Pseudoscalar Transition Form Factors and the Hadronic Light-by-Light Contribution to a_μ

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Introduction

- *Christian* introduced the importance of the muon $g-2$ (a_μ).
- Theory error on a_μ is currently dominated by two hadronic loop corrections:
  1. Hadronic Vacuum Polarization (HVP) [$\mathcal{O}(\alpha^2)$]
  2. Hadronic Light-by-Light (HLbL) scattering [$\mathcal{O}(\alpha^3)$].

- The different contributions (Aoyama et al., 2020):

Contributions	**Value** $\times 10^{11}$
Experiment	116 592 089(63)
QED	116 584 718.931(0.104)
Electroweak	153.6(1.0)
HVP	6845(40)
HLbL	92(19)
Total SM value	116 591 810(43)
Difference: $\Delta a_\mu \equiv a_\mu^{\text{exp}} - a_\mu^{\text{SM}}$	279(76)

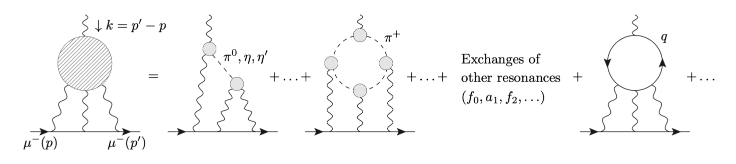
→ HLbL is a small contribution *but* with a relatively large error.

→ Theory error needs to be reduced by a factor of two to meet future experimental error.

Hadronic Light-by-Light Contribution

- The Hadronic Light-by-Light scattering has different sub-contributions (Aoyama et al., 2020):

Contributions	**Value** $\times 10^{11}$
π^0, η, η'-poles	93.8(4.0)
π, K-loops/boxes	-16.4(0.2)
$\pi\pi$ scattering	-8(1)
scalars + tensors	-1(3)
axial vectors	6(6)
u, d, s-loops / short distance	15(10)
c-loop	3(1)
Total	92(19)



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→ Pseudoscalar (π^0, η, η') poles form the largest contribution to HLbL diagram.

→ π^0-pole estimated using lattice (Gérardin et al., 2019) + dispersive framework (data-driven) (Hoferichter et al., 2018).

→ η, η'-pole has no lattice/dispersive result (yet).

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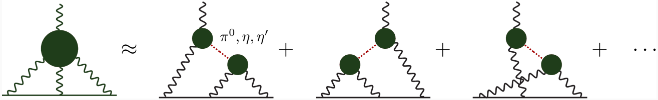
Project Goal: Calculate the sum of π^0, η, η'-pole contributions from lattice QCD methods with $< 10\%$ precision.



Contributions from the pseudoscalar (ps) poles to a_μ have been shown to be (Knecht and Nyffeler, 2002)

$$a_\mu^{ps-pole} = \left(\frac{\alpha_e}{\pi}\right)^3 \int_0^\infty dQ_1 \int_0^\infty dQ_2 \int_{-1}^{1} d\tau \left[w_1(Q_1,Q_2,\tau)\mathscr{F}_{ps\gamma^*\gamma^*}(-Q_1^2,-Q_3^2)\mathscr{F}_{ps\gamma^*\gamma^*}(-Q_2^2,0) + w_2(Q_1,Q_2,\tau)\mathscr{F}_{ps\gamma^*\gamma^*}(-Q_1^2,-Q_2^2)\mathscr{F}_{ps\gamma^*\gamma^*}(-Q_3^2,0)\right],$$

where $Q_3^2 = Q_1^2 + Q_2^2 + 2\tau Q_1 Q_2$ and $\tau = \cos\theta$ with θ the angle between Q_1 and Q_2.

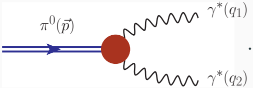
$$a_\mu^{ps-pole} = \left(\frac{\alpha_e}{\pi}\right)^3 \int_0^\infty dQ_1 \int_0^\infty dQ_2 \int_{-1}^1 d\tau \, \Big[w_1(Q_1, Q_2, \tau) \mathscr{F}_{ps\gamma^*\gamma^*}(-Q_1^2, -Q_3^2) \mathscr{F}_{ps\gamma^*\gamma^*}(-Q_2^2, 0) + w_2(Q_1, Q_2, \tau) \mathscr{F}_{ps\gamma^*\gamma^*}(-Q_1^2, -Q_2^2) \mathscr{F}_{ps\gamma^*\gamma^*}(-Q_3^2, 0) \Big] .$$

Here, we recognize two different objects:

1. $\mathscr{F}_{ps\gamma^*\gamma^*}(q_1^2, q_2^2)$ are the **Transition Form Factors (TFFs)**,
2. $w_i(q_1, q_2, \tau)$ are weight functions (known analytically).

The TFF encodes the interaction between a pseudoscalar and two photons. E.g. for the pion

$$\mathscr{F}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) =$$



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a_μ^{ps} and the Transition Form Factor

$$a_\mu^{ps-pole} = \left(\frac{\alpha_e}{\pi}\right)^3 \int_0^\infty dQ_1 \int_0^\infty dQ_2 \int_{-1}^{1} d\tau \left[w_1(Q_1,Q_2,\tau)\mathscr{F}_{ps\gamma^*\gamma^*}(-Q_1^2,-Q_3^2)\mathscr{F}_{ps\gamma^*\gamma^*}(-Q_2^2,0) \right.$$
$$\left. + w_2(Q_1,Q_2,\tau)\mathscr{F}_{ps\gamma^*\gamma^*}(-Q_1^2,-Q_2^2)\mathscr{F}_{ps\gamma^*\gamma^*}(-Q_3^2,0)\right].$$

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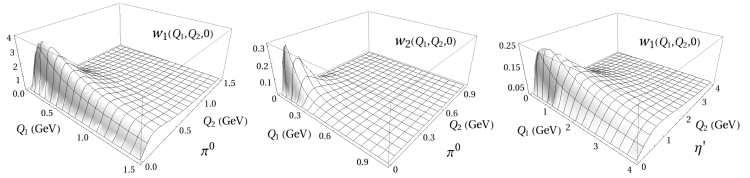
Weight functions are peaked at low spacelike Q^2 so lattice QCD is the perfect method.



Transition Form Factor from the Lattice

The TFF for a pseudoscalar meson is extracted from matrix elements $M_{\mu\nu}$

$$M_{\mu\nu}(p,q_1) = i\int d^4x\, e^{iq_1\cdot x}\langle\Omega|\,T\{J_\mu(x)J_\nu(0)\}\,|PS(\vec{p})\rangle$$
$$= \varepsilon_{\mu\nu\alpha\beta}\, q_1^\alpha q_2^\beta\, \mathscr{F}_{PS\gamma^*\gamma^*}(q_1^2,q_2^2),$$

where J_μ is the EM current. (Euclidean) Matrix elements are related to 3-point correlation function $C_{\mu\nu}^{(3)}$ on lattice through (Ji and Jung, 2001)

$$M_{\mu\nu}^E = \frac{2E_{PS}}{Z_{PS}}\int_{-\infty}^{\infty} d\tau\, e^{\omega_1\tau}\tilde{A}_{\mu\nu}(\tau),$$

where τ is the time-separation between the two EM currents and

1. $\tilde{A}_{\mu\nu}(\tau)$ can be related to a three-point correlation function $C_{\mu\nu}^{(3)}$, calculable on the lattice

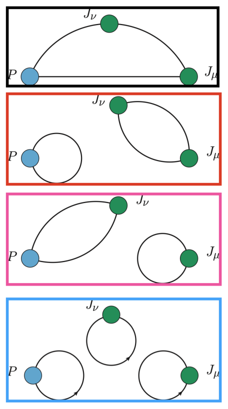
$$\tilde{A}_{\mu\nu}(\tau) \sim C_{\mu\nu}^{(3)}(\tau,t_{PS}) = a^6\sum_{\vec{x},\vec{z}}\langle J_\mu(\vec{z},t_i)J_\nu(\vec{0},t_f)P^\dagger(\vec{x},t_0)\rangle e^{i\vec{p}\cdot\vec{x}}e^{-i\vec{q}_1\cdot\vec{z}}.$$

2. E_{PS}, Z_{PS} energy and overlap of the pseudoscalar that are extracted from two-point correlations functions.
3. $q_1 = (\omega_1,\vec{q}_1)$ and $q_2 = (E_{PS}-\omega_1,\vec{p}-\vec{q}_1)$

Wick Contractions Correlation Function

$$C^{(3)}_{\mu\nu}(\tau, t_{PS}) = a^6 \sum_{\vec{x},\vec{z}} \langle J_\mu(\vec{z}, t_i) J_\nu(\vec{0}, t_f) P^\dagger(\vec{x}, t_0) \rangle e^{i\vec{p}\cdot\vec{x}} e^{-i\vec{q}_1\cdot\vec{z}}.$$

Correlation function receives contributions from (potentially) four different Wick contractions



1. - **For the π^0**

     $$P_{\pi^0}(x) = \frac{1}{\sqrt{2}} \left(\overline{u}\gamma_5 u(x) - \overline{d}\gamma_5 d(x) \right).$$

   - We work in the isospin limit.
     Consequently disconnected pseudoscalar loop is formally zero.
   - Two diagrams contribute.
   - Disconnected contribution is small $\mathscr{O}(1-2\%)$.
     (Gérardin et al., 2019).

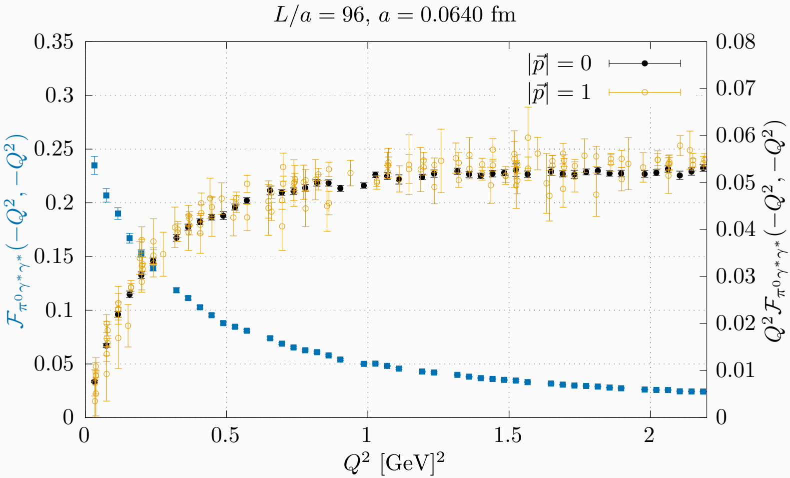
2. - **For the η, η'**

     $$P_{\eta_8}(x) = \frac{1}{\sqrt{6}} \left(\overline{u}\gamma_5 u(x) + \overline{d}\gamma_5 d(x) - 2\overline{s}\gamma_5 s(x) \right),$$

     $$P_{\eta_0}(x) = \frac{1}{\sqrt{3}} \left(\overline{u}\gamma_5 u(x) + \overline{d}\gamma_5 d(x) + \overline{s}\gamma_5 s(x) \right).$$

   - All four diagrams contribute.
   - η^8 and η^0 mix to create physical η, η'.

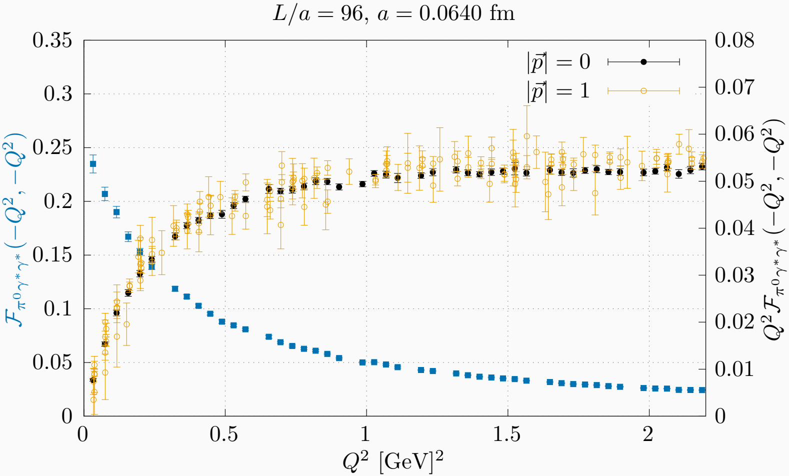
π^0 TFF: Result on a Single Ensemble



- Good agreement between $\vec{p} = \vec{0}$ & $\vec{p} = \frac{2\pi}{L}(0,0,+1)$.
- Error on $\mathscr{F}_{\pi^0\gamma^*\gamma^*}$ grows with decreasing Q^2.

π^0 TFF: Result on a Single Ensemble

$$a_\mu^{ps-pole} = \left(\frac{\alpha_e}{\pi}\right)^3 \int_0^\infty dQ_1 \int_0^\infty dQ_2 \int_{-1}^{1} d\tau \left[w_1(Q_1,Q_2,\tau) \mathscr{F}_{ps\gamma^*\gamma^*}(-Q_1^2,-Q_3^2) \mathscr{F}_{ps\gamma^*\gamma^*}(-Q_2^2,0) + w_2(Q_1,Q_2,\tau) \mathscr{F}_{ps\gamma^*\gamma^*}(-Q_1^2,-Q_2^2) \mathscr{F}_{ps\gamma^*\gamma^*}(-Q_3^2,0) \right].$$



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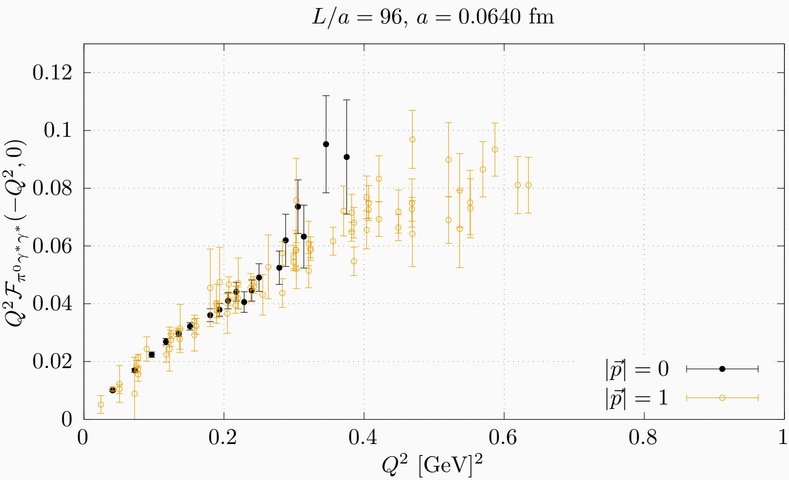
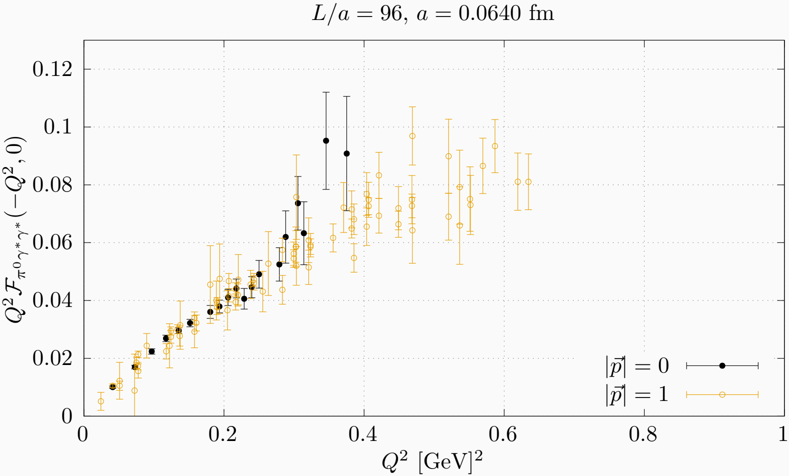


- Good agreement between $\vec{p} = \vec{0}$ & $\vec{p} = \frac{2\pi}{L}(0, 0, +1)$.
- Smaller extent in Q^2 than $\mathscr{F}(-Q^2, -Q^2)$ (previous slide).

π^0 TFF: Result on a Single Ensemble

$$a_\mu^{ps-pole} = \left(\frac{\alpha_e}{\pi}\right)^3 \int_0^\infty dQ_1 \int_0^\infty dQ_2 \int_{-1}^{1} d\tau \left[w_1(Q_1,Q_2,\tau)\mathscr{F}_{ps\gamma^*\gamma^*}(-Q_1^2,-Q_3^2)\mathscr{F}_{ps\gamma^*\gamma^*}(-Q_2^2,0) + w_2(Q_1,Q_2,\tau)\mathscr{F}_{ps\gamma^*\gamma^*}(-Q_1^2,-Q_2^2)\mathscr{F}_{ps\gamma^*\gamma^*}(-Q_3^2,0) \right].$$



Fitting the TFF

The TFF can be fitted in a model-independent way using the $z-$**expansion**,

$$P(Q_1^2,Q_2^2)\mathscr{F}_{\pi^0\gamma^*\gamma^*}(-Q_1^2,-Q_2^2)=\sum_{n,m=0}^{N}c_{nm}\left(z_1^n-(-1)^{N+n+1}\frac{n}{N+1}z_1^{N+1}\right)\times \left(z_2^m-(-1)^{N+m+1}\frac{m}{N+1}z_2^{N+1}\right),$$

where z_k are conformal variables

$$z_k=\frac{\sqrt{t_c+Q_k^2}-\sqrt{t_c-t_0}}{\sqrt{t_c+Q_k^2}+\sqrt{t_c-t_0}},\quad k=1,2,$$

- c_{nm} symmetric coefficients
- t_0 free parameter

  - $t_c=4m_\pi^2$
  - $P(Q_1^2,Q_2^2)=1+\frac{Q_1^2+Q_2^2}{M_V^2}$

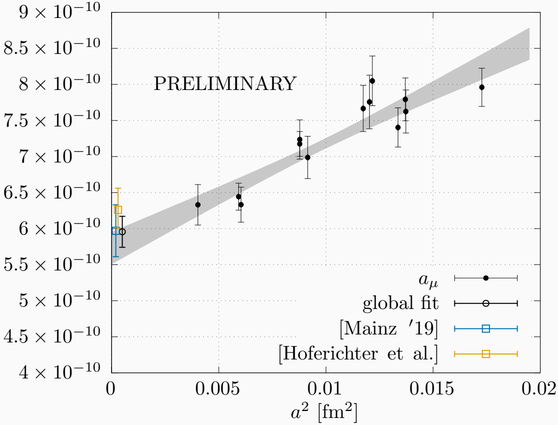
Advantages:

→ Fit is model-independent, only systematic is choice of N.

→ Obtain TFF in whole (Q_1^2,Q_2^2)-plane.

Continuum Extrapolation $a_\mu^{\pi^0-\text{pole}}$

- We have 13 ensembles on 6 different lattice spacings.
- Preliminary result agrees well with previous estimates: lattice (Gérardin et al., 2019) and dispersive (Hoferichter et al., 2018)



$\rightarrow$ Next step: estimate systematics & disconnected contribution (small).

Difficulties for the η, η' TFFs

- For the η, η'

$$P_{\eta_8}(x) = \frac{1}{\sqrt{6}} \left(\overline{u}\gamma_5 u(x) + \overline{d}\gamma_5 d(x) - 2\overline{s}\gamma_5 s(x) \right),$$

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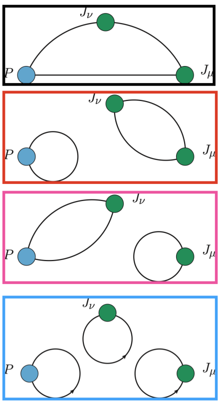
- **All four diagrams contribute.**
- **Mixing between η^8 and η^0 to create physical η, η'**
- Reminder:

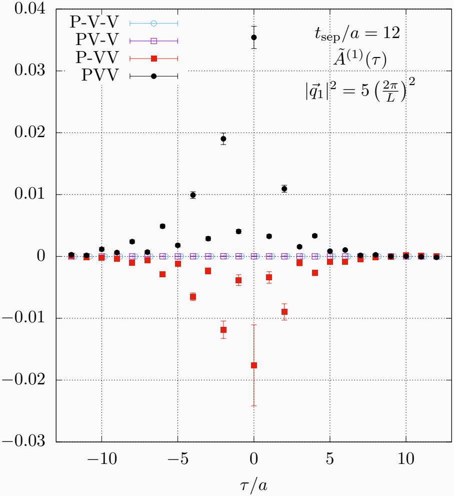
$$\mathscr{F}_{ps\gamma^*\gamma^*} \sim M^E_{\mu\nu} = \frac{2E_{PS}}{Z_{PS}} \int_{-\infty}^{\infty} d\tau \, e^{\omega_1 \tau} \tilde{A}_{\mu\nu}(\tau),$$

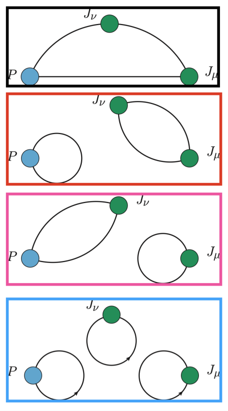
where τ is the time-separation between the two EM currents.

→ We first look at an example of the integrand.



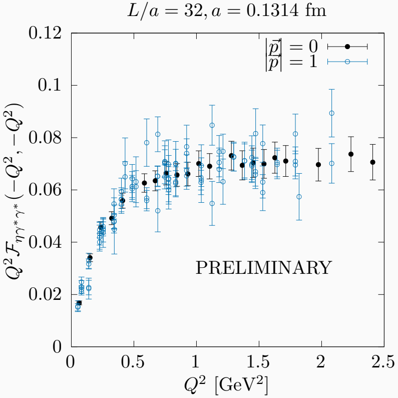
$\eta^{(\prime)}$ Transition Form Factor Integrand

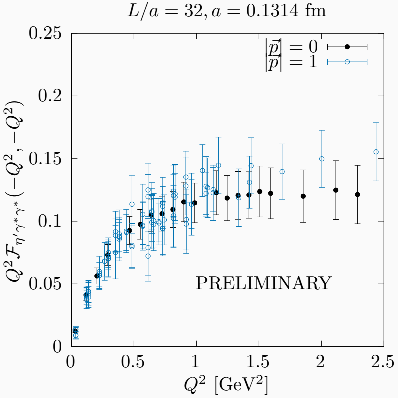




- Largest contributions from fully connected PVV and disconnected $P-VV$ diagram (with opposite signs).

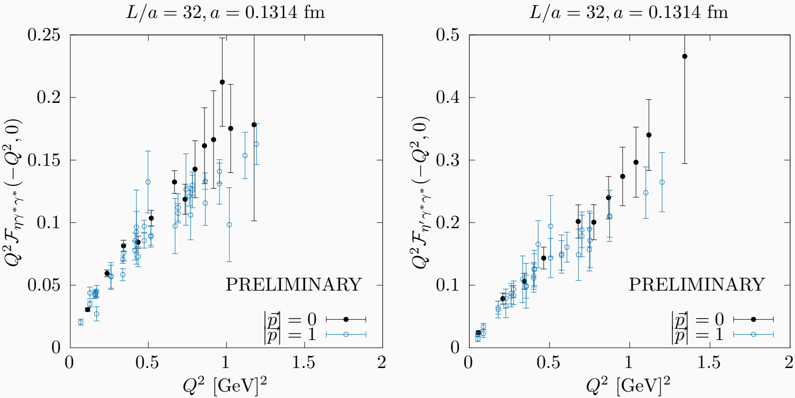
η, η' TFF: Result on a Single Ensemble





- Good agreement between $\vec{p} = \vec{0}$ & $\vec{p} = \frac{2\pi}{L}(0,0,+1)$.
- Errors larger than for π^0 because of difficulties mentioned before.

η, η' TFF: Result on a Single Ensemble



- Good agreement between $\vec{p} = \vec{0}$ & $\vec{p} = \frac{2\pi}{L}(0, 0, +1)$.

Summary

- π^0: Preliminary results in agreement with previously published calculation on lattice (Gérardin et al., 2019) and dispersive/data-driven (Hoferichter et al., 2018)
  To do:
  - Estimate contribution from disconnected diagram.
  - Estimate systematics.

  Note:
  - Here we already work at physical pion mass, so no chiral extrapolation.
- First calculation η, η' TFF on the lattice.
  - Preliminary data looks good in different kinematical regimes.
  - Analyze more ensembles.
- π^0, η, η': Goal of $< 10\%$ error on a_μ should be attainable judging from preliminary fits.

References

Aoyama, T. et al. (2020). The anomalous magnetic moment of the muon in the standard model. *Physics Reports*, 887:1–166.

Gérardin, A., Meyer, H. B., and Nyffeler, A. (2019). Lattice calculation of the pion transition form factor with $N_f = 2+1$ Wilson quarks. *Phys. Rev. D*, 100(3):034520.

Hoferichter, M., Hoid, B.-L., Kubis, B., Leupold, S., and Schneider, S. P. (2018). Dispersion relation for hadronic light-by-light scattering: pion pole. *Journal of High Energy Physics*, 2018(10).

Ji, X. and Jung, C. (2001). Studying hadronic structure of the photon in lattice qcd. *Physical Review Letters*, 86(2):208–211.

Knecht, M. and Nyffeler, A. (2002). Hadronic light-by-light corrections to the muon $g-2$: The pion-pole contribution. *Physical Review D*, 65(7).

Sampling of (Q_1^2, Q_2^2) plane