\rho] = 4(K_\mu P_\nu + K_\nu P_\mu - g_{\mu\nu} K \cdot P) . \tag{352}$$

In the following we need to treat the following three cases separately,

$$\Pi_{00}(Q) \;=\; 4e^2\frac{T}{V}\sum_K \frac{k_0p_0+\mathbf{k}\cdot\mathbf{p}}{(k_0^2-k^2)(p_0^2-p^2)}\,, \tag{353a}$$

$$\Pi_{0i}(Q) \;=\; \Pi_{i0}(Q) = 4e^2\frac{T}{V}\sum_K \frac{k_0p_i+k_ip_0}{(k_0^2-k^2)(p_0^2-p^2)}\,, \tag{353b}$$

$$\Pi_{ij}(Q) \;=\; 4e^2\frac{T}{V}\sum_K \frac{k_ip_j+k_jp_i+\delta_{ij}(k_0p_0-\mathbf{k}\cdot\mathbf{p})}{(k_0^2-k^2)(p_0^2-p^2)}\,. \tag{353c}$$

The Matsubara sum over k_0 is taken over fermionic Matsubara frequencies since the loop is an electron loop. Since the external momentum Q belongs to the photon, the Matsubara frequencies in q_0 are bosonic. Therefore we can use the result from Exercise 4. With the relations

$$1-f_F(\epsilon_1)-f_F(\epsilon_2) \;=\; \frac{f_F(\epsilon_1)f_F(\epsilon_2)}{f_B(\epsilon_1+\epsilon_2)} = -\frac{f_F(-\epsilon_1)f_F(-\epsilon_2)}{f_B(-\epsilon_1-\epsilon_2)}\,, \tag{354a}$$

$$f_F(\epsilon_1)-f_F(\epsilon_2) \;=\; \frac{f_F(-\epsilon_1)f_F(\epsilon_2)}{f_B(-\epsilon_1+\epsilon_2)} = -\frac{f_F(\epsilon_1)f_F(-\epsilon_2)}{f_B(\epsilon_1-\epsilon_2)}\,, \tag{354b}$$

we can write Eq. (168) as (changing q_0 to $-q_0$)

$$\begin{aligned} T\sum_{k_0}\frac{(k_0+\xi_1)(k_0-q_0+\xi_2)}{(k_0^2-\epsilon_1^2)[(k_0-q_0)^2-\epsilon_2^2]} \;=\; \frac{1}{4\epsilon_1\epsilon_2}\Bigg\{&\left[\frac{(\epsilon_1-\xi_1)(\epsilon_2-\xi_2)}{q_0+\epsilon_1-\epsilon_2}-\frac{(\epsilon_1+\xi_1)(\epsilon_2+\xi_2)}{q_0-\epsilon_1+\epsilon_2}\right][f(\epsilon_1)-f(\epsilon_2)]\\ &+\left[\frac{(\epsilon_1+\xi_1)(\epsilon_2-\xi_2)}{q_0-\epsilon_1-\epsilon_2}-\frac{(\epsilon_1-\xi_1)(\epsilon_2+\xi_2)}{q_0+\epsilon_1+\epsilon_2}\right][1-f(\epsilon_1)-f(\epsilon_2)]\Bigg\}\,. \end{aligned} \tag{355}$$

Since eventually only fermionic distribution functions occur we have abbreviated $f\equiv f_F$. Consequently, since the dispersions in our massless approximation are $\epsilon_k=k$, Eqs. (353) become

$$\begin{aligned} \Pi_{00} \;=\; e^2\int\frac{d^3k}{(2\pi)^3}\Bigg[&\left(\frac{1}{q_0+k-p}-\frac{1}{q_0-k+p}\right)(1+\hat{\mathbf{k}}\cdot\hat{\mathbf{p}})(f_k-f_p)\\ &+\left(\frac{1}{q_0-k-p}-\frac{1}{q_0+k+p}\right)(1-\hat{\mathbf{k}}\cdot\hat{\mathbf{p}})(1-f_k-f_p)\Bigg]\,, \end{aligned} \tag{356a}$$

$$\begin{aligned} \Pi_{0i} \;=\; -e^2\int\frac{d^3k}{(2\pi)^3}\Bigg[&\left(\frac{1}{q_0+k-p}+\frac{1}{q_0-k+p}\right)(\hat{p}_i+\hat{k}_i)(f_k-f_p)\\ &+\left(\frac{1}{q_0-k-p}+\frac{1}{q_0+k+p}\right)(\hat{p}_i-\hat{k}_i)(1-f_k-f_p)\Bigg]\,, \end{aligned} \tag{356b}$$

$$\begin{aligned} \Pi_{ij} \;=\; e^2\int\frac{d^3k}{(2\pi)^3}\Bigg[&\left(\frac{1}{q_0+k-p}-\frac{1}{q_0-k+p}\right)[\delta_{ij}(1-\hat{\mathbf{k}}\cdot\hat{\mathbf{p}})+\hat{k}_i\hat{p}_j+\hat{k}_j\hat{p}_i](f_k-f_p)\\ &+\left(\frac{1}{q_0-k-p}-\frac{1}{q_0+k+p}\right)[\delta_{ij}(1+\hat{\mathbf{k}}\cdot\hat{\mathbf{p}})-\hat{k}_i\hat{p}_j-\hat{k}_j\hat{p}_i](1-f_k-f_p)\Bigg]\,. \end{aligned} \tag{356c}$$

where we have abbreviated $f_k\equiv f(k)$ and the unit vector $\hat{\mathbf{k}}\equiv\mathbf{k}/k$. We now apply the so-called "Hard Thermal Loop (HTL)" approximation [11], where the dominant contribution comes from fermion momenta $k\sim T$ (called "hard"), and the photon momentum is of the order of $q_0,q\sim eT$ (called "soft"). In this spirit, we approximate the denominators in Eqs. (356) as (remember $p=|\mathbf{k}-\mathbf{q}|$)

$$q_0\pm k\mp p\simeq q_0\pm\mathbf{q}\cdot\hat{\mathbf{k}}\,,\qquad q_0\pm k\pm p\simeq\pm 2k\,, \tag{357}$$

and the distribution functions as

$$f_k - f_p \simeq \mathbf{q}\cdot\hat{\mathbf{k}}\,\frac{\partial f_k}{\partial k}\,, \qquad 1 - f_k - f_p \simeq 1 - 2f_k\,. \tag{358}$$

Now, to compute Π_{00} first note that $\hat{\mathbf{k}}\cdot\hat{\mathbf{p}} = 1 + \mathcal{O}(q^2/k^2)$, hence we can neglect the second line in Eq. (356a). Thus,

$$\begin{aligned}\Pi_{00} &\simeq 2e^2\int\frac{d^3k}{(2\pi)^3}\left(\frac{1}{q_0+\mathbf{q}\cdot\hat{\mathbf{k}}}-\frac{1}{q_0-\mathbf{q}\cdot\hat{\mathbf{k}}}\right)\mathbf{q}\cdot\hat{\mathbf{k}}\,\frac{\partial f_k}{\partial k}\\ &= 4e^2\int\frac{d^3k}{(2\pi)^3}\frac{\mathbf{q}\cdot\hat{\mathbf{k}}}{q_0+\mathbf{q}\cdot\hat{\mathbf{k}}}\frac{\partial f_k}{\partial k}\\ &= 4e^2\int\frac{d^3k}{(2\pi)^3}\left(1-\frac{q_0}{q_0+\mathbf{q}\cdot\hat{\mathbf{k}}}\right)\frac{\partial f_k}{\partial k}\,.\end{aligned} \tag{359}$$

In the second step we have used that the angular integral gives the same result for both terms: with θ being the angle between $\hat{\mathbf{k}}$ and $\hat{\mathbf{q}}$, and $x \equiv \cos\theta$, one changes the integration variable $x \to -x$ in the first term to obtain the second term. Hence we can drop the second term and obtain a factor 2. In the third step we have separated the $q_0 = 0$ part.[8]

To compute Π_{i0} in the HTL approximation, we can again neglect the second line in Eq. (356b) since $\hat{p}_i - \hat{k}_i$ is of the order of q/k. The remaining term, being of order one, is

$$\begin{aligned}\Pi_{i0} &\simeq -2e^2\int\frac{d^3k}{(2\pi)^3}\left(\frac{1}{q_0+\mathbf{q}\cdot\hat{\mathbf{k}}}+\frac{1}{q_0-\mathbf{q}\cdot\hat{\mathbf{k}}}\right)\mathbf{q}\cdot\hat{\mathbf{k}}\,\hat{k}_i\,\frac{\partial f_k}{\partial k}\\ &= 4e^2\int\frac{d^3k}{(2\pi)^3}\frac{q_0}{q_0+\mathbf{q}\cdot\hat{\mathbf{k}}}\,\hat{k}_i\,\frac{\partial f_k}{\partial k}\,.\end{aligned} \tag{360}$$

In Π_{ij}, the second line cannot be neglected. The first term is analogous to the first term in Π_{00} and we obtain

$$\Pi_{ij} = 2e^2\int\frac{d^3k}{(2\pi)^3}\left[2\left(1-\frac{q_0}{q_0+\mathbf{q}\cdot\hat{\mathbf{k}}}\right)\hat{k}_i\hat{k}_j\,\frac{\partial f_k}{\partial k} - \frac{1}{k}(\delta_{ij}-\hat{k}_i\hat{k}_j)(1-2f_k)\right]. \tag{361}$$

The 1 in the factor $1 - 2f_k$ is the vacuum contribution which has to be subtracted as discussed for the case of the self-energy in ϕ^4 theory. Then, with

$$\int\frac{d\Omega}{4\pi}\hat{k}_i\hat{k}_j = \frac{\delta_{ij}}{3}\,, \tag{362}$$

we see via partial integration that only one term in Eq. (361) survives,

$$\int\frac{d^3k}{(2\pi)^3}\left[\hat{k}_i\hat{k}_j\frac{\partial f_k}{\partial k}+\frac{1}{k}(\delta_{ij}-\hat{k}_i\hat{k}_j)f_k\right] = \frac{\delta_{ij}}{6\pi^2}\int_0^\infty dk\left(k^2\frac{\partial f_k}{\partial k}+2kf_k\right) = 0\,. \tag{363}$$

Consequently, we are left with

$$\Pi_{ij} = -4e^2\int\frac{d^3k}{(2\pi)^3}\frac{q_0}{q_0+\mathbf{q}\cdot\hat{\mathbf{k}}}\hat{k}_i\hat{k}_j\frac{\partial f_k}{\partial k}\,. \tag{364}$$

In all three results (359), (360), and (364) we have the same k integral which we can perform exactly,

$$\int_0^\infty dk\,k^2\frac{\partial f_k}{\partial k} = -\frac{T^2}{2}\int_0^\infty dx\frac{x^2}{1+\cosh x} = -\frac{T^2\pi^2}{6}\,. \tag{365}$$

---

[8] Note that the limits $q_0 \to 0$, $q \to 0$ do not commute. We shall see in the subsequent sections that different limits correspond to different physics.

Consequently, we have

$$\Pi_{00} = -2m^2 \left(1 - \int \frac{d\Omega}{4\pi} \frac{q_0}{q_0 + \mathbf{q}\cdot\hat{\mathbf{k}}}\right), \tag{366a}$$

$$\Pi_{0i} = -2m^2 \int \frac{d\Omega}{4\pi} \frac{q_0 \hat{k}_i}{q_0 + \mathbf{q}\cdot\hat{\mathbf{k}}}, \tag{366b}$$

$$\Pi_{ij} = 2m^2 \int \frac{d\Omega}{4\pi} \frac{q_0 \hat{k}_i \hat{k}_j}{q_0 + \mathbf{q}\cdot\hat{\mathbf{k}}}, \tag{366c}$$

where we defined

$$m^2 \equiv \frac{e^2 T^2}{6}. \tag{367}$$

B. Photon propagator

After having computed the polarization tensor, we want to use the result to determine the photon modes in the plasma. To this end, we need the photon propagator $D(Q)$. In a covariant gauge $\partial_\mu A^\mu = 0$ with gauge-fixing parameter ρ the free inverse propagator is

$$D^{-1}_{0,\mu\nu}(Q) = Q^2 g_{\mu\nu} - \left(1 - \frac{1}{\rho}\right) Q_\mu Q_\nu. \tag{368}$$

(Cf. the inverse propagator in axial gauge in Eq. (214).) Physical quantities must of course be independent of ρ. Inversion gives

$$D_{0,\mu\nu}(Q) = \frac{g_{\mu\nu}}{Q^4} - (1-\rho)\frac{Q_\mu Q_\nu}{Q^4}. \tag{369}$$

(One can easily check that $D_0^{\mu\nu} D^{-1}_{0,\nu\sigma} = g^\mu{}_\sigma = \delta^\mu_\sigma$.) Let us now introduce projection operators P_L, P_T via

$$P_T^{00} = P_T^{0i} = P_T^{i0} = 0, \tag{370a}$$

$$P_T^{ij} = \delta^{ij} - \hat{q}^i \hat{q}^j, \tag{370b}$$

and

$$P_L^{\mu\nu} = \frac{Q^\mu Q^\nu}{Q^2} - g^{\mu\nu} - P_T^{\mu\nu}. \tag{371}$$

Both P_T and P_L are 4-transverse to Q, i.e., $Q_\mu P_L^{\mu\nu} = Q_\mu P_T^{\mu\nu} = 0$. The projector P_T is 3-transverse, while P_L is 3-longitudinal. We have $P_L^{\mu\nu} P_{L,\nu\sigma} = -P^\mu_{L,\sigma}$, $P_T^{\mu\nu} P_{T,\nu\sigma} = -P^\mu_{T,\sigma}$, $P_L^{\mu\nu} P_{T,\nu\sigma} = P_T^{\mu\nu} P_{L,\nu\sigma} = 0$. In terms of these projectors, the photon self-energy can be written as

$$\Pi_{\mu\nu}(Q) = F(Q) P_{L,\mu\nu} + G(Q) P_{T,\mu\nu}, \tag{372}$$

with scalar functions F and G. This follows from rotational invariance and the tranversality property of the self-energy $Q_\mu \Pi^{\mu\nu} = 0$ (in non-abelian gauge theories, the structure of the self-energy is more complicated). For the full inverse propagator we then find

$$D^{-1}_{\mu\nu} = D^{-1}_{0,\mu\nu} + \Pi_{\mu\nu} = (F - Q^2) P_{L,\mu\nu} + (G - Q^2) P_{T,\mu\nu} + \frac{Q_\mu Q_\nu}{\rho}, \tag{373}$$

such that

$$D_{\mu\nu}(Q) = \frac{P_{L,\mu\nu}}{F(Q) - Q^2} + \frac{P_{T,\mu\nu}}{G(Q) - Q^2} + \rho \frac{Q_\mu Q_\nu}{Q^4}. \tag{374}$$

(Again, one checks $D^{-1}_{\mu\nu} D^{\nu\sigma} = g_\mu{}^\sigma$.) To obtain an explicit form for F we may for instance consider the (00)-component of Eq. (372), which yields[9]

$$F(Q) = \frac{Q^2}{q^2} \Pi_{00}(Q) \,. \tag{375}$$

To obtain G, we multiply Eq. (372) with $P_T^{\sigma\mu}$ and take the $\sigma = i, \nu = j$ component. This yields

$$G(Q) = \frac{1}{2}(\delta_{jk} - \hat{q}_j \hat{q}_k)\Pi_{kj} = \Pi_{xx}(Q) \,, \tag{376}$$

where, in the second step, we have chosen a reference frame in which $\mathbf{q}$ points into the z-direction, in which case we also have $\Pi_{xx} = \Pi_{yy}$.

We can now use our results from the previous subsection. With Eq. (366a) we find

$$\begin{aligned} F(Q) &= -2m^2 \frac{Q^2}{q^2} \left(1 - \frac{1}{2} \int_{-1}^{1} dx \frac{q_0}{q_0 + qx} \right) \\ &= -2m^2 \frac{Q^2}{q^2} \left(1 - \frac{q_0}{2q} \ln \frac{q_0 + q}{q_0 - q} \right) , \end{aligned} \tag{377}$$

and with Eq. (366c)

$$\begin{aligned} G(Q) &= \frac{m^2}{2\pi} \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \, \sin\theta \frac{q_0 \sin^2\theta \cos^2\varphi}{q_0 + q\cos\theta} \\ &= \frac{m^2 q_0}{2} \int_{-1}^{1} dx \, \frac{1 - x^2}{q_0 + qx} \\ &= \frac{m^2 q_0}{q} \left(\frac{q_0}{q} - \frac{Q^2}{2q^2} \ln \frac{q_0 + q}{q_0 - q} \right) \\ &= m^2 - \frac{1}{2} F(Q) \,. \end{aligned} \tag{378}$$

In both functions F and G the Legendre function of the second kind

$$Q_0(x) = \frac{1}{2} \ln \frac{x+1}{x-1} \tag{379}$$

appears. This function is defined in the complex plane, cut from -1 to 1. Consequently, $F(Q)$ and $G(Q)$ are defined in the complex q_0 plane, cut from $-q$ to q. For timelike $Q^2 > 0$ (q_0 real), F and G are real, while for spacelike $Q^2 < 0$ they become complex.

C. Debye screening

In order to discuss the physical meaning of the photon propagator with the functions F and G, we first consider a static point charge $\mathcal{Q}$ in the plasma. The resulting potential in position space then is given by the function F (see chapter 6.3 in Ref. [2] for more details)

$$V(r) = \mathcal{Q} \int \frac{d^3q}{(2\pi)^3} \frac{e^{i\mathbf{q}\cdot\mathbf{r}}}{q^2 + F(q_0 = 0, \mathbf{q})} \,. \tag{380}$$

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[9] We may also express F in terms of Π_{0i} by multiplying both sides of Eq. (372) with $P_L^{\sigma\mu}$ and take the $\sigma = i, \nu = 0$ component of the resulting equation. After a few lines of algebra one finds $F(Q) = \frac{Q^2}{q q_0} \hat{q}_j \Pi_{j0}$.

For $F = 0$ we recover the usual Coulomb potential $V(r) = \mathcal{Q}/r$. The effect of F is to screen this potential,

$$V(r) = \frac{\mathcal{Q}}{r} e^{-r m_D} \,, \tag{381}$$

with the so-called *Debye screening mass*

$$m_D^2 \equiv F(q_0 = 0, q \to 0) = -\Pi_{00}(q_0 = 0, q \to 0) = \frac{e^2 T^2}{3} \,. \tag{382}$$

This screening is easy to understand: since we are at finite temperature, there are electrons end positrons in the system. Say $\mathcal{Q}$ is a negative charge, then positrons will be attracted and screen the charge. They do so on a length scale given by the inverse Debye mass. In other words, looking from far away (from distances much larger than m_D^{-1}) one cannot see the charge. One has to come closer (up to distances $\sim m_D^{-1}$ or closer) to resolve the charge.

There is no magnetic screening mass, $G(q_0 = 0, q \to 0) = 0$. The magnetic screening mass becomes non-vanishing in a superconductor ($\to$ *Meissner mass*), where magnetic fields are screened.

D. Plasma oscillations and Landau damping

Next we discuss the collective excitations in the plasma. (See, besides the textbooks by LeBellac and Kapusta, Ref. [12] for details.) To this end, we consider the *spectral density*, given by

$$\rho_{L,T}(Q) \equiv \frac{1}{\pi} \mathrm{Im}\, D_{L,T}(Q) \,, \tag{383}$$

where the longitudinal and transverse parts of the photon propagator are defined as

$$D_L(Q) \;=\; \frac{Q^2}{q^2} \frac{1}{F(Q) - Q^2} = \frac{1}{\Pi_{00}(Q) - q^2} \,, \tag{384a}$$

$$D_T(Q) \;=\; \frac{1}{G(Q) - Q^2} \,, \tag{384b}$$

where we have used Eq. (375). The imaginary part in Eq. (383) has to be understood as the imaginary part of the retarded propagator, $\lim_{\epsilon \to 0} \mathrm{Im}\, D_{L,T}(q_0 + i\epsilon, \mathbf{q})$, $\epsilon > 0$.

Expanding

$$\Pi_{00}(q_0 + i\epsilon, \mathbf{q}) \simeq \Pi_{00}(q_0, \mathbf{q}) + i\epsilon \partial \Pi_{00}(q_0, \mathbf{q}) - \frac{\epsilon^2}{2} \partial^2 \Pi_{00}(q_0, \mathbf{q}) \,, \tag{385}$$

(with the abbreviation $\partial \equiv \frac{\partial}{\partial q_0}$), we can write

$$\begin{aligned} \mathrm{Im}\, D_L(q_0 + i\epsilon, \mathbf{q}) \;\simeq\; & -\Big[\mathrm{Im}\, \Pi_{00} + \epsilon \mathrm{Re}\, \partial \Pi_{00} - \frac{\epsilon^2}{2} \mathrm{Im}\, \partial^2 \Pi_{00} \Big] \\ & \times \Big\{ (\mathrm{Re}\, \Pi_{00} - q^2)^2 + (\mathrm{Im}\, \Pi_{00})^2 + 2\epsilon [\mathrm{Im}\, \Pi_{00}\, \mathrm{Re}\, \partial \Pi_{00} - (\mathrm{Re}\, \Pi_{00} - q^2) \mathrm{Im}\, \partial \Pi_{00}] \\ & + \epsilon^2 [(\mathrm{Re}\, \partial \Pi_{00})^2 + (\mathrm{Im}\, \partial \Pi_{00})^2 - (\mathrm{Re}\, \Pi_{00} - q^2) \mathrm{Re}\, \partial^2 \Pi_{00} - \mathrm{Im}\, \Pi_{00} \mathrm{Im}\, \partial^2\, \Pi_{00}] \Big\}^{-1} \,. \end{aligned} \tag{386}$$

As discussed above, the imaginary parts of F and G are nonzero if and only if the four-momentum Q is spacelike, $q^2 - q_0^2 > 0$. In this case, the limit $\epsilon \to 0$ can be taken without any subtleties: all terms proportional to any order of ϵ can simply be discarded, and we obtain

$$q^2 - q_0^2 > 0 : \qquad \lim_{\epsilon \to 0} \frac{1}{\pi} \mathrm{Im}\, D_L(q_0 + i\epsilon, \mathbf{q}) \;=\; -\frac{1}{\pi} \frac{\mathrm{Im}\, \Pi_{00}}{(\mathrm{Re}\, \Pi_{00} - q^2)^2 + (\mathrm{Im}\, \Pi_{00})^2} \,. \tag{387}$$

This is different for timelike momenta, $q_0^2 - q^2 > 0$, where the imaginary part of Π_{00} vanishes. In this case, we have

$$\begin{aligned} q_0^2 - q^2 > 0 : \qquad \lim_{\epsilon \to 0} \frac{1}{\pi} \mathrm{Im}\, D_L(q_0 + i\epsilon, \mathbf{q}) \;&=\; -\lim_{\epsilon \to 0} \frac{1}{\pi} \frac{\epsilon \partial \Pi_{00}}{(\Pi_{00} - q^2)^2 + \epsilon^2 [(\partial \Pi_{00})^2 - (\Pi_{00} - q^2) \partial^2 \Pi_{00}]} \\ &=\; -\frac{1}{\partial \Pi_{00}} \delta \left(\frac{\Pi_{00} - q^2}{\partial \Pi_{00}} \right) \\ &=\; -\mathrm{sgn}(\partial \Pi_{00})\, \delta(\Pi_{00} - q^2) \,, \end{aligned} \tag{388}$$

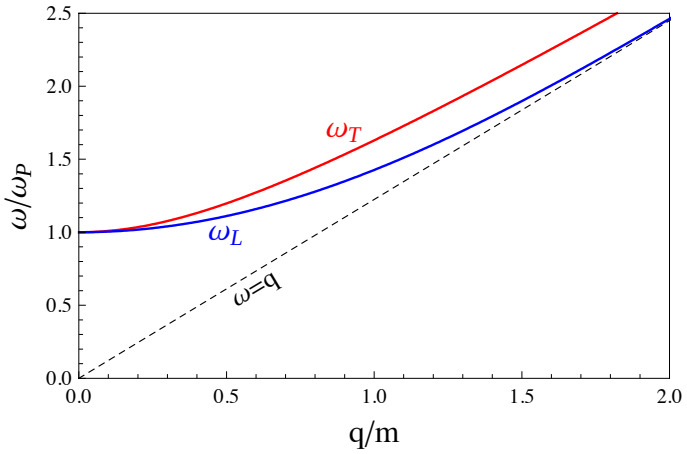

FIG. 5: Longitudinal (plasmon) and transverse modes in a QED plasma. $\omega_P \equiv eT/3$, $m^2 \equiv e^2T^2/6$.

where we have used

$$\lim_{\epsilon\to 0}\frac{1}{\pi}\frac{\epsilon}{x^2+\epsilon^2} = \delta(x)\,. \tag{389}$$

For the transverse part we proceed analogously. As a result, the longitudinal and transverse parts of the spectral function are

$$\rho_L(Q) \;=\; -\mathrm{sgn}\left(\frac{\partial\Pi_{00}}{\partial q_0}\right)\,\delta(\Pi_{00}-q^2) - \Theta(q^2-q_0^2)\frac{1}{\pi}\frac{\mathrm{Im}\,\Pi_{00}}{(\mathrm{Re}\,\Pi_{00}-q^2)^2+(\mathrm{Im}\,\Pi_{00})^2}\,, \tag{390a}$$

$$\rho_T(Q) \;=\; -\mathrm{sgn}\left(\frac{\partial(G-q_0^2)}{\partial q_0}\right)\,\delta(G-Q^2) - \Theta(q^2-q_0^2)\frac{1}{\pi}\frac{\mathrm{Im}\,G}{(\mathrm{Re}\,G-Q^2)^2+(\mathrm{Im}\,G)^2}\,. \tag{390b}$$

Since Π_{00} and G are both even in q_0 we can write

$$\Pi_{00}-q^2 = (q_0-\omega_L)(q_0+\omega_L)\,, \qquad G-Q^2 = (q_0-\omega_T)(q_0+\omega_T)\,. \tag{391}$$

The zeros $\omega_{L,T}(q)$ have to be determined numerically, see Fig. 5. They correspond to quasiparticle dispersions. We see that besides the transverse photon there is another, longitudinal degree of freedom. This is due to the plasma in which the photon propagates, hence the name *plasmon* for this quasiparticle.

We can rewrite the quasiparticle contribution to the spectral density with the help of the general formula

$$\delta[f(x)] = \sum_{x_0}\frac{\delta(x-x_0)}{|f'(x_0)|}\,, \tag{392}$$

where x_0 are the zeros of the function $f(x)$. We thus first compute

$$\frac{\partial\Pi_{00}}{\partial q_0} = \frac{1}{q_0}\left(\Pi_{00}-\frac{3\omega_P^2q^2}{Q^2}\right) = \frac{1}{q_0}\left(q_0^2-\omega_L^2+q^2-\frac{3\omega_P^2q^2}{Q^2}\right)\,, \tag{393}$$

where we have defined the plasma frequency

$$\omega_P^2 = \frac{2}{3}m^2 = \frac{e^2T^2}{9}\,. \tag{394}$$

This yields

$$\left.\frac{\partial\Pi_{00}}{\partial q_0}\right|_{q_0=\pm\omega_L} = \mp\frac{q^2(3\omega_P^2-\omega_L^2+q^2)}{\omega_L(\omega_L^2-q^2)}\,. \tag{395}$$

One can check that the result is negative (positive) for the upper (lower) sign for all q.

And, analogously, for the transverse component,

$$\frac{\partial(G-q_0^2)}{\partial q_0} = \frac{1}{q_0}\left[\frac{(G-m^2)(3q_0^2-q^2)}{Q^2} - 2q_0^2 + m^2\right], \tag{396}$$

and thus

$$\left.\frac{\partial(G-q_0^2)}{\partial q_0}\right|_{q_0=\pm\omega_T} = \mp\frac{3\omega_T^2\omega_P^2 - (\omega_T^2-q^2)^2}{\omega_T(\omega_T^2-q^2)}\,. \tag{397}$$

Consequently, the spectral densities from Eqs. (390) become

$$\rho_L(Q) = \frac{\omega_L(\omega_L^2-q^2)}{q^2(3\omega_P^2+q^2-\omega_L^2)}[\delta(q_0-\omega_L)-\delta(q_0+\omega_L)] - \frac{1}{\pi}\frac{\Theta(q^2-q_0^2)\,\mathrm{Im}\,\Pi_{00}}{(\mathrm{Re}\,\Pi_{00}-q^2)^2+(\mathrm{Im}\,\Pi_{00})^2}\,, \tag{398a}$$

$$\rho_T(Q) = \frac{\omega_T(\omega_T^2-q^2)}{3\omega_P^2\omega_T^2-(\omega_T^2-q^2)^2}[\delta(q_0-\omega_T)-\delta(q_0+\omega_T)] - \frac{1}{\pi}\frac{\Theta(q^2-q_0^2)\,\mathrm{Im}\,G}{(\mathrm{Re}\,G-Q^2)^2+(\mathrm{Im}\,G)^2}\,. \tag{398b}$$

The spacelike part of the spectral functions describes *Landau damping*. This is related to scattering processes of the photon off electrons and positrons in the plasma. Through these processes energy is dissipated, i.e., the photon is "damped".

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