Soft gluon threshold resummation

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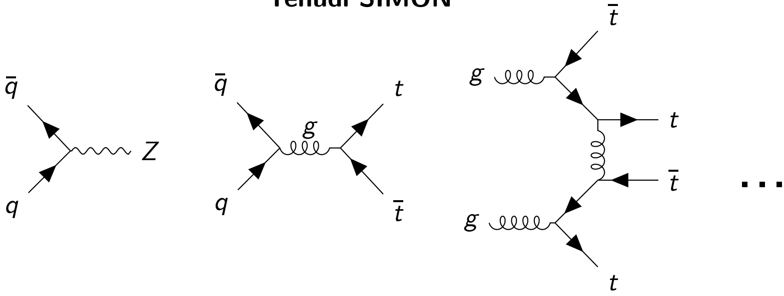

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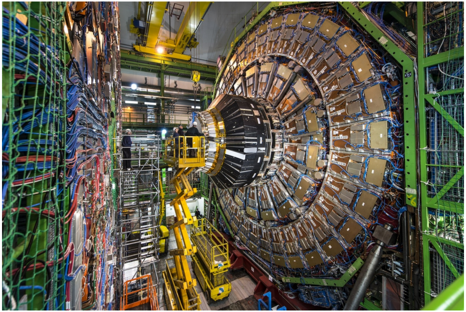
Content

Precision computation

Motivations

- LHC Run 3 soon: more data $\rightarrow$ less statistical error
- Experimental precision getting better and better
- Theory must adapt and be even more accurate: "Precision era"



QCD corrections can be the most important in some processes: we need to take them into account. To do so we can use *resummation* techniques!

Motivational example: top pair production

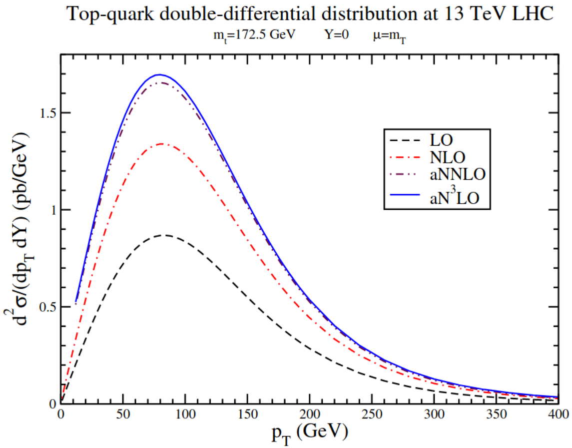
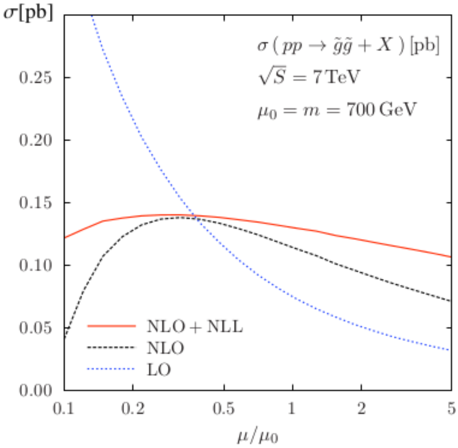


Figure: Fixed order comparison for top pair [3]

Motivational example: scale dependance

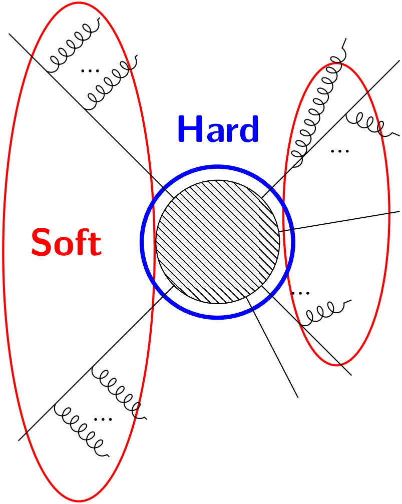


(c) Gluino-gluino production.

Figure: Cross section scale dependance, PhD thesis Irene Niessen

Content

Factorization of scales



- Factorization theorem: Soft scale vs. Hard scale
- Unlike QED the emissions are correlated → color structure
- $d\sigma \propto H.S$ in Mellin space

Hadronic to partonic

- The link between hadronic and partonic is the convolution by PDFs:

$$\sigma_{h_1 h_2 \to \dots} = \int f_{p_1/h_1} \otimes f_{p_1/h_1} \otimes d\hat{\sigma}_{p_1 p_2 \to \dots}$$

- Mellin space ($\sim$ Laplace transform):

$$\mathscr{M}[f](N) = f^N = \int_0^1 dz f(z) z^{N-1}$$

$$\sigma^N_{h_1 h_2 \to \dots} = \int f^N_{p_1/h_1} . f^N_{p_1/h_1} . d\hat{\sigma}^N_{p_1 p_2 \to \dots}$$

N.B.: Invariant-mass threshold definition: $\hat{s} = P^2_{in} \to P^2_{out} = Q^2 \neq$ absolute threshold: $\sqrt{\hat{s}} \to \sum m_{out}$

Large logs

$$d\hat{\sigma}^{N}_{p_1 p_2 \to \dots} = \frac{d\Phi_n}{F} \overline{\sum} |\mathscr{M}^{N}_{p_1 p_2 \to \dots}|^2$$

For the $q\bar{q} \to Z$ process at NLO:

$$d\hat{\sigma}^{N}_{q\bar{q} \to Z} = d\sigma_0 \left(1 + \frac{\alpha_s}{2\pi}\left[4C_F \ln^2(\overline{N}) - 4C_F \ln(\overline{N}) \ln(m_Z^2/\mu_F^2) + \tilde{C}^{(1)}_{q\bar{q}\to Z}\right]\right) \quad (1)$$

N Mellin conjugate to $z = \frac{Q^2}{\hat{s}}$, $|N| \to +\infty \longleftrightarrow z \to 1$

Generic logarithms coming from real emissions of *(collinear-)soft* gluons appear to all orders: can be *resummed*.

Expansion of logarithms

Resumming and exponentiation of logarithms produces a new power series in $\alpha_s L$ at large $L = \ln(N)$: Leading-Logarithms (LL), NLL, $\mathrm{N}^k\mathrm{LL}$

$\alpha_s^m L^k$	LO	NLO	...	$\mathrm{N}^m\mathrm{LO}$
LL	$m = k = 0$	$k = 2$	...	$m + 1 \leq k \leq 2m$
NLL	$\emptyset$	$m = k = 1$	...	$m \leq k \leq 2m - 1$
...	...	...	...	...
$\mathrm{N}^p\mathrm{LL}$	$\emptyset$	$\emptyset$	...	$m + 1 - p \leq k \leq 2m - p$

General formalism

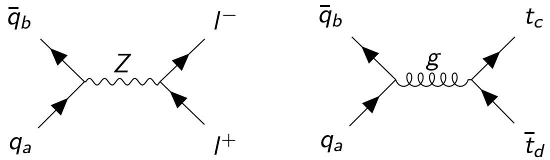
In a general way, we can write the factorized formula:

$$\boxed{d\hat{\sigma}^{N,\,res.}_{ij\to\dots} \propto \mathrm{Tr}\left(\mathbf{H} e^{\int \Gamma^\dagger} \mathbf{S} e^{\int \Gamma}\right) \Delta_i \Delta_j} \tag{2}$$

- **H**: "hard" matrix, high energy process part
- **S**: "soft" matrix, low energy emissions and color structure
- Γ: soft anomalous dimension color matrix controlling the evolution over RG of **S**
- Δ_i: (colinear-)soft radiations from initial state massless partons

$$\Delta_i = e^{g_1(\alpha_s \ln(\bar{N}))\ln(\bar{N}) + g_2(\alpha_s \ln(\bar{N})) + \dots}$$

Color basis

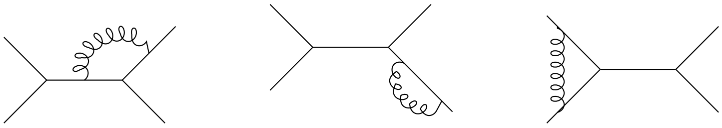


$\mathbf{\overline{3}} \otimes \mathbf{3} = \mathbf{8} \oplus \mathbf{1}$, singlet-octet decomposition. Color tensor basis depends on the incomming and outgoing particles, not unique.

We can choose the following color tensor bases for the example processes:

$$\left\{ c_{DY} = \frac{1}{\sqrt{N_c}} \delta_{ab} \right\} \qquad \left\{ c_1 = \frac{1}{N_c} \delta_{ab} \delta_{cd},\; c_8 = \sqrt{\frac{2}{C_F N_c}} T^i_{ba} T^i_{cd} \right\}$$

Hard function



Actual hard scale process at fixed loop order, we can expand in α_s

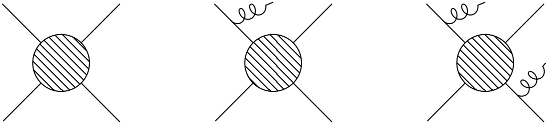
$$\mathbf{H} = \sum_{k=0} \mathbf{H}^{(k)} \left(\frac{\alpha_s}{4\pi}\right)^k$$

The (finite) virtual contribution for NLO is encoded in:

$$\mathbf{H}^1 \propto \overline{\sum} 2\Re(\mathscr{M}_0^* . \mathscr{M}_1)$$

understood as colored renormalized matrix elements, expandable on the color tensor basis relevant for the process.

Soft functions

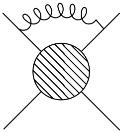


S can also be expanded in α_s, if the color bases is orthonormal (like the examples) the first term is:

$$\mathbf{S}^0 \propto \left(c_I . c_J \right) = \mathbb{I}_d$$

Higher orders of **S** are needed for precise treatment of the soft limit and consists in considering all possible soft emissions from external legs.

Γ computation



Γ is the color matrix which drives the evolution of the soft matrix through RG:

$$\frac{d\mathbf{S}}{d\ln(\mu)} = -\Gamma^\dagger \mathbf{S} - \mathbf{S}\Gamma$$

To compute it we need to consider loop contributions in the eikonal approximation and sum over all possible leg combination:

$$\Gamma^{(1)}_{IJ} = -\sum_{k,l} C^{kl}_{IJ} \lim_{\epsilon\to 0} \epsilon\,\omega^{(1)}_{kl}$$

Content

Matching

We can also expand $d\hat{\sigma}^{N,\,res.}$ to NLO in α_s and compare it to the usual NLO cross section (obtained with `MG5_aMC@NLO`).

Fixed order

$d\sigma_{|NLO}$

Valid away from threshold

Resummed

$d\sigma^{res.}$

Valid at threshold

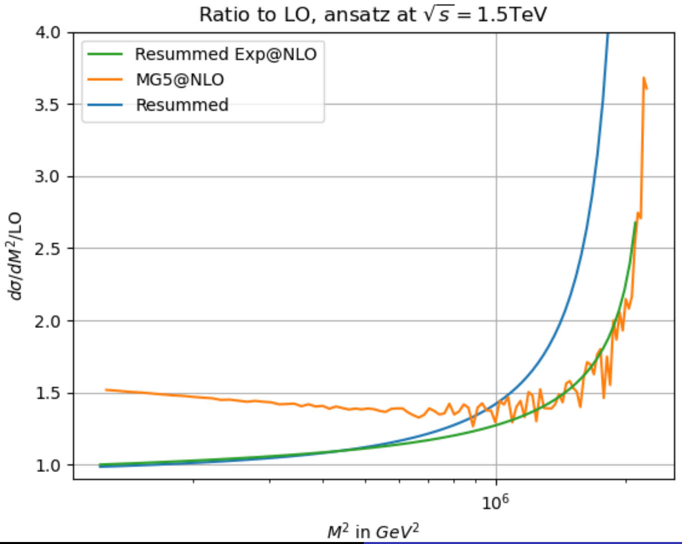
Resummed expanded at f.o.

$d\sigma^{res.}_{|NLO}$

Double counting

Matching: $d\sigma_{|NLO} + d\sigma^{res.} - d\sigma^{res.}_{|NLO}$ valid everywhere

Comparison of cross sections behaviour: $q\bar{q} \to t\bar{t}$



Expected behaviours

- We expect the ratio $\frac{1}{d\sigma^0/dQ^2}\left(\frac{d\sigma^{res.}}{dQ^2} - \frac{d\sigma^{res.}}{dQ^2}\Big|_{NLO}\right) \underset{Q^2 \ll S_h}{\longrightarrow} 0$

  Away from threshold the logarithmic terms are not important and the behaviour is captured by the first orders of the expansion.

- We expect also $\frac{1}{d\sigma^0/dQ^2}\left(\frac{d\sigma^{NLO}}{dQ^2} - \frac{d\sigma^{res.}}{dQ^2}\Big|_{NLO}\right) \underset{Q^2 \to S_h}{\longrightarrow} 0$

  In the threshold regime, the resummed expanded reproduces the behaviour of original cross section.

To obtain a sensible cross section in all ranges we may consider the combination: $\sigma_{|_{NLO}} + \sigma^{res.} - \sigma^{res.}_{|_{NLO}}$

To be done

Top pair production

- Apply on $gg \to t\bar{t}$
- Color basis of $\mathbf{8} \otimes \mathbf{8} = \mathbf{8}_S \oplus \mathbf{8}_A \oplus \mathbf{1} \oplus \ldots$ is 3 dimensional but similar treatment

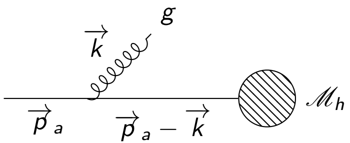
4 top

- Apply on $gg/q\bar{q} \to t\bar{t}t\bar{t}$
- Statistics greatly improved with Run 3 and HL-LHC
- Color structure and kinematics more complex

Thank you for your attention!

Backup

Eikonal approximation

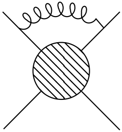


$$\mathcal{M}_e = \mathcal{M}_h \frac{i(\not{p}_a - \not{k} + m)}{(p_a - k)^2 - m^2 + i\epsilon}(-ig_s \mathbf{T}^a \gamma^\mu) u(p_a) \epsilon^*_\mu(k)$$

$$\mathcal{M}_e \underset{k \ll p}{\rightarrow} \mathcal{M}_h u(p_a) g_s \mathbf{T}^a \frac{-p_a^\mu}{p_a . k + i\epsilon} \epsilon^*_\mu(k)$$

Effective Feynman rules for soft ($k \ll p$) gluon radiation and generators depending of particle nature and if it's incomming/outgoing.

Γ computation



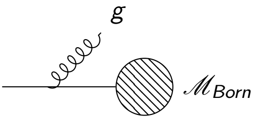
$$\Gamma_{IJ}^{(1)} = -\sum_{k,l} C_{IJ}^{kl} \lim_{\epsilon \to 0} \epsilon\, \omega_{kl}^{(1)}$$

Where $\omega_{kl}^{(1)}$ contains the kinematics at 1-loop in eikonal approximation:

$$\omega_{kl}^{(1)} = g_s^2 \int \frac{d^d q}{(2\pi)^d} \frac{-i}{q^2 + i\epsilon} \left(\frac{\Delta_k \Delta_l p_k . p_l}{(\delta_k p_k . q + i\epsilon)(\delta_l p_l . q + i\epsilon)} + \ldots \right)$$

And the color factor: $C_{IJ}^{kl} = \dfrac{\langle c_I | \mathbf{t}^k \mathbf{t}^l | c_J \rangle}{\sqrt{\langle c_I | c_I \rangle \langle c_J | c_J \rangle}}$

Δ building



$$\mathscr{M}_e \underset{Eikonal}{=} \mathscr{M}_{Born} u(p_a) g_s \mathbf{T}^a \frac{-p_a^\mu}{p_a.k + i\epsilon} \epsilon^*_\mu(k)$$

$$d\sigma_e \propto \frac{2p_a.p_b}{E_a E_b k_0^2 (1 - \cos^2(\theta))} d\sigma_{Born}$$

Phase space factorization:

$$d\Phi_2 = \frac{d^{d-1}k}{(2\pi)^{d-1} 2k_0} \frac{2\pi}{\hat{s}} \delta(\frac{2k}{\sqrt{\hat{s}}} - 1 + z)$$

Regularization and resummation

When we add the virtual contribution in the same eikonal approximation to regulate $z \to 1$ divergences and integrate over phase space we get for each massless leg:

$$I = 2C_i \int_0^1 dz \frac{z^{N-1} - 1}{1-z} \int_{-1}^1 \frac{d\cos(\theta)}{1-\cos^2(\theta)} \frac{\alpha_s}{\pi}$$

In the collinear limit $\cos(\theta) \approx 1 - \frac{\theta^2}{2}$ and we can approximate $\frac{d\theta^2}{\theta^2} \approx \frac{dk^2}{k^2}$ where k represents the momentum taken away by the gluon.

$$I = C_i \int_0^1 dz \frac{z^{N-1} - 1}{1-z} \int_{\mu_F^2}^{(1-z)^2 Q^2} \frac{dk^2}{k^2} \frac{\alpha_s(k^2)}{\pi}$$

We can extrapolate this for multiple emissions, decoupled for this LL integral:

$$\Delta_i^{LL} = \sum_{n=0}^{+\infty} \frac{I^n}{n!} = e^I$$

H computation

$\mathbf{H}^{(1)}$ contains all the hard process dependent part of the cross section.
Since it's hard, it doesn't depend on $\log(\overline{N})$ which encapsulate the soft limit.
It can be extracted from the full NLO computation or *via* dipole formalism together with virtual contutribution.
Schematically:

$$\mathbf{H}^{(1)} = \mathbf{V}\Big|_{reg.} + \Big(\mathbf{P} + \mathbf{K} + \mathbf{I}\Big)\Big|_{N-indep.}$$

P Operator

At 1^{st} order for a process $a+b\to\sum_{l=1}^{n}p_l$, we have:

$$\boldsymbol{P}^{aa'}_{n+a/b}(\{p_l\},\ xp_a,\ p_b,\ x)=\frac{\alpha_s}{2\pi}P_{aa'}(x)\sum_{l\neq a,a'}\frac{\mathbf{T}_l.\mathbf{T}_{a'}}{\mathbf{T}^2_{a'}}\ln\left(\frac{\mu_F^2}{xs_{al}}\right)$$

It has to be convoluted with the partonic differential colored cross section at tree level: $d\hat{\sigma}^0_{a'b}$ and summed over all possible a' (wa have both contributions of $\boldsymbol{P}^{aa'}$ and $\boldsymbol{P}^{bb'}$ of course.
After Mellin transform we get:

$$\boldsymbol{P}^{aa'}_{n+a/b}(N)=\frac{\alpha_s}{2\pi}\delta^{aa'}\left(\gamma_a-2C_a\ln(\overline{N})\right)\sum_{l\neq a,a'}\frac{\mathbf{T}_l.\mathbf{T}_{a'}}{C_{a'}}\ln\left(\frac{\mu_F^2}{s_{al}}\right)+o(1) \quad (3)$$

With $s_{ij}=2p_i.p_j$

K Operator

$$
\begin{aligned}
&\boldsymbol{K}^{aa'}(\{q_n\},\ \{p_m\},\ p_a,\ p_b,\ x) = \frac{\alpha_s}{2\pi}\Big\{\overline{K}^{aa'}(x) - K^{aa'}_{F.S.}(x) - \sum_{i\neq a,b} \mathbf{T}_i.\mathbf{T}_{a'}\mathscr{K}_i^{aa'}(x, s_{ia}, m_i) \\
&- \frac{1}{\mathbf{T}^2_{a'}}\sum_{i\neq a,b}\mathbf{T}_i.\mathbf{T}_{a'}\Big[P^{reg}_{aa'}(x)\ln\left(\frac{(1-x)s_{ia}}{(1-x)s_{ia}+m_i^2}\right) + \\
&\delta^{aa'}\delta(1-x)\gamma_a\left(\ln\left(\frac{s_{ia}-2m_i\sqrt{m_i^2+s_{ia}}+2m_i^2}{s_{ia}}\right)+\frac{2m_i}{\sqrt{s_{ia}+m_i^2}+m_i}\right)\Big] \\
&- \frac{\mathbf{T}_b.\mathbf{T}_{a'}}{\mathbf{T}^2_{a'}}\left[P^{reg}_{aa'}(x)\ln(1-x) + \delta^{aa'}\mathbf{T}^2_{a'}\left(2\frac{\ln(1-x)}{1-x}\right)\Big|_+ - \frac{\pi^2}{3}\delta(1-x)\right]\Big\}
\end{aligned}
$$

K Operator

After Mellin transform:

$$\boldsymbol{K}^{aa'}(N) = \frac{\alpha_s}{2\pi}\Bigg\{\delta^{aa'} C_a \left(\ln^2(\overline{N}) - \frac{\pi^2}{6}\right) - \sum_{i\neq a,b} \mathbf{T}_i.\mathbf{T}_{a'} \mathscr{K}_i^{aa'}(N, s_{ia}, m_i)$$

$$- \frac{1}{\mathbf{T}_{a'}^2} \sum_{i\neq a,b} \mathbf{T}_i.\mathbf{T}_{a'} \left[\delta^{aa'} \gamma_a \left(\ln\left(\frac{s_{ia} - 2m_i\sqrt{m_i^2 + s_{ia}} + 2m_i^2}{s_{ia}}\right) + \frac{2m_i}{\sqrt{s_{ia} + m_i^2} + m_i}\right)\right]$$

$$- \mathbf{T}_b.\mathbf{T}_{a'} \delta^{aa'} \left(\ln^2(\overline{N}) - \frac{\pi^2}{6}\right) + o(1)\Bigg\}$$

$\mathscr{K}_i^{aa'}(N)$

- If i refers to a massive (anti-)quark or gluino:

$$\mathscr{K}_{q_i}^{aa'}(N) = \delta^{aa'}\Big[\ln^2(\overline{N}) + 2\ln(\overline{N})\Big(1+\ln(m_i^2/s_{ia})\Big) - 1 - \frac{\pi^2}{6} + 2\mathrm{Li}_2(1+s_{ia}/m_i^2) + \ln\left(\frac{m_i^2}{m_i^2+s_{ia}}\right)\left(\frac{1}{2}+\frac{m_i^2}{s_{ia}}+2\ln(m_i^2/s_{ia})\right)\Big]$$

- If it's massless:

$$\mathscr{K}_i^{aa'}(N) = \delta^{aa'}\frac{\gamma_i}{C_i}\Big[\ln(\overline{N}) - 1\Big]$$

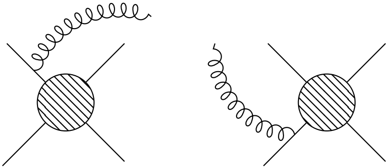
If there are some massive quarks in the model, the gluonic $\mathscr{K}_g^{aa'}(N)$ picks up an other term depending on the quark masses

I Operator

$$\mathbf{I}_{m+a+b} = -\frac{\alpha_s}{2\pi}\frac{(4\pi)^\varepsilon}{\Gamma(1-\varepsilon)}\sum_I \frac{1}{\mathbf{T}_I^2}\sum_{J\neq I}\mathbf{T}_I.\mathbf{T}_J\Big[K_I+\gamma_I\Big(1+\ln\Big(\frac{\mu^2}{s_{IJ}}\Big)\Big) - \frac{\pi^2}{3}\mathbf{T}_I^2+\Gamma_I(\varepsilon,\mu,m_i)+\mathbf{T}_I^2\Big(\frac{\mu^2}{s_{IJ}}\Big)^\varepsilon \mathscr{V}_I(\varepsilon,s_{IJ},\kappa)\Big]\delta(1-x)$$

$\mathscr{V}_I$ contains a singular part in ε symetric in the input particles I and J and a non-singular part non totally symetric and depending on the massiveness of the particles *c.f.* [1] Sec. 6.2.

$S^{(1)}$



$$\mathbf{S}^{(1)} = \frac{\alpha_s}{2\pi}\frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)}\left(\frac{\mu_F^2}{Q^2}\right)^\epsilon \sum_{i,j} \mathbf{T}^i.\mathbf{T}^j \left[2\Omega^{ij}_{-1}\left(\ln^2(\overline{N}) + \frac{\pi^2}{6}\right) + 2\Omega^{ij}_0 \ln(\overline{N}) + \Omega^{ij}_1 + o(1)\right]$$

	$i=j\ m_i=0$	$i=j\ m_i\neq 0$	$m_i=0=m_j$	$m_i=0\ m_j\neq 0$	$m_i m_j \neq 0$
Ω^{ij}_{-1}	0	0	-1	$-1/2$	0
Ω^{ij}_0	0	1	$\ln\frac{p_i.p_j}{2E_iE_j} \underset{c.m.}{=} 0$	$\ln\frac{p_i.p_j}{E_i m_j}$	$-L_\beta$
					if $m_i = m_j$
Ω^{ij}_1	0	$\frac{1}{\beta_i}\ln(\frac{1+\beta_i}{1-\beta_i})$	$\frac{\pi^2}{6}$	cf. N.B.	0

N.B.

From F.K.S paper [2] we have the expressions of the eikonal integrals:

$$\mathcal{E}^{ij} = -\frac{\xi^{-2\epsilon}}{2\epsilon}\frac{2^{2\epsilon}}{(2\pi)^{1-2\epsilon}}\left(\frac{s}{\mu^2}\right)^{-\epsilon}\int d\Omega_k^{(d-1)} E_k^2 \frac{p_i.p_j}{(p_i.k)(p_j.k)}$$
$$= \frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)}\left(\frac{\mu^2}{Q^2}\right)^\epsilon\left[\frac{\mathcal{E}^{ij}_{-2}}{\epsilon^2} + \frac{\mathcal{E}^{ij}_{-1}}{\epsilon} + \mathcal{E}^{ij}_0\right]$$

$$\Omega_1^{ij} = -\mathcal{E}_0^{ij} - \mathcal{E}_{-1}^{ij}\ln\left(\frac{s\xi^2}{Q^2}\right) - \frac{\mathcal{E}_{-2}^{ij}}{2}\ln^2\left(\frac{s\xi^2}{Q^2}\right)$$
$$\Omega_0^{ij} = -\mathcal{E}_{-1}^{ij} - \mathcal{E}_{-2}^{ij}\ln\left(\frac{s\xi^2}{Q^2}\right)$$
$$\Omega_{-1}^{ij} = -\mathcal{E}_{-2}^{ij}$$

N.B.

- For $m_i = m_j = 0$, we have:
$$\Omega_1^{ij} = \mathrm{Li}_2\left(\frac{p_i.p_j}{2E_iE_j}\right) + \frac{1}{2}\ln^2\left(\frac{p_i.p_j}{2E_iE_j}\right) - \ln\left(\frac{p_i.p_j}{2E_iE_j}\right)\ln(1-\frac{p_i.p_j}{2E_iE_j}) \underset{c.m.}{=} \frac{\pi^2}{6}$$
- For $m_j = 0 \neq m_i$:
$$\Omega_1^{ij} = \frac{\pi^2}{12} + \frac{1}{4}\ln^2(\frac{1+\beta_i}{1-\beta_i}) - \frac{1}{2}\ln^2(\frac{p_i.p_j}{E_iE_j(1-\beta_i)}) + \mathrm{Li}_2(1-\frac{E_iE_j(1+\beta_i)}{p_i.p_j}) - \mathrm{Li}_2(1-\frac{p_i.p_j}{E_iE_j(1-\beta_i)})$$
- For $m_i \neq 0,\ m_j \neq 0,\ a \neq b$:
$$\Omega_1^{ij} = \frac{(1+v_{ij})(p_i.p_j)^2}{2m_i^2}\left(J^{(A)}(\alpha_{ab}E_i, \alpha_{ab}E_i\beta_i) - J^{(A)}(E_j, E_j\beta_j)\right) \underset{m_i=m_j}{=} 0$$

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