Heavy-ion collisions in the QCD phase diagram
Jean-Yves Ollitrault
June 27, 2022

**Solution of the hands-on session on:
Boosted thermal distributions**

Ideal gas at rest

1. For massless particles, $p^0 = p$. Integrating over solid angle, one replaces $\int d^3p = 4\pi p^2 dp$. One obtains

$$\frac{dN}{d^3x} = \frac{4\pi}{(2\pi)^3}\int_0^\infty p^2 \exp(-p/T)dT = \frac{1}{\pi^2}T^3. \tag{1}$$

2. In the limit $m \gg T$, we expand $p^0 \simeq m + \frac{1}{2m}p^2$, and we still use $\int d^3p = 4\pi p^2 dp$ for the integration over solid angle:

$$\frac{dN}{d^3x} = \frac{4\pi}{(2\pi)^3}\exp\left(-\frac{m}{T}\right)\int_0^\infty p^2 \exp\left(-\frac{p^2}{2mT}\right)dp. \tag{2}$$

We then define the dimensionless integration variable x by $p = x(mT)^{1/2}$. We obtain

$$\frac{dN}{d^3x} = \frac{4\pi}{(2\pi)^3}\exp\left(-\frac{m}{T}\right)(mT)^{3/2}\int_0^\infty x^2 \exp\left(\frac{x^2}{2}\right)dx \propto (mT)^{3/2}\exp\left(-\frac{m}{T}\right). \tag{3}$$

3. The above approximation is best for the heaviest particles. We can use it for protons, with mass m_p, and deuterons, with mass $m_d \sim 2m_p$. For each of these species, we must multiply Eq. (3) with the number of spin states $2S+1$, with $S = \frac{1}{2}$ for protons and $S = 1$ for deuterons. The volume d^3x cancels by taking the ratio N_d/N_p, which only depends on T. One obtains after simplifications

$$\frac{N_d}{N_p} = \frac{3}{2}2^{3/2}\exp\left(-\frac{m_p}{T}\right). \tag{4}$$

ALICE measures (Fig.16 of `https://arxiv.org/pdf/1506.08951.pdf`) $N_d/N_p \simeq 0.0036$ in Pb+Pb collisions at 2.76 TeV. Inverting the above equation with $m_p = 938$ MeV, one obtains $T = 133$ MeV.

A more careful investigation must take into account that roughly half of the observed protons are decay products of, typically, Δ resonances. Hence, the number of "primordial" protons, as given by our statistical model, is smaller than the measured number (see Fig. 2 of `https://arxiv.org/pdf/1710.09425.pdf` for an illustration). Taking this effect into account results in a temperature higher by $\sim 15\%$.

Boosted ideal gas

4. dN/d^3xd^3p is maximum when $p^\mu u_\mu$ is minimum. We locate the minimum by imposing that the derivative with respect to $\mathbf{p}$ vanishes:

$$\frac{\partial}{\partial \mathbf{p}}(p^0u^0 - \mathbf{p}\cdot\mathbf{u}) = u^0\frac{\partial p^0}{\partial \mathbf{p}} - \mathbf{u} = 0. \tag{5}$$

The derivative of the energy with respect to momentum is the velocity, according to Hamilton's equation: $\partial p^0/\partial \mathbf{p} = \mathbf{v} = \mathbf{p}/p^0$. We thus obtain

$$\frac{\mathbf{p}}{p^0} = \frac{\mathbf{u}}{u^0}. \tag{6}$$

The most probable value of the momentum is that for which the particle velocity is equal to the fluid velocity. In other terms, the particle is at rest in the rest frame of the fluid. Then, the particle 4-momentum in the laboratory frame is $p^\mu = mu^\mu$, and its energy in the rest frame of the fluid is $p^\mu u_\mu = m$.

5. If the fluid velocity is along the x axis, the expression of the momentum distribution simplifies to:

$$\frac{dN}{d^3x d^3p} = \frac{1}{(2\pi)^3} \exp\left(-\frac{p^0u^0 - p_x u}{T}\right). \tag{7}$$

If the temperature is small, the particle momentum is almost in the direction of the fluid velocity, that is, p_y and p_z are typically much smaller than p_x. We expand p^0 to the leading non-trivial order in p_y and p_z:

$$p^0 = (m^2 + p_x^2 + p_y^2 + p_z^2)^{1/2} = m_t + \frac{p_y^2 + p_z^2}{2m_t}, \tag{8}$$

where we have used the notation $m_t \equiv \sqrt{m^2 + p_x^2}$. Inserting Eq. (8) in Eq. (7), we obtain:

$$\frac{dN}{d^3x d^3p} = \frac{1}{(2\pi)^3} \exp\left(-\frac{m_t u^0 - p_x u}{T}\right) \exp\left(-\frac{p_y^2 + p_z^2}{2(m_t T/u^0)}\right). \tag{9}$$

The distribution of (p_y, p_z) is a Gaussian with width $(2m_t T/u^0)^{1/2}$. Note that for the most probable value of p_x determined in the previous question, $m_t = mu^0$, hence the width reduces to $(2mT)^{1/2}$, which is the same result as for a nonrelativistic ideal gas. Integrating Eq. (9) over p_y and p_z, we obtain the distribution of p_x:

$$\frac{dN}{d^3x dp_x} = \frac{1}{(2\pi)^2} \frac{m_t T}{u^0} \exp\left(-\frac{m_t u^0 - p_x u}{T}\right). \tag{10}$$

If the mass m of the particle is large enough, the distribution of p_x is a Gaussian. This can be seen by expanding the exponent to second order in p_x, around the maximum $p_x = mu$ determined in question 4. One must simply evaluate the second derivative d^2m_t/dp_x^2 for this particular value of p_x. One obtains (after some algebra, for which one can use a symbolic calculator)

$$-\frac{m_t u^0 - p_x u}{T} = -\frac{m}{T} - \frac{(p_x - mu)^2}{2mT(u^0)^2}, \tag{11}$$

where the first term is the maximum determined in question 4, and the last term is the second-order correction which, once inserted in the exponential, results in a Gaussian dispersion of p_x around the maximum. The width of this distribution is $(2mT)^{1/2}u^0$. Note the dependence on u^0, which is a relativistic effect associated with Lorentz contraction. The distribution of p_x is broader than the distribution of p_y and p_z by a factor u^0, so that the thermal dispersion appears elongated along the direction of motion of the fluid.

6. We have seen that the most probable value of the momentum is mu, directly proportional to the fluid 4-velocity. The dispersion around this most probable value is $(2mT)^{1/2}u^0$ according to

the previous question. The larger the mass, the smaller the relative dispersion. Massive particles essentially follow the fluid, and we have seen in question 4 that their momentum is proportional to m. In contrast, in a fluid at rest, the thermal momentum is proportional to $(mT)^{1/2}$. The more massive the particle, the larger the effect of the collective motion. This effect is usually referred to as "radial flow" in the heavy-ion literature.

7. If $m = 0$, then $m_t = |p_x|$ and we simplify Eq. (10) accordingly. For positive p_x, we obtain:

$$\frac{dN}{d^3x dp_x} = \frac{1}{(2\pi)^2} \frac{p_x T}{u^0} \exp\left(-\frac{p_x(u^0 - u)}{T}\right). \tag{12}$$

This is, up to a normalization constant, the probability distribution of p_x. It depends on u and T only through the combination $T/(u^0-u) = T(u^0+u)$, where we have used the identity $(u^0)^2-u^2 = 1$. As T decreases, the distribution of p_x does not become narrower, which is the essential difference with massive particles studied in question 5. The reason is that massless particles cannot move with the velocity of the fluid.

8. The pion is almost massless, in the sense that its mass is smaller (by a factor ~ 4 at the LHC) than its mean transverse momentum. The distribution of p_x determined in the previous question is also that of the transverse momentum p_t, since p_y is much smaller than p_x in the considered limit. Hence, the only quantity that one can extract from the transverse momentum distribution is $T(u^0 + u)$. In practice, a smaller temperature can be compensated by a larger fluid velocity u. This is a well-known degeneracy of blast-wave fits.

Anisotropic flow at low p_t

9. We replace p_x with $p_t \cos\varphi$ in Eq. (10):

$$\frac{dN}{d^3x dp_x} = \frac{1}{(2\pi)^2} \frac{m_t T}{u^0} \exp\left(-\frac{m_t u^0 - p_t u \cos\varphi}{T}\right). \tag{13}$$

Note that $m_t = \sqrt{m^2 + p_t^2}$ is independent of φ. Hence the distribution only depends on $\cos\varphi$, which has $\varphi \to -\varphi$ symmetry.

10. One can expand this expression in powers of p_t. Since the dependence on φ is only through the combination $p_t \cos\varphi$, any term in $\cos^k\varphi$ comes with a factor p_t^k. Now, the Fourier series expansion of $\cos^k\varphi$ only involves harmonics of order $n \le k$. Hence, a term in $\cos n\varphi$ can only result from the expansion of $\cos^k\varphi$ with $k \ge n$, which goes with a factor p_t^k. In the limit $p_t \to 0$, only the smallest value $k = n$ contributes. Hence, $v_n(p_t) \propto p_t^n$ at low p_t.

Anisotropic flow at low temperature

11. We evaluate the φ distribution using Eq. (10), where the dependence on φ comes from the fluid velocity. In question 5, the x-axis had been chosen parallel to the fluid velocity. Now that the fluid velocity has an angle φ with the x axis, we replace p_x with the transverse momentum p_t in the result. In addition, in the limit of low temperature, the exponential dominates and we neglect the φ dependence from the pre-exponential factor:

$$\frac{dN}{d\varphi} \propto \exp\left(-\frac{m_t u^0(\varphi) - p_t u(\varphi)}{T}\right). \tag{14}$$

We then assume $u(\varphi) = \langle u\rangle + \varepsilon \cos n\varphi$. The φ dependence of $u^0(\varphi)$ is then given by the relation $u^0(\varphi) = (1 + u(\varphi)^2)^{1/2}$, which we expand to leading order in ε:

$$u^0(\varphi) = (1 + \langle u\rangle^2)^{1/2} + \frac{\langle u\rangle}{(1 + \langle u\rangle^2)^{1/2}} \varepsilon \cos n\varphi = \langle u^0\rangle + \langle v\rangle \varepsilon \cos n\varphi, \tag{15}$$

where, in the last equality, we have used the notations $\langle u^0\rangle \equiv (1+\langle u\rangle^2)^{1/2}$ and $\langle v\rangle \equiv \langle u\rangle/\langle u^0\rangle$. Note that $\langle v\rangle$ is the fluid velocity averaged over φ. We insert these expressions into Eq. (14) and we only keep the terms which depend on φ, i.e., we omit the overall factor which is independent of φ:

$$\frac{dN}{d\varphi} \propto \exp\left(-\frac{m_t\langle v\rangle\varepsilon\cos n\varphi - p_t\varepsilon\cos n\varphi}{T}\right). \tag{16}$$

We finally expand to first order in ε:

$$\frac{dN}{d\varphi} \propto 1 + \left(\frac{-m_t\langle v\rangle + p_t}{T}\right)\varepsilon\cos n\varphi. \tag{17}$$

Finally, $v_n(p_t)$ is defined as the average value of $\cos n\varphi$ for a fixed p_t:

$$v_n(p_t) \equiv \frac{\int_0^{2\pi}\cos n\varphi\frac{dN}{d\varphi}d\varphi}{\int_0^{2\pi}\frac{dN}{d\varphi}d\varphi} = \frac{\varepsilon}{2T}\left(-m_t\langle v\rangle + p_t\right). \tag{18}$$

For a massless particle (typically pions with not too low p_t) , $m_t = p_t$, and Eq. (18) predicts that v_n is proportional to p_t. It also predicts that at a given p_t, $v_n(p_t)$ decreases as the mass m increases. This property is referred to as the *mass ordering of anisotropic flow*. This mass ordering is only present for p_t of order m or smaller, and disappears for larger values of p_t for which $m_t \simeq p_t$. For a more detailed derivation of this result, see `https://arxiv.org/pdf/nucl-th/0506045.pdf`.

Understanding LHC data

12. The result of question 10 explains the observation that the variation of v_n with p_t is flatter for larger n. One then sees that, away from the low p_t region, the pion v_n increases linearly with p_t, as explained in question 11. The mass ordering predicted by Eq. (18) is also clearly observed for $p_t \simeq m$. Specifically, the ordering of the mass is pion<kaon<proton, and the ordering of v_n is opposite for fixed p_t. Except for the low p_t region, $v_n(p_t)$ has a similar p_t dependence for all n, which is also in agreement with the result of question 11. Only the overall normalization varies with n. This variation comes from the Fourier coefficient ε of the fluid velocity, which turns out to decrease with harmonic order n, which is often the case for Fourier coefficients.

The only feature in the data which is in clear disagreement with our simple calculation is the observation that the proton v_n becomes slightly larger than the pion v_n for the highest value of p_t shown in the figure. Generally, hydrodynamics fails to describe data for p_t larger than 2 GeV. Note, however, that this only corresponds to a very small fraction of the produced particles.

13. We use successively Eq. (16) and Eq. (18), in which we set $n = 2$, :

$$\frac{dN}{d\varphi} \propto \exp\left(-\frac{m_t\langle v\rangle\varepsilon\cos 2\varphi - p_t\varepsilon\cos 2\varphi)}{T}\right) = \exp\left(2v_2(p_t)\cos 2\varphi)\right). \tag{19}$$

We then expand this equation to order ε^2 or, equivalently, $v_2(p_t)^2$, and we use the identity $\cos^2 2\varphi = (1+\cos 4\varphi)/2$:

$$\frac{dN}{d\varphi} \propto 1 + 2v_2(p_t)\cos 2\varphi + \left(v_2(p_t)\right)^2(1+\cos 4\varphi). \tag{20}$$

We finally evaluate $v_4(p_t)$ as in Eq. (18) and we obtain, to leading order:

$$v_4(p_t) = \frac{1}{2}\left(v_2(p_t)\right)^2. \tag{21}$$

This has been recently verified, see Fig. 5 of `https://arxiv.org/pdf/2005.12217.pdf` by the HADES collaboration.