Brief introduction to gravitational waves: theory, production and detection

Lecture Notes, in progress

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Abstract

Based on lectures given to M2/PhD students at the 2022 1st MaNiTou Summer School, Aix-Marseille University, with extra details and some additional material. To be completed with references. Work in progress, please let me know if you find any mistakes, typos or unclear paragraphs. I will make an updated version available on my homepage.

Contents

Preamble

References:

- Robert Wald, *General Relativity* (UCP 1984), the best reference for GR, but very little on GW
- Steven Weinberg, *Gravitation and Cosmology* (Wiley 1974), old but still quite good for the general background material including GW
- Eric Poisson and Clifford Will, *Gravity* (CUP 2014), for a more detailed description of gravitational waves and for going further than these lecture notes. Focuses on LL approach
- Michele Maggiore, *Gravitational waves* (OUP 2008), for a more detailed description of gravitational waves and for going further than these lecture notes. Focuses on averaging approach, but gives relation to the LL approach and effective-one-body
- Luc Blanchet, *Gravitational Radiation from Post-Newtonian Sources and Inspiralling Compact Binaries* (2013), `https://arxiv.org/abs/1310.1528`, for the advanced reader who knows already the PN expansion algorithm (not treated here)

1 Introduction

It is a great pleasure to start these lectures just a couple of days after the announcement by NanoGRAV of GW signals through PTA.

When the famous apple drops on Newton's head, the mass distribution of the Earth changes. Accordingly, the gravitation field created by the mass distribution having the apple on the tree is different than the one created when the apple has fallen. In Newton's time, this variation, however negligible, was assumed to be the effect of some instantaneous 'action at a distance'. After the discovery that the speed of light is finite, and that all causal effects in our universe seem to be subject to this limitation, it seems natural to expect that also the variations of the gravitational field will not be felt instantaneously in the whole universe, but will rather be propagated at the speed of light - or less.[1] The propagation of this perturbation of the gravitational field is intuitively what we call a gravitational wave. Conceptually, it is the same thing as a water wave or an electromagnetic wave propagating a modification in the height of water or the intensities of the electromagnetic field, respectively. What makes these waves physical is that they carry energy. Just like a charged particle emits electromagnetic waves when moving along a closed trajectory, the Earth emits gravitational waves when orbiting the sun, these carry away energy and make the orbit decay.

[1] We will shortly see that Einstein's GR predicts that they propagate at exactly the speed of light, and if some future experiment shows that they propagate at a lesser speed, than this would be an explicit violation of GR.

But in practise, it is a much subtler effect. First of all, because the absence of fixed background structure in GR makes the concept of a propagating way potentially ambiguous, and many decades passed before a consensus was reached. Famously, Einstein himself initially doubted the physical existence of GW, for reasons that we will briefly review and clarify below. Second, because if we think them as waves propagating a medium, this medium is extraordinarily rigid: therefore they go very fast and have very tiny amplitudes. To give an idea, the power emitted by the Earth in the form of GW is around 200W a year. This rigidity has to do with the weakness of gravity. You may think that gravity is strong when you are trying to beat that high jump record, or when you are skydiving. But this strength is ridiculously small compared to the much much stronger electro-magnetic force. Example of the chalk. Background independence and weakness of the signal are typical issues that one has to face when studying GWs.

2 Linearized general relativity

Einstein's equation of general relativity are given by

$$R_{\mu\nu} - \frac{R}{2}g_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{\kappa^2}{2}T_{\mu\nu}, \qquad \kappa^2 = \frac{16\pi G}{c^4}. \tag{2.1}$$

The value of the cosmological constant Λ can be determined from the observed acceleration of the cosmological expansion assuming homogeneity and isotropy on large scales, and turns out to be $\Lambda \simeq 10^{-52}m^{-2}$. This scale is largely irrelevant to the perturbative description of GWs we will be concerned with. Therefore, we will set $\Lambda = 0$ and ignore this term in the following.

The gravitational coupling $\kappa^2 \simeq 10^{-43}kg^{-1}m^{-1}s^{-2}$ is stupendously small, and this is the origin of the rigidity mentioned in the preamble. The smallness of this parameter has on the other hand a positive side: the gravitational force is so weak that many of the observed phenomena, in particularly virtually all solar system experiments, can be studied using the weak field approximation. This is quite helpful because the Einstein's equations are non-linear, and there is no general solution known. Strong gravity effects occur only near very compact objects, and to treat them, one has to resort to numerical techniques, or hope to be able to resum many order of the perturbative treatment.

Before discussing the weak field approximation, there are two fundamental facts about Einstein's theory that is useful to recall, to better appreciate what follows. First of all, its invariance under general coordinate transformations, also known as diffeomorphisms. This invariance is very similar to the gauge invariance of Maxwell and Yang-Mills theories. And indeed just like these theories, the resulting field equations have a three-sided structure: part of the field is completely arbitrary because of the gauge freedom, part is completely constrained because of the constraints associated with the gauge symmetry via Noether's theorem, and part describes independent degrees of freedom. This three-sided structure is better exposed doing the Hamiltonianian analysis of the dynamics, but for linear theories it can also be easily seen at the covariant level, as we will show below.

The second fact that is useful to recall is a direct consequence of diffeomorphism invariance, which prevents the existence of any fixed background structure. In particular, there is no universal notion of time, nor well-defined notion of energy density of the gravitational field. We will see that the weak field approximation, by means of the flat Minkowski background it introduces and its associated class of Cartesian observers, permits to select their proper time as preferred time. But the lack of well-defined notion of energy density is a subtlelty that persists also at the weak-field level.

2.1 Linearization of Einstein's equations

After this preamble, let us discuss the weak-field approximation. It is based on assuming that spacetime is on average flat, and only small departures from it are allowed. Accordingly, we write

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \qquad |h_{\mu\nu}| \ll 1 \tag{2.2}$$ {glin}

in some chosen coordinate system.[2] We use the background metric to raise and lower indices, e.g.

$$h^{\mu\nu} = \eta^{\mu\rho}\eta^{\nu\sigma}h_{\rho\sigma}, \qquad h := \eta^{\mu\nu}h_{\mu\nu}. \tag{2.3}$$

The inverse metric at leading order is

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}. \tag{2.4}$$ {ginvlin}

Using the expressions (2.2) and (2.4), we can systematically expand all other geometric quantities. For instance, the background Levi-Civita connection vanishes if we use Cartesian coordinates, and the first order is

$$\Gamma^{\mu}_{\nu\rho} = \frac{1}{2}\eta^{\mu\sigma}(2\partial_{(\nu}h_{\rho)\sigma} - \partial_\sigma h_{\nu\rho}). \tag{2.5}$$

Similarly,

$$R_{\mu\nu\rho\sigma} = -\partial_\rho\partial_{[\mu}h_{\nu]\sigma} + \partial_\sigma\partial_{[\mu}h_{\nu]\rho}. \tag{2.6}$$ {Riem1}

Contracting with the background metric, we obtain the expressions for the linearized Ricci tensor and Ricci scalar,

$$R_{\mu\nu} = -\frac{1}{2}\Box h_{\mu\nu} + \partial_{(\mu}\partial_\rho h^\rho{}_{\nu)} - \frac{1}{2}\partial_\mu\partial_\nu h, \qquad R = \partial_\mu\partial_\nu h^{\mu\nu} - \Box h, \tag{2.7}$$ {Ric1}

where $\Box$ is the flat space d'Alambertian. Using these results, the linearized Einstein's equations (multiplied by an overall -2 for convenience to have a unit coefficient in front of the d'Alambertian) give

$$\Box h_{\mu\nu} - 2\partial_{(\mu}\partial_\rho h^\rho{}_{\nu)} + \partial_\mu\partial_\nu h + \eta_{\mu\nu}(\partial_\rho\partial_\sigma h^{\rho\sigma} - \Box h) = -\kappa^2 T_{\mu\nu}. \tag{2.8}$$ {linE}

These equations are Poincaré covariant, a symmetry inherited from the chosen background. The presence of the d'Alambertian suggests that wave solutions are indeed possible. However, there is an intricate tensorial structure that needs to be dealt with. In particular, notice that the equations are invariant under the transformation

$$h_{\mu\nu} \mapsto h_{\mu\nu} + 2\partial_{(\mu}\xi_{\nu)}. \tag{2.9}$$ {hdiffeo}

By analogy with the similar invariance of Maxwell's equations, we can refer to (2.9) as a gauge transformation. This gauge transformation is nothing but an infinitesimal diffeomorphism: in fact, recall that the transformation of the metric under infinitesimal diffeos is $g_{\mu\nu} \mapsto g_{\mu\nu} + \pounds_\xi g_{\mu\nu}$. This preserves the decomposition (2.2) and induces (2.9) at leading order. Therefore, the gauge-invariance of (2.8) is nothing but the left-over of the diffeo-invariance of the non-linear theory. We will come back to this point below when talking about gravitational energy.

This means that four of the components of the matrix $h_{\mu\nu}$ are not dynamical: they are not determined by the field equations and can therefore take arbitrary values. Furthermore, four will be fixed by the constraints, the linearized version of the diffeomorphism constraints of the full theory. A linearized canonical analysis, which we report in Appendix B, exposes the three-sided nature of the linearized equations, and establishes that there are only two propagating degrees of freedom, just like in the full theory.

[2] This can be made covariant writing $g_{\mu\nu} = \bar{g}_{\mu\nu} + \kappa h_{\mu\nu}$ and expanding in powers of κ, but the results are equivalent and comparative factors of κ can be restored easily with dimensional analysis.

2.2 De Donder gauge

The invariance can be exploited to simplify the equations, and bring them closer to the form of a standard wave equation. To that end, we require the De Donder gauge

$$\partial_\mu h^\mu{}_\nu - \frac{1}{2}\partial_\nu h = 0. \tag{2.10}$$

(in the literature, it is also called Lorenz gauge by analogy with electromagnetism, or harmonic gauge because of its derivation from the non-linear theory, see App. A) It is immediate to see that this gauge can always be reached, choosing

$$\Box\xi^\mu = -\partial_\nu h^{\mu\nu} + \frac{1}{2}\partial^\mu h. \tag{2.11}$$ {DDc}

In this gauge, the field equations simplify to

$$\Box(h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h) = -\kappa^2 T_{\mu\nu}. \tag{2.12}$$

It is then convenient to define

$$\bar{h}_{\mu\nu} := h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h, \tag{2.13}$$

so that the linearized equations and De Donder gauge can be written more compactly as follows,

$$\Box\bar{h}_{\mu\nu} = -\kappa^2 T_{\mu\nu}, \qquad \partial_\mu \bar{h}^{\mu\nu} = 0. \tag{2.14}$$ {boxbarh}

The field equations are now in the form of a wave equation for the ten components of the matrix $\bar{h}_{\mu\nu}$. However, not all components are allowed to freely propagate, because of the constraints that have to be imposed to satisfy the De Donder gauge. Furthermore, the De Donder condition does not fix completely the gauge, since (2.11) admits infinitely many solutions. In other words, once the De Donder condition is satisfied, there remains a residual freedom of gauge transformations that satisfy $\Box\xi^\mu = 0$. We will see below how to further specialize the De Donder gauge to remove any residual arbitrariness.

2.3 Static Newtonian sources

Let's consider static solutions with respect to a Newtonian time t. In this case $T_{\mu\nu} = \rho(\vec{x})t_\mu t_\nu$, and (2.14) reduce to

$$\partial^2 \bar{h}_{00} = -\kappa^2 \rho. \tag{2.15}$$

Recalling the the Newtonian potential satisfies $\partial^2 V = 4\pi G\rho$, we identify $V = -\bar{h}_{00}/4$. All remaining components satisfy the vacuum Poisson equation, therefore they vanish with suitable boundary conditions. It follows that $h = -\bar{h}_{00}$ hence $h_{00} = \frac{1}{2}\bar{h}_{00} = -2V$. Then from the geodesic equation we get

$$\ddot{x}^\mu = -\Gamma^\mu_{00} = \begin{cases} 0 \\ \frac{1}{2}\partial_a h_{00} = -\partial_a V. \end{cases} \tag{2.16}$$

2.4 Spin and helicities of gravitational waves

A direct comparison of linearized gravity with electromagnetism is useful first to review many of the same physical concepts in the algebraically simpler case of a vector field as opposed to a tensorial field, and second to enhance the different physical behaviour due to the fact that one field has spin 1 (dipole moment) and the other has spin 2 (quadrupole moment). Let's start first by understanding why we talk about spin, and what is the difference between spin and helicity.

Recall that tensors in Minkowski space provide finite-dimensional representations of the Lorentz group. This property can be used to decompose each tensor into irreducible parts, namely the decomposition is preserved and not mixed up by a Lorentz transformation. Furthermore, one can restrict attention to a rotation subgroup of the Lorentz group, and decompose the tensor into different *spin* representations. For instance, a 4-vector or a 4-form like the Maxwell potential A_μ transforms under the irreducible Lorentz representation $(\mathbf{\frac{1}{2}}, \mathbf{\frac{1}{2}})$. Under the rotation little group preserving the time axis, this splits into two irreducible representations, $\mathbf{1} \oplus \mathbf{0}$, given by the spatial vector A^a and the spatial scalar A_0. We can interpret these components of the field as spin-1 and spin-0 modes respectively. Introducing the time direction $t^\mu = (1, 0, 0, 0)$, $t^2 = -1$, we can write this decomposition in terms of two projectors

$$P^{(\mathbf{1})} = \delta^\mu_\nu + t^\mu t_\nu, \qquad P^{(\mathbf{0})} = -t^\mu t_\nu, \qquad \mathbb{1} = P^{(\mathbf{1})} + P^{(\mathbf{0})}. \tag{2.17}$$

The spin-1 mode can be further decomposed into three different helicities: this is nothing but the familiar decomposition of spherical harmonics in spins l and magnetic numbers m given by the projection of the angular momentum along a given direction. In the case of mass-less waves (or particles), there is a preferred direction given by the spatial momentum. The projection along this direction defined what is called helicity of the wave (or particle). The decomposition of the spin-1 field into two components, one including the pair of helicities ± 1, and one of helicity 0, is a special case of the Helmholtz decomposition.

Recall the Helmholtz decomposition of a 3d vector field into solenoidal and irrotational parts:

$$A^a = A^a_{\mathrm{T}} + A^a_{\mathrm{L}}, \qquad \partial_a A^a_{\mathrm{T}} = 0, \qquad A^a_{\mathrm{L}} = \partial^a A, \qquad \nabla \times \vec{A}_{\mathrm{L}} = 0. \tag{2.18}$$ {AHelmholtz}

This decomposition is unique up to harmonic functions, $\partial^2 f = 0$, which can be freely traded from one piece to the other. Since the Poisson equation admits a unique solution for suitable boundary conditions, the Helmholtz decomposition is also unique up to boundary conditions. In the simplest case these are given by vanishing fields at the spatial boundary, and then all harmonic functions vanish exactly. Then we can do the usual Fourier transform, and for each mode this decomposition is nothing but the standard resolution of the identity into projectors,

$$\mathbb{1}^{(\mathbf{1})} = P^{(\mathbf{1})}_{\mathrm{T}} + P^{(\mathbf{1})}_{\mathrm{L}}, \qquad P^{(\mathbf{1})}_{\mathrm{T}}{}^a{}_b = T^a_b := \delta^a_b - \frac{p^a p_b}{\vec{p}^2}, \qquad P^{(\mathbf{1})}_{\mathrm{L}}{}^a{}_b = L^a_b := \frac{p^a p_b}{\vec{p}^2}, \tag{2.19}$$

so that

$$A^a_{\mathrm{T}} = P^{(\mathbf{1})}_{\mathrm{T}}{}^a{}_b A^b, \qquad A^a_{\mathrm{L}} = P^{(\mathbf{1})}_{\mathrm{L}}{}^a{}_b A^b = \frac{p^a p_b}{\vec{p}^2} A^b, \qquad A = \frac{\vec{p} \cdot \vec{A}}{\vec{p}^2}. \tag{2.20}$$

Remark: often one gives up the extra step of working in Fourier space and writes instead the projectors in configuration space, e.g.

$$\delta^a_b - \frac{\partial^a \partial_b}{\partial^2}. \tag{2.21}$$

This expression is somewhat implicit because one needs to specify boundary conditions in order to have a well-defined inverse of ∂^2. With the understanding that the boundary conditions are those of vanishing fields at spatial infinity, then the projector is equivalent to the momentum space definition. The latter is more explicit since unambiguous, and manifestly well-defined since $\vec{p} \neq 0$ for a null vector, but you can find both in the literature.

The notation T and L stand respectively for transverse and longitudinal, and should be obvious from the explicit expressions. The shorthand notation T^a_b and L^a_b will be useful below. To check that these correspond to helicities ± 1 and 0, we consider a wave propagating along the z axis with positive frequency ω. Then $\vec{p} = \omega(0,0,1)$, and the rotation matrix of the little group is given by

$$R^a{}_b = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.22}$$ {Rz}

Furthermore the projectors take the simple form

$$P^{(\mathbf{1})a}_{\mathrm{T}}{}_b = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad P^{(\mathbf{1})a}_{\mathrm{L}}{}_b = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.23}$$

Next we take the eigenvectors of the projectors and study how they behave under this rotation:

$$P^{(\mathbf{1})a}_{\mathrm{T}\;\;b} e^b_{\pm} = e^a_{\pm}, \qquad e^a_{\pm} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \pm i \\ 0 \end{pmatrix}, \qquad R^a{}_b e^b_{\pm} = e^{\pm i\theta} e^a_{\pm}, \tag{2.24}$$

$$P^{(\mathbf{1})a}_{\mathrm{L}\;\;b} e^b_0 = e^a_0, \qquad e^a_{\pm} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \qquad R^a{}_b e^b_0 = e^a_0. \tag{2.25}$$

The story is exactly the same for a gravitational perturbation. The symmetric matrix $h_{\mu\nu}$ transforms under the reducible Lorentz representation $(\mathbf{1},\mathbf{1}) \oplus (\mathbf{0},\mathbf{0})$. Under the rotation subgroup $SO(3)$, this splits into four spin-irreps, $\mathbf{2} \oplus \mathbf{1} \oplus \mathbf{0} \oplus \mathbf{0}$, given respectively by

$$h_{ab} - \frac{1}{3}\delta_{ab} h^c_c, \quad h_{0a}, \quad h_{00}, \quad h^c_c. \tag{2.26}$$

For a given direction of propagation of the wave, we can also introduce the notion of helicity, namely the projection of the spin along the axis of propagation. This amounts to looking at the component $\vec{J} \cdot \hat{p}$ of the rotational generators. A spin-2 mode then splits into 5 helicities $\pm 2, \pm 1, 0$ and a spin-1 mode into 3 helicities $\pm 1, 0$.

To see how this is done, the spin 0 and spin 1 parts work as before. For the spin-2 part, one needs a generalization of the Helmholtz decomposition, which is given by

$$\mathbb{1}^{(\mathbf{2})} = P^{(\mathbf{2})}_{\mathrm{TT}} + P^{(\mathbf{2})}_{\mathrm{L}} + P^{(\mathbf{2})}_{\mathrm{LL}}, \tag{2.27}$$

$$P^{(\mathbf{2})ab}_{\mathrm{TT}\;cd} = T^a_{(c} T^b_{d)} - \frac{1}{2} T^{ab} T_{cd}, \qquad P^{(\mathbf{2})ab}_{\mathrm{L}\;\;cd} = T^a_{(c} L^b_{d)} + L^a_{(c} T^b_{d)}, \qquad P^{(\mathbf{2})ab}_{\mathrm{LL}\;cd} = L^{\langle a}_{\langle c} L^{b\rangle}_{d\rangle}. \tag{2.28}$$

Here $\langle ab \rangle$ is the symmetric and traceless combination. One can easily check that these are projectors and that they are orthogonal to one another. Furthermore thanks to having defined them as traceless, their action on $h_{ab} - \frac{1}{3}\delta_{ab}h$ is the same as the action on h_{ab}. Explicitly, using these projectors and suitable fall-off conditions at spatial boundaries to define the inverse of ∂^2/Fourier transform, we have

$$h^{\rm TT}_{ab} = P^{(\mathbf{2})ab}_{\rm TT\,cd} h^{cd} = \left(T^a_c T^b_d - \frac{1}{2}T^{ab}T_{cd}\right) h^{cd}, \tag{2.29}$$

$$P^{(\mathbf{2})ab}_{\rm L\ \,cd} h^{cd} = 2p^{(a}\left(\delta^{b)}_c - \frac{p^{b)}p_c}{\vec{p}^2}\right)\frac{p_d h^{cd}}{\vec{p}^2} = 2\partial_{(a}B_{b)}, \qquad B_a = 2\left(\delta^b_a - \frac{\partial^b\partial_a}{\partial^2}\right)\frac{\partial_d}{\partial^2}h^{cd}, \tag{2.30}$$

$$P^{(\mathbf{2})ab}_{\rm LL\,cd} h^{cd} = \left(\frac{p^a p^b}{\vec{p}^2} - \frac{1}{3}\delta^{ab}\right)\frac{p_c p_d}{\vec{p}^2}h^{cd} = (\partial_a\partial_b - \frac{1}{3}\delta_{ab}\partial^2)B, \qquad B = \frac{\partial_c\partial_d}{\partial^4}h^{cd}. \tag{2.31}$$

One can then write

$$h_{ab} = [(P^{(\mathbf{2})}_{\rm TT} + P^{(\mathbf{2})}_{\rm L} + P^{(\mathbf{2})}_{\rm LL} + P^{(\mathbf{0})}_{\rm TR})h]_{ab} = h^{\rm TT}_{ab} + 2\partial_{(a}B_{b)} + (\partial_a\partial_b - \frac{1}{3}\delta_{ab}\partial^2)B + \frac{1}{3}\delta_{ab}h_{cc}. \tag{2.32}$$

Next, to study the different helicities of the spin-2 mode, we proceed as before and consider their transformation under a rotation along the axis of propagation. Because the spin-2 mode belongs to the 5-dimensional irrep of the rotational group, we would need to use the 5x5 version of $R_{\hat{z}}$. However no need to work that out, because this irrep is built simply as the symmetric and traceless tensor product of two spin-1 irreps, hence the 5d action is isomorphic to the action on the vector space of symmetric-traceless 3x3 matrices $A_{ab} \mapsto (RAR^T)_{ab}$. The action of the projectors is

$$P^{(\mathbf{2})ab}_{\rm TT\,cd} A^{cd} = (TAT)^{ab} - \frac{1}{2}{\rm Tr}(TA)T^{ab} = \begin{pmatrix} \frac{1}{2}(A_{11}-A_{22}) & A_{12} & 0 \\ A_{12} & -\frac{1}{2}(A_{11}-A_{22}) & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{2.33}$$ {PTTA}

and similarly for the other two. Then it is easy to check that the eigenvectors of the projectors are

$$e^+_{ab} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad e^\times_{ab} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{2.34}$$

$$f^1_{ab} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad f^2_{ab} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad e^{\rm LL}_{ab} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.35}$$

Defining as before the circular polarizations,

$$e^\pm_{ab} := (e^+_{ab} \pm i e^\times_{ab}), \qquad f^\pm_{ab} := 2p_{(a}(e^1_{b)} \pm i e^2_{b)}), \tag{2.36}$$

we have under the rotation (2.22)

$$R^c{}_a R^d{}_b\, e^\pm_{cd} = e^{\pm 2i\theta} e^\pm_{ab}, \qquad R^c{}_a R^d{}_b\, f^\pm_{cd} = e^{\pm i\theta} f^\pm_{ab}, \qquad R^c{}_a R^d{}_b\, e^{00}_{cd} = e^{00}_{ab}, \qquad R^c{}_a R^d{}_b\, e^{\rm tr}_{cd} = e^{\rm tr}_{ab}.$$

In the following, we will use often the TT projector, and just denote it $P^{\rm TT}$ removing the spin-2 label for short. Now the question is how many of these 10 different helicity modes are truly dynamical.

2.5 Gauge-invariant description: many Newton potentials

Consider first Maxwell's equations

$$\Box A_\mu - \partial_\mu \partial_\nu A^\nu = j_\mu. \tag{2.37}$$

They are invariant under the gauge transformation $A_\mu \mapsto A_\mu + \partial_\mu \lambda$. Accordingly, the covariant divergence $\partial_\mu A^\mu$ drops out, meaning that this component of the field is undetermined. In other words, it is a pure gauge quantity. This means that even fixing initial conditions, the PDE operator cannot be inverted, and the Green's function does not exist. This fact can be solved decomposing the equations in gauge-invariant quantities that can be inverted for, or by introducing a choice of gauge. Let us see one procedure at a time.

The Helmholtz decomposition seen earlier is useful to identify the gauge-invariant components of the magnetic vector potential, and make the 'three-sided' structure of Maxwell's equations manifest without recurring to the Hamiltonian analysis. Starting from (2.18), we see that A^a_{T} and $\Phi := A_0 - \dot{A}$ are gauge-invariant, and Maxwell's equations can be split into

$$\partial^2 \Phi = j_0, \qquad \Box A^a_{\mathrm{T}} + \partial^a \dot{\Phi} = j^a, \qquad \Box A^a_{\mathrm{T}} = j^a_{\mathrm{T}} \tag{2.38}$$

We see that one part of the field, Φ, is dynamical but its dynamics is entirely constrained by the matter content; one part, A_{T} has independent propagating degrees of freedom; and one part has dropped out and is left arbitrary. This is the 'three-sided' structure we where referring to earlier. Notice also that the distinction between independent degrees of freedom and constrained ones is non-local in configuration space. Therefore it is not possible with any local measurement to distinguish between the two. [But with a full Cauchy slice at disposal then yes, see the recent gendanken experiments by Bob]

In the static limit, it is possible to neglect radiation and study the Coulombic part alone. However one should be careful that this is a non-relativistic description: the Coulombic interaction is described as an 'action-at-a-distance' which violates causality. For instance, if I switch on a localized elctric source at a given instant of time, the electric field outside the light-cone will remain zero, but to see this one needs explicitly the radiative part of the field. Using the Coulombic part only, one gets an electric field instantaneously changing in the whole universe.

The fact that the gauge-invariant quantities are non-local functions of the fundamental variables makes them often unpractical for performing explicit calculations, and one is then better off working with a fixed gauge instead. Consider for instance the Coulomb gauge $\partial_a A^a = 0$. This sets $A = 0$ and leads to important simplifications: the gauge-invariant potential coincides with the temporal potential A_0, and the transverse modes coincide with the vector potential, thus eliminating the non-locality. Another common gauge choice is the temporal gauge $A_0 = 0$, which corresponds to the reduced phase space most commonly used in the canonical analysis. This choice also simplifies the description of the gauge-invariant potential by relating it to the single component $\dot{A}$. However the resulting potential A is then time dependent also when the source is static. This time dependence is clearly a gauge artefact, hence the Coulomb gauge is clearly preferable when working on static problems.

For radiative problems on the other hand, neither of these choices are preferable. The problem is that they are manifestly not Lorentz-invariant, whereas the field equations are. As a consequence, one ends up working with more cumbersome Green's functions. If we look again at the Maxwell's equations, we see that there is a Lorentz-invariant choice that is particularly helpful: the Lorenz gauge $\partial_\mu A^\mu = 0$, which leads to

$$\Box A_\mu = j_\mu. \tag{2.39}$$

One can now solve for the 4 components of A_μ using a simple and covariant Green function, and the solution will be valid provided it satisfies the Lorenz gauge condition. However, notice that there are infinitely many solutions to the Lorenz gauge condition. Given a solution, any other is obtained performing a further gauge transformation whose parameter satisfies the wave equation, $\Box\lambda = 0$.

One may wonder whether the Coulomb or temporal gauges are part of this family. The answer depends on whether there are sources or not: the Coulomb is compatible with the Lorenz gauge only in vacuum or with static sources, and temporal only in vacuum (everywhere).

To see this, notice that we can always achieve $A_0 = 0 = \partial_a A^a$ on some initial slice $t = 0$, by suitably choosing $\dot\lambda(0,\vec{x}) = -A_0$ and $\lambda(0,\vec{x}) = -1/\partial^2\partial_a A^a$. This fixes a unique solution of the wave equation for λ, and thus exhausts the gauge freedom. Then by the Lorenz gauge, $\dot A_0(0,\vec{x}) = 0$. Then from the Maxwell's equations in Lorenz gauge we find $\Box A_0 = j_0$, which fixes A_0 everywhere uniquely in terms of j_0 and the initial data $A_0(0) = \dot A_0(0) = 0$. In particular $A_0 = 0$ in vacuum. But if we are not in vacuum, then $A_0 \neq 0$ as a consequence of the Lorenz gauge.

Similarly, the Coulomb gauge everywhere is only compatible with the Lorenz gauge in vacuum or at most with static sources, since imposing $\partial_a A^a = \partial_\mu A^\mu = 0$ everywhere implies $\dot A_0 = 0$ and $\partial^2 A_0 = j_0$.

In conclusion, we see that two components of the electromagnetic field A_μ are non-physical, therefore the theory only propagates two independent degrees of freedom. Further, the nature of the non-propagating degrees of freedom is of two different type: one is arbitrary, and one is fully constrained.

These considerations apply also to the gravitational field. The Newtonian approximation given by Newton's equation alone is only meaningful in the static limit. It describes an instantaneous interaction, and causality is restored restoring the radiative dofs.

We can do the same for the gravitational perturbations, introducing gauge-invariant potentialas was studied initially by York and Bardeen. We denote

$$h_{00} = -2V, \tag{2.40}$$
$$h_{0a} = W_a = W_a^{\mathrm{T}} + \partial_a W \tag{2.41}$$
$$h_{ab} = h_{ab}^{\mathrm{TT}} + 2\partial_{(a}B_{b)} + (\partial_a\partial_b - \frac{1}{3}\delta_{ab}\partial^2)B + \frac{1}{3}\delta_{ab}h_s. \tag{2.42}$$

From these we can extract 6 gauge-invariant quantities:

$$h_{ab}^{\mathrm{TT}}, \tag{2.43}$$
$$\Phi := V + \dot W - \frac{1}{2}\ddot B = -\frac{1}{2}h_{00} + \partial_a \dot h_{0a} - \frac{3}{4}\frac{\partial_a\partial_b}{\partial^4}h^{ab}, \tag{2.44}$$
$$\Phi_a = W_a^{\mathrm{T}} - \dot B_a = (\delta_{ab} - \frac{\partial_a\partial_b}{\partial^2}h^{ab}) - \frac{1}{\partial^2}\partial_b \dot h_{ab} + \partial_a\frac{\partial_b\partial_c}{\partial^4}\dot h^{bc}, \tag{2.45}$$
$$\Psi = -h_s + \partial^2 B = \frac{3}{2}(\frac{\partial_a\partial_b}{\partial^2}h^{ab} - h_{aa}). \tag{2.46}$$

In terms of these variables,

$$G_{00} = \frac{1}{3}\partial^2\Psi, \qquad G_{0a} = \frac{1}{3}\partial_a\dot\Psi - \frac{1}{2}\partial^2\Phi_a, \tag{2.47}$$ {linconstrai

$$G_{ab} = -\frac{1}{2}\Box h_{ab}^{\mathrm{TT}} + \ldots \tag{2.48}$$

Decomposing $T_{\mu\nu}$ also in irreducible parts,

$$T_{00} = \rho, \qquad T_{0a} = s_a + \partial_a s, \qquad T_{ab} = \sigma_{ab} + 2\partial_{(a}\sigma_{b)} + (\partial_a\partial_b - \frac{1}{3}\delta_{ab}\partial^2)\sigma + \frac{1}{3}\delta_{ab}\tau, \tag{2.49}$$

one can rewrite the linear equations in terms of the following six independent ones,

$$\Box h^{\rm TT}_{ab} = -\kappa^2 T^{\rm TT}_{ab}, \tag{2.50}$$

$$\partial^2\Psi = \frac{3}{2}\kappa^2 T_{00}, \tag{2.51}$$

$$\partial^2\Phi_a = -\kappa^2 T^{\rm T}_{0a}, \tag{2.52}$$

$$\partial^2(\Phi - \frac{1}{6}\Psi) = \frac{3}{4}\kappa^2(s - \frac{1}{3}\tau). \tag{2.53}$$

Check static source: $\Psi = 6\Phi$ thus $\partial^2\Phi = \kappa^2/4\rho$ ok.

Why so many 'Newton potentials'? Similar situation in e.m: relativistic invariance forces us to go from one potential to two. Consider two charged wires and the analogy with Biot-Savart. Here we go from one to 4, as a consequence of the higher tensorial nature of the field. The new relativistic effects include precessions of equinoxes, light bending, and frame dragging (Lense-Thirring effect).

$$\ddot{\vec{x}} = -\vec{\nabla}V - \dot{\vec{W}} + \vec{v}\times\nabla\times\vec{W} - 2(\dot{h}\vec{v}) + ... \tag{2.54}$$

whose first two terms look like a Lorentz force.

This rewriting of the linearized equations in terms of gauge-invariant quantities makes theirs three-sided nature manifest: We have the four constrained modes, the two independent degrees of freedom, and four pure gauge quantities that dropped out of the equations. One solves first the equations for the dynamical modes, then fixes a gauge to determine the whole metric tensor.

In spite of being conceptually appealing, this decomposition turns out to be not very practical for studying perturbation theory. As in the Maxwell case, it is non-local and non-Lorentz covariant.

Considering gauge-fixing, the best option to study radiation is to use a covariant one like the De Donder. This is however not unique. It turns out that it can be uniquely specified requiring $h = h_{0a} = 0$. This is called the transverse-traceless gauge, and can always be achieved in vacuum. Once this is done, h_{00} is fully constrained by the matter content, in particular it vanishes only if there is no matter anywhere.

If we have a solution in a generic De Donder gauge, we can put it in TT gauge applying the projector

$$h^{\rm TT}_{ab} = P^{\rm TT}{}_{ab}^{cd} h^{\rm DeD}_{cd}, \qquad P^{\rm TT}{}_{ab}^{cd} := (P^c_{(a}P^d_{b)} - \frac{1}{2}P_{ab}P^{cd}), \qquad P_{ab} = \delta_{ab} - \frac{p_a p_b}{\vec{p}^2}. \tag{2.55}$$ {PTT}

The last term can also be written in terms of a unit vector on the sphere, representing the direction of propagation of the wave. Notice that this projector is defined in Fourier space; it is non-local, and fully gauge-invariant.

2.6 Polarization tensors

We can solve the vacuum wave equation decomposing it in plane waves,

$$\bar{h}_{\mu\nu}(x) = \int d^4p\, e_{\mu\nu}(p) e^{ip\cdot x} + cc. \tag{2.56}$$

where $e_{\mu\nu}(p)$ is the polarization tensor, whose form is determined by the field equations and the gauge choice. To simplify the notation in the following, we can without loss of generality consider a single monochromatic plane-wave,

$$\bar{h}_{\mu\nu}(x) = e_{\mu\nu}(p)\cos(p\cdot x). \tag{2.57}$$

Imposing the vacuum equations in the DeD gauge we get

$$p^2 = 0, \qquad p^\mu e_{\mu\nu} = 0. \tag{2.58}$$

From the first, we have that $p^0 = \pm|\vec{p}|$. We choose p^μ to be future-pointing, and accordingly denote $\omega := p^0 > 0$. This convention permits to identify the positive-frequency modes.[3] We can then write

$$p = \omega(1,\hat{p}), \qquad \hat{p}^a\hat{p}^b\delta_{ab} = 1 \tag{2.59}$$

The second condition gives 4 algebraic equations constraining the 10 components of the 4×4 symmetric matrix $e_{\mu\nu}$. There are six independent solutions, that can be parametrized in terms of p^μ and a choice of spatial orthonormal basis e^i, $i=1,2$, in the plane orthogonal to $\vec{p}$ (namely $t^\mu e^i_\mu = 0 = p^\mu e^i_\mu$): {6sols}

$$p_\mu p_\nu, \qquad p_{(\mu}e^i_{\nu)}, \qquad e^1_\mu e^1_\nu + e^2_\mu e^2_\nu, \tag{2.60a}$$ {4sols}

$$e^1_{(\mu}e^2_{\nu)}, \qquad e^1_\mu e^1_\nu - e^2_\mu e^2_\nu. \tag{2.60b}$$ {2sols}

[MAP these to the previous notation, show which of the ten e_{ab} we have eliminated.] It is straightforward to check that these are all the independent solutions. It is however often convenient to be more explicit, and pick a specific direction for the momentum. For instance, let's suppose that the 4-vector p is aligned with the z-direction, namely

$$p^\mu = \omega(1,0,0,1). \tag{2.61}$$ {pz}

There is no loss of generality in this choice, because we can always achieve this configuration by a Lorentz transformation, and the theory is Lorentz covariant: all results obtained in this frame can be trivially mapped to results in any other frame using a Poincare transformation. Then the equation to be solved is $e_{0\nu} + e_{3\nu} = 0$, and this is done by (2.60) with $e^i_\mu = \delta^i_\mu$, $i=1,2$. In the following, we will write both the general form of the solution, and the one specialized to (2.61).

Notice that we have chosen the independent solutions (2.60) so that they are all traceless, except the last one of (2.60a). However, not all solutions are physical. In fact, we still have the freedom to perform residual gauge transformations. For instance, consider $\xi^\mu := -2e^{i\mu}\sin(p\cdot x)$. This is admissible since $\Box\xi^\mu = 2p^2\epsilon^{i\mu}\sin(p\cdot x) = 0$, and will set to zero the second of (2.60). Another example is $\xi^\mu := p^\mu\sin(p\cdot x)$, which will set to zero the first solution. In fact, the four solutions (2.60a) can be simultaneously put to zero with residual gauge transformations.

To see this, consider a general linear combination of (2.60a), with arbitrary coefficients a,b,c,d:

$$h_{\mu\nu}(x) = \begin{pmatrix} a\omega^2 & -b\omega & -c\omega & -a\omega^2 \\ -b\omega & d & 0 & b\omega \\ -c\omega & 0 & d & c\omega \\ -a\omega^2 & b\omega & c\omega & a\omega^2 \end{pmatrix} \cos\big(\omega(t-z)\big). \tag{2.62}$$ {PWgauge}

[3] This is one instance where the Minkowski background structure plays a key role, thanks to its time translation symmetry. In a general dynamical spacetime, there is no time-like Killing vector that can be used to identify positive-frequency modes.

But as shown above, in vacuum it is always possible to reach the TT gauge in which $h_{0\mu} = h = 0$.[4] In terms of this basis of polarization tensors, this implies that in the TT gauge $a = b = c = d = 0$. In other words, (2.62) is a pure gauge solution in vacuum, and can be set to vanish identically without loss of physical information. For the same reason, it is also possible to make another diffeomorphism which will replace the $t-z$ in the cosine with $t-vz$ for an arbitrary constant v. Hence the pure gauge modes don't really propagate, and if a gauge is chosen so that they look like they are propagating, well one can do this with an arbitrary speed, the speed is not constrained in any way by the dynamics. Only for the physical modes, the propagation speed is fixed to be the speed of light by Einstein's equations. To use Eddington's words, the non-physical gauge modes propagate at the "speed of thought".

The only gauge-invariant ones are the the polarizations (2.60b),

$$e^{+}_{\mu\nu} := 2e_1^{(\mu}e_2^{\nu)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad e^{\times}_{\mu\nu} := e_1^{\mu}e_1^{\nu} - e_2^{\mu}e_2^{\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{2.63}$$ {polz}

As a result, the general physical plane-wave solution has the form

$$h_{\mu\nu}(x) = (h_+ e^{+}_{\mu\nu} + h_\times e^{\times}_{\mu\nu})\cos(p\cdot x) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cos\big(\omega(t-z)\big), \tag{2.64}$$ {PWsol}

for some constants h_+ and $h_\times$. In the second equality we used the polarization tensors given by (2.63) when the wave propagates in the z direction. Notice that we removed the bar since the solution is traceless, hence $\bar{h}_{\mu\nu} = h_{\mu\nu}$.

The solution (2.64) was found in a special case of the De Donder gauge. We see that the resulting wave tensor is transverse to the direction of propagation, and traceless. For this reason, this gauge is referred to as transverse-traceless gauge. Here we showed it for a special monochromatic ansatz. But this gauge can always be reached for vacuum solutions. On the other hand, it is not possible to achieve this gauge inside sources. This is quite analogue to the electromagnetic case, where the Lorenz gauge is only compatible with the Coulomb gauge in vacuum, and not inside sources.

The resulting perturbed metric is

$$ds^2 = -dt^2 + (1 + h_+\cos\omega u)dx^2 + (1 - h_+\cos\omega u)dy^2 + (1 + 2h_\times\cos\omega u)dxdy + dz^2, \tag{2.65}$$ {linmetric}

in terms of retarded time $u := t - z$ to shorten the notation. This metric is *not* a solution of the exact Einstein's equations; but it is a solution of the linearized theory. In other words, there are vacuum solution the take approximately the form (2.65) in some regions of spacetime, but none that has that exact form everywhere in spacetime.

Since we have ∞^3 choices of possible momenta for a plane wave, we conclude that a general superposition of plane waves will be described by $2 \times \mathbb{R}^3$ arbitrary numbers. These are the physical degrees of freedom of gravitational waves. The linearized approximation has allowed a complete characterization of the physical degrees of freedom of the theory. Notice that it is the same number of the Maxwell field, and of two scalar fields. However what changes between these three examples

[4] If we were not in vacuum, (2.60) are not solutions.

is the behaviour under Lorentz transformations of the plane waves. It is clear from the pictures that the two modes of a GW are related by a $\pi/4$ rotation. The electromagnetic modes on the other hand occur on orthogonal planes and are thus related by a $\pi/2$ rotation, see Fig. 2. This difference can be characterized in terms of a physical property of the waves called helicity, as we discussed earlier.

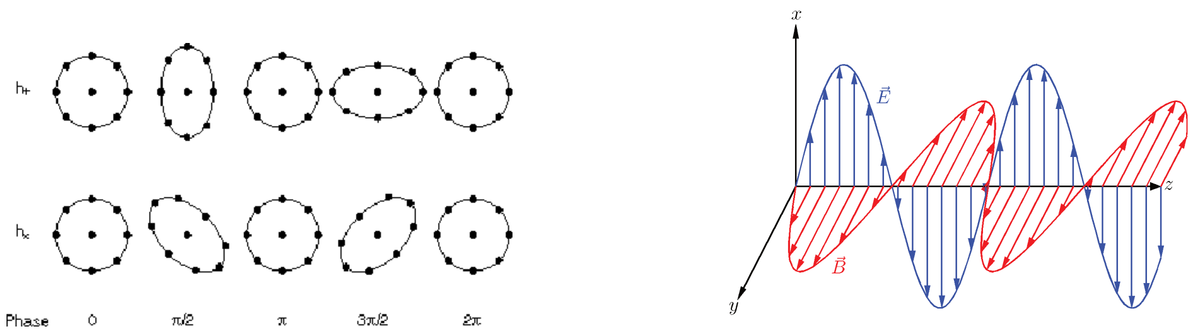


Figure 1: Figpols Left panel: . Right panel: *By comparison if it were a circular distribution of charged test particles and there is an electromagnetic field passing through, in the linearized approximation in which we neglect their self-interaction and the magnetic interactions, the whole circle will just move up and down along the axis of polarization of the wave, without deforming.*

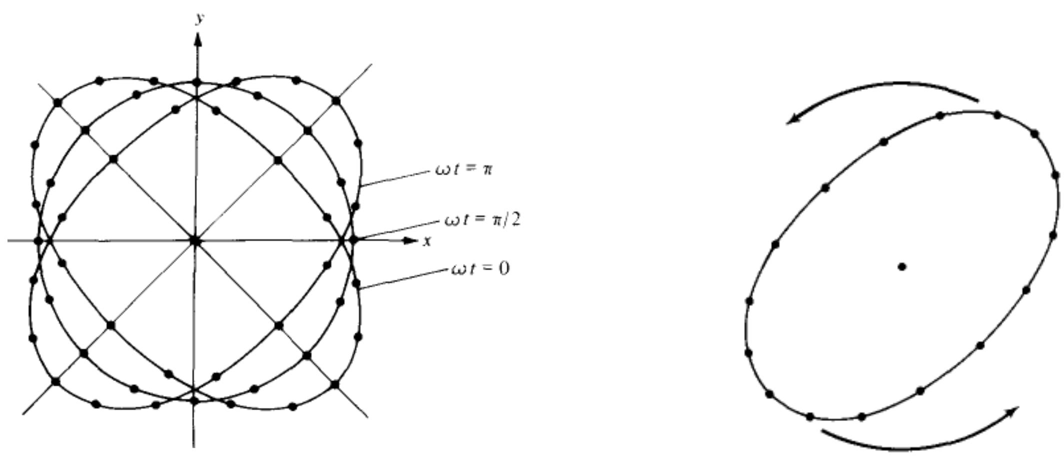


Figure 2: Figpols Left panel: *x polarization.* Right panel: *Circular polarization, right-handed if direction of propagation is towards the viewer.*

The mapping between notations:

$$e^{\pm}_{ab} := (e^{+}_{ab} \pm i e^{\times}_{ab}), \qquad f^{\pm}_{ab} := 2p_{(a}(e^{1}_{b)} \pm i e^{2}_{b)}). \tag{2.66}$$

Under the rotation (2.22) we obtain

$$R^c{}_a R^d{}_b\, e^{\pm}_{cd} = e^{\pm 2i\theta} e^{\pm}_{ab}, \qquad R^c{}_a R^d{}_b\, f^{\pm}_{cd} = e^{\pm i\theta} f^{\pm}_{ab}, \qquad R^c{}_a R^d{}_b\, e^{00}_{cd} = e^{00}_{ab}, \qquad R^c{}_a R^d{}_b\, e^{\rm tr}_{cd} = e^{\rm tr}_{ab}.$$

The last two above are shorthand notations for the *pp* and trace modes in (2.60a). We conclude that the physical modes have helicity 2, whereas the gauge modes have helicities 1 and 0.

The fact that these modes are pure gauge and can be set to zero will see has the important implication that GW cannot have neith monopole nor dipole radiation terms, similar to how the pure gauge nature of $A_{\rm L}$ means that e-m radiation has no monopole component. Since this is strictly related to the gauge invariance of the system, it would be different for massive theories, and it can be understood in physical terms as a consequence of the symmetries of the theory, associated to the invariances by Noether's theorem.

The fact that only maximal helicities are propagated by the Maxwell and linearized Einstein's equations leaves an imprint also in quantum field theory: massless particles must always be in eigenstates of maximal helicities, unlike massive particles.

2.7 Covariant gauge-invariant description

For the reader familiar with QFT, it is probably more customary to revisit the discussion using covariant projectors, which don't rely on a 3+1 splitting. Pro: covariant description of the spin-irreps, achieved without selecting a given time direction. Cons: off-shell description.

Short-hand notation:

$$\omega_{\mu\nu} := \frac{p_\mu p_\nu}{p^2}, \qquad \theta_{\mu\nu} := \delta_{\mu\nu} - \omega_{\mu\nu}. \tag{2.67}$$

Spin projectors:

$$P^{(2)} = \frac{1}{2}(\theta_{\mu\rho}\theta_{\nu\sigma} + \theta_{\mu\sigma}\theta_{\nu\rho}) - \frac{1}{3}\theta_{\mu\nu}\theta_{\rho\sigma} \tag{2.68}$$

$$P^{(1)} = \frac{1}{2}(\theta_{\mu\rho}\omega_{\nu\sigma} + \theta_{\mu\sigma}\omega_{\nu\rho} + \theta_{\nu\rho}\omega_{\mu\sigma} + \theta_{\nu\sigma}\omega_{\mu\rho}) \tag{2.69}$$

$$P^{(0)}_s = \frac{1}{3}\theta_{\mu\nu}\theta_{\rho\sigma}, \qquad P^{(0)}_\omega = \omega_{\mu\nu}\omega_{\rho\sigma}, \qquad P^{(0)}_{s\omega} = \frac{1}{\sqrt{3}}\theta_{\mu\nu}\omega_{\rho\sigma}, \qquad P^{(0)}_{\omega s} = \frac{1}{\sqrt{3}}\omega_{\mu\nu}\theta_{\rho\sigma} \tag{2.70}$$

Using these, we have (adding a source term)

$$\Box\left(P^{(2)} - 2P^{(0)}\right)h_{\mu\nu} = -\kappa^2 T_{\mu\nu}. \tag{2.71}$$

This formulation is used in perturbative quantum gravity, and it is also the starting point of the EFT application to classical gravitational perturbation theory Notice the mixing with the spin 0: this is ultimately responsible of the Van Damme discontinuity that rules out applications of linearized massive gravity to solar system physics. We see that one spin-0 and the spin-1 modes drop out: they are arbitrary, pure gauge. Of the remaining 6, only two propagate independently. However it is slightly harder to see this here, because going on-shell introduces divergences in the propagator. So what one has to do is to contract the propagator with conserved sources, then compute the residues at the pole. Would be nice to add a reference where this is done explicitly.

3 Detection of GWs

3.1 Gauge and reference frames

There is another difference between e-m gauge, which is not observable, and gravitational gauge, which has instead a physical interpretation. Different coordinate systems can in fact be thought of as describing different type of reference frames attached to different observers.

For instance in flat spacetime, Cartesian coordinates describe Newtonian inertial observers, and rotating coordinates e.g. those obtained from (2.22) with $\theta = \omega t$ describe a non-interntial observer on a rotating frame of constant angular velocity.

More in general in curved spacetime Cartesian coordinates don't exist, but they can be introduced locally and this is an implementation of the equivalence principle in Riemannian geometry. For instance, Riemann normal coordinates satisfy $g_{\mu\nu} = \eta_{\mu\nu}$ and $\Gamma^\mu_{\nu\rho} = 0$ at one chosen point. These coordinates describe a local inertial frame. One can explicitly construct such coordinates, and prove that

$$g_{\mu\nu} = \eta_{\mu\nu} - \frac{1}{3}R_{\mu\rho\nu\sigma}x^\rho x^\sigma + O(x^3), \qquad \Gamma^\mu_{\nu\rho} = -\frac{2}{3}R^\mu{}_{(\nu\rho)\sigma} \quad \textit{check} \tag{3.1}$$

Another example is Fermi normal coordinates, that satisfy $\Gamma^\mu_{\nu\rho} = 0$ along a time-like geodesics. These coordinates describe a freely falling frame, and are relevant to describe the physics of Lisa to a first approximation.

[5]

How about the TT gauge used to simplify the solution of the GW equation in vacuum? In this gauge

$$\Gamma^\mu_{00} = \partial_0 h^\mu{}_0 - \frac{1}{2}\partial^\mu h_{00} + O(h^2) = 0 + O(h^2). \tag{3.2}$$ {Gm00}

This implies that at first order the *coordinate distance* between two nearby time-like geodetics remains the same, and also that their coordinate time delay is the same. To see this, let us look at the geodesics equation

$$\frac{du^\mu}{d\tau} + \Gamma^\mu_{\nu\rho}u^\nu u^\rho = 0, \tag{3.3}$$

where

$$u^\mu = \frac{dx^\mu}{d\tau}, \qquad u^2 = -1 \tag{3.4}$$

describes time-like trajectories in proper time. If the velocity is constant, then $u^0 = \gamma$ and $u^a = \gamma v^a$ where $v^a = dx^a/dt$ is the velocity wrt lab time. Now assume that these geodesics define a collection of particles which are initially at rest: $u^a = 0$. Then

$$\dot{u}^\mu = -\Gamma^\mu_{00}u^0 u^0 = 0 \tag{3.5}$$

at first order, thanks to (3.2). It follows that test particles at rest before the passage of the wave remain at rest even doing the passage of the wave, at first order. Then using these test masses to build the coordinate system, we obtain one that remains constant across a gravitational wave. This is kind of like the temporal gauge that gives us spatial coordinates which are Lie-dragged by the normal, and a constant lapse function.

3.2 Coordinate displacements versus physical displacements

To study the effect of a gravitational wave, let us look at the geodesics equation

$$\frac{du^\mu}{d\tau} + \Gamma^\mu_{\nu\rho}u^\nu u^\rho = 0. \tag{3.6}$$

[5] Uses a FW transported tetrad and proper time coordinate τ, $\frac{d}{d\tau}v^\mu = 2u^{[\mu}a^{\nu]}v_\nu$. Then $g_{00} = -1 - 2a_a x^a + O(x^2)$, $g_{00} = O(x^2)$, $g_{ab} = \delta_{ab} + O(x^2)$ on the whole trajectory, and hence linear order vanish for a geodesics. With a more general transport, other inertial effects can be included.

At first order, using our solution (2.64), we find that the spatial coordinates are constant:

$$\frac{du^a}{d\tau} = -(\partial_0 h_0{}^a - \frac{1}{2}\partial^a h_{00})u^0 u^0 = 0. \tag{3.7}$$ {geo0}

Therefore, the *coordinate distance* between two points x and x' remains the same.

So this gives a meaning to the TT gauge: it is the gauge in which coordinates oscillate together with the wave, so that any two points remain at the same coordinate distance at first order. We could have chosen another gauge, than coordinate distance would change arbitrarily.

However, recall that the *physical distance* between two points is not merely the coordinate distance, but it includes the effect of the curving of spacetime described by the metric,

$$L = \int_x^{x'} ds \sqrt{g_{\mu\nu}u^\mu u^\nu}, \tag{3.8}$$ {Lxx}

where s is an affine parameter for the curve connecting the two points, and $u^\mu = dx^\mu/ds$ its tangent vector. For instance, let us consider two points along the x axis, say x_1 and x_2, and a monochromatic plane wave going in the z direction. By (3.7), the coordinate distance $\Delta x := x_2 - x_1$ is constant even if a GW like (2.64) is passing through. But the physical distance changes and oscillates in time:

$$L = \int_{\Delta x} dx \sqrt{g_{xx}} = \int_{\Delta x} dx \sqrt{1 + h_+ \cos\omega t} = \sqrt{1 + h_+ \cos\omega t}\,\Delta x \simeq (1 + \frac{1}{2}h_+ \cos\omega t)\Delta x. \tag{3.9}$$ {DL}

The physical distance, also known as *proper* distance, which can be measured e.g. bouncing light back and forth, changes. This is how we can detect GWs!

Notice the important role played by the difference between coordinate and physical distances. It is something that you should be already familiar with, for instance if you took a course in cosmology. There, it is often convenient to choose a coordinate system such that the values of the coordinate grid represent galaxies, so that their coordinate distance does not change, but the physical distance does.

It is in other words a good example of the most fundamental lesson of general relativity: coordinates are absolutely void of any physical meaning, and one can always find coordinates systems in which things look like nothing is happening. To understand the true dynamics, one has to look at physical quantities, which are geometric observables built in a coordinate-invariant way, such as proper distances.

This does not mean that coordinates are useless! They are very often necessary to perform calculations, and it is important to choose a clever coordinate system adapted to the physical system under scrutiny in order to simplify the calculations. If no mistakes are made, all physical results will be independent of such choices in the end. In this specific example, it was important to choose coordinates preserving the location of the test particle, so to be able to use (3.8) at all values of t without having to change the extrema of the integral. Had we chosen a different coordinate gauge, we would have needed to incorporate the t dependence of the extrema. Calculation harder, but same result in the end.

Let us further elaborate on (3.9). We can dispose of the coordinate distance Δx if we look at the relative change in physical distance

$$\frac{\delta L}{L_0} := \frac{L - L_0}{L_0} \simeq \frac{1}{2}h_+ \cos\omega t, \tag{3.10}$$

where L_0 is the unperturbed physical distance. To see the difference between the h_+ and $h_\times$ components, let us consider a circular distribution of point masses in the $z = 0$ plane. Using (2.64), it is clear that h_+ causes a shear in the xy directions, hence the label "+", whereas $h_\times$ causes a shear in directions tilted by 45 degrees, hence the label " $\times$ ". We can then deduce that the effect for two test masses initially at distance L_0 along the direction e^a in the plane orthogonal to the wave is

$$\frac{\delta L}{L_0} \simeq \frac{1}{2} h^{\mathrm{TT}}_{ab} e^a e^b. \tag{3.11}$$

We have assumed that the wave is perfectly perpendicular to the distribution of test masses. If it is not, integration will be more complicated. However it remains trivial if we assume that $\lambda \gg L$. Then the formula is still valid, and e^a can be any spatial direction for any given direction of propagation of the wave.

This result can also be derived from the geodesic deviation equation.

We have seen that gravitational waves change the physical distance between bodies. And if we have a circular distribution of test masses, their spin-2 nature shows up in the quadrupolar deformation of the distribution. So the most direct way to detect a GW would be to measure the relative acceleration of two test masses, namely the time-dependence of the gravitational tidal force. For two nearby free-falling test masses, this force is described by the geodesic deviation equation,

$$\frac{d^2 \xi^\mu}{d\tau^2} = R^\mu{}_{\nu\rho\sigma} u^\nu u^\rho \xi^\sigma, \tag{3.12}$$

where ξ^μ is a vector connecting neighbouring geodesics, chosen orthogonal to u^μ. If moving slowly,

$$\frac{d^2 \xi^a}{d\tau^2} = R^a{}_{00b} \xi^b = \frac{1}{2} \ddot{h}^{ab}_{\mathrm{TT}} \xi_b \tag{3.13}$$

hence if we have a detector that can detect tidal effects, it will be sourced precisely by the physical components of the GW, and not by the gauge ones.

If the spacetime curvature is caused by a GW, and the if the distance between the masses is smaller than the wavelength, then Riemann is roughly constant in space and we can trivially integrate this equation to get

$$\xi^a(t) = L_0 (e^a + \frac{1}{2} h^{\mathrm{TT}}_{ab} e^b), \qquad L = e_a \xi^a = L_0 (1 + \frac{1}{2} h^{\mathrm{TT}}_{ab} e^a e^b) \tag{3.14}$$

as before.

This result can be used to detect waves measuring the distance between test masses. Via laser bemas, or connecting them say with a resonant bar (a solid bar would be set into oscillation by the stresses, and one could look for resonant frequencies)

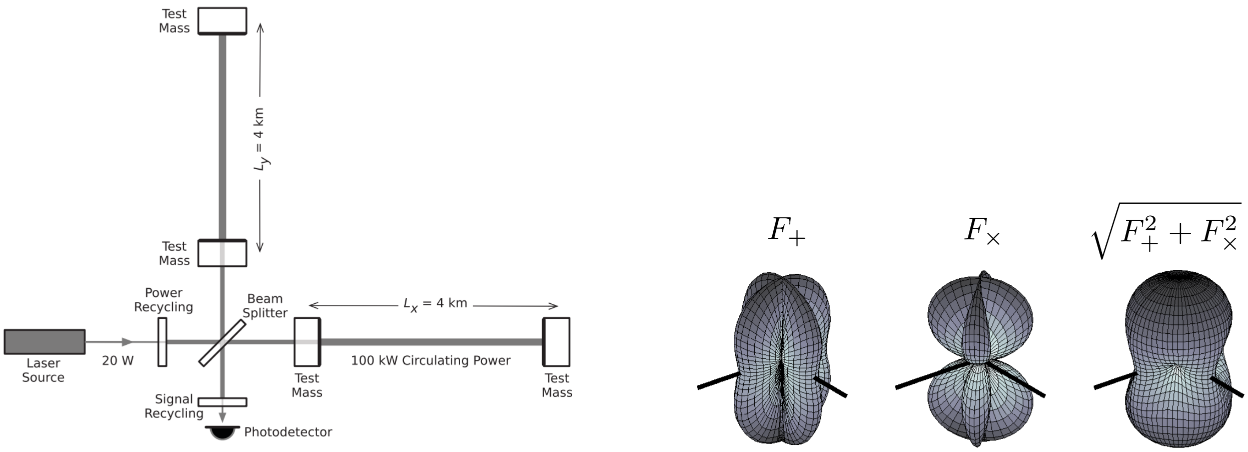
If we set the lasers so that the phases at the beam splitter are identical, the phase shift after the travel to and back from the mirrors will be

$$\Delta\phi = \frac{2\pi\nu}{c} N_p (2L_1 - 2L_2), \tag{3.15}$$

where $N_p = 1$ for a Michelson device and up to 300 for the Fabry-Pérot type used in Ligo/Virgo. Plugging in the previous result we arrive at

$$\Delta\phi = \frac{2\pi\nu}{c} N_p L_0 h^{\mathrm{TT}}_{ab} (e^a_1 e^b_1 - e^a_2 e^b_2), \tag{3.16}$$

where now e^a are the unit vectors giving the direction of each arm.



FigScheme
Figure 3: Left panel: *Schematics of an interferometer like LIGO/Virgo.* Right panel: *Antenna pattern functions for LIGO in the long-wavelength approximation, from [?].*

This equation is only valid for $\lambda \gg L$, otherwise further terms are needed. This is the case for resonant bars and Ligo but not Lisa, so further effects need to be taken into account.

It is possible to express the result in terms of the two wave polarizations, if we introduce a rotation from $R(\theta, \phi) := R_{\hat{z}}(\phi) R_{\hat{x}}(\theta)$ from the detector's frame to the frame of propagation, plus a reflection. This leads to

$$\Delta\phi = \frac{4\pi\nu}{c} N_p L_0 (F_+ h_+ + F_\times h_\times), \tag{3.17}$$

where

$$F_+ = \frac{1}{2}(1 + \cos^2\theta)\cos 2\phi \cos 2\psi - \cos\theta \sin 2\phi \sin 2\psi \tag{3.18}$$

$$F_\times = \frac{1}{2}(1 + \cos^2\theta)\cos 2\phi \sin 2\psi + \cos\theta \sin 2\phi \cos 2\psi \tag{3.19}$$

are called detector's pattern functions. Here ψ is the angle between the $\hat{x}, \hat{y}$ axis obtained after the rotation and whatever basis is used to define the two polarizations. This is interesting because we see that there are directions in which the detector is blind, like $(\theta, \phi) = (\frac{\pi}{2}, \frac{\pi}{4})$.

So with 3 detectors we can deduce θ, ϕ and ψ and then we are sensible to polarization.

For non-orthogonal arms, F multiplied by an overall $\sin\chi$, provided we orient them symmetrically with respect to the initial frame.

4 Generation of GWs: Introducing sources

At leading order we have (4.1). Hence the sources follow geodesics on Minkowski. Test bodies cannot be affected by gravity at linear order. To see the effect of gravity, we need to go beyond lowest order, and look at the geodesic equation, which is associated with $\nabla_\mu T^{\mu\nu} = 0$. So recovering Newton's law from the geodesics already requires going beyond the linearized approximation. In other words, the linearized approximation still treats GR as a force on flat spacetime, really. It includes however special relativistic effects not present in the Newtonian description if we go beyond the slow-motion approximation.

We recover Newton's equations. However recall that we have a very different perspective here: notion of inertia are different. We now go beyond the static limit. Beyond the static limit, it is not meaningful to look at the Poisson equation alone: it describes an instantaneous action at a distance, incompatible with causality. As in electromagnetism, restoration of causality is restored appreciating that radiative modes are necessarily excited in non-static configurations.

4.1 Source multipoles

Let us go back to the linearized field equation (**??**), and discuss now the situation in the presence of a non-vanishing right-hand side. The first thing to notice is that by the linearized Bianchi identities on the left-hand side, the source energy-momentum must be conserved with respect to the background metric,

$$\partial_\mu T^{\mu\nu} = \dot{T}^{0\nu} + \partial_a T^{a\nu} = 0. \tag{4.1}$$ {pT0}

This is nothing but the zeroth order of the covariant conservation law $\nabla_\mu T^{\mu\nu} = 0$. Recall from your GR classes that this equation implies that test masses follow geodesics. Hence, in the linearized approximation the matter sources follow geodesics of flat spacetime. This means that matter can interact with itself, but not with the gravitational field: all bodies must move on geodesics of the Minkowski metric. To include sources whose energy has a non-negligible gravitational origin, we must go beyond this approximation. Therefore the linearized theory is valid in weak field, but also neglecting gravitational self-interactions. In other words, the linearized theory still describes gravity in the Newtonian way, namely as a force acting in flat spacetime. However, it already contains departures from Newton's theory, since it includes the special relativistic effects such as the gravito-magnetic interaction and radiation.

A student asked if these quantities are gauge invariant. The answer is yes, but in a trivial way: this $T_{\mu\nu}$ is the lowest-order one, it is evaluated wth metric η. Therefore it is not a diffeo tensor (but only a Poincare tensor), and does not transform under diffeos.

This standard flat spacetime conservation law immediately implies conservation of the total energy and momentum,

$$M = \int d^3x\, T^{00}, \quad \dot{M} = 0, \qquad P^a = \int d^3x\, T^{0a}, \quad \dot{\vec{P}} = 0. \tag{4.2}$$

where we used integration by parts and vanishing boundary conditions. Although the first is truly the total energy, it is customary to denote it as a mass, using Newtonian language. Next, we look at the center of mass position,

$$M^a = \int d^3x\, x^a T^{00}, \qquad \dot{M}^a = -P^a. \tag{4.3}$$

Thus recovering the familiar result that the total momentum vanishes in the center-of-mass frame. The quantity M^a can be also called mass-dipole moment. The term moment comes because if we see $\rho = T_{00}$ as a distribution, this quantity represent the first moment of that distribution. We define all higher moments as

$$M^{ab\ldots} = \int d^3x T_{00} x^a x^b \ldots, \tag{4.4}$$

and similarly for P^a,

$$P^{ab\ldots} = \int d^3x\, T^{0a} x^b \ldots. \tag{4.5}$$

Another useful identity

$$x^a x^b \ddot{T}^{00} = -x^a x^b \partial_c \dot{T}^{c0} = \dot{T}^{c0}\partial_c(x^a x^b) = 2x^{(a}\dot{T}^{b)0} = -2x^{(a}\partial_c T^{b)c} = 2\partial_c x^{(a}T^{b)c} = 2T^{ab}. \qquad (4.6)$$

from which it follows that

$$\int d^3x\, T^{ab} = \frac{1}{2}\int d^3x\, \ddot{T}^{00} x^a x^b. \qquad (4.7)$$ {TabtoT00}

The trace-less part of this second momentum of the energy density is the quadrupole moment.

4.2 Solving the wave equation with sources

Having found vacuum solutions of the wave equations in the previous Section, let us go back to (**??**) and look for inhomogeneous solutions. For this we can use Green's method for solving differential equations. First we look for a solution to the distributional problem

$$\Box_x G(x,x') = \delta^{(4)}(x-x'). \qquad (4.8)$$

There are two independent solutions, called retarded and advanced, and characterised by whether they vanish respectively for x in the past or the future of x',

$$G_\pm(x,x') = -\frac{\delta(t-t' \mp |\vec{x}-\vec{x}'|)}{4\pi|\vec{x}-\vec{x}'|} = -\frac{1}{2\pi}\Theta\big(\pm(t-t')\big)\delta\big((t-t')^2 - |\vec{x}-\vec{x}'|^2\big). \qquad (4.9)$$

The retarded solution imposes no-incoming radiation boundary conditions, and it is the one relevant to study the emission of waves from a source. Then, in De Donder gauge we have

$$\bar{h}_{\mu\nu} = -\kappa^2\int d^4x'\, G(x,x')T_{\mu\nu}(x') = \frac{\kappa^2}{4\pi}\int d^3x'\, \frac{T_{\mu\nu}(t-|\vec{x}-\vec{x}'|,\vec{x}')}{|\vec{x}-\vec{x}'|}. \qquad (4.10)$$ {hwithT}

Recall that outside the source, we can achieve the TT gauge without loss of generality, so $h=0$ and $h_{0a}=0$, but h_{00} will not be zero, but rather constrained in terms of the matter distribution. This integral is in general very complicated and we don't have an analytic solution. So we resort to approximation schemes.

We now introduce two independent approximations:

(*i*) Wave-zone approximation: we assume to be very far away from the sources, hence $r := |\vec{x}| \gg |\vec{x}\,'|$. This is an expansion in powers of $1/r$, where

$$\frac{1}{|\vec{x}-\vec{x}'|} = \frac{1}{r} + \frac{\vec{x}\cdot\vec{x}'}{r^3} + \frac{3}{2}(x'_a x'_b - \frac{r'^2}{3}\delta_{ab})\frac{x^a x^b}{r^5} + \dots \qquad (4.11)$$ {rinvexp}

with $r := |\vec{x}|$, $\vec{n} = \vec{x}/r$.

Furthermore, the direction of propagation of the wave coincides with the direction from the source $\hat{r}^a := x^a/r$ (assuming the origin inside the source). Hence TT projector can be written in terms of $\hat{n}$.

From

$$|\vec{x}-\vec{x}\,'| = r - \frac{\vec{x}\cdot\vec{x}\,'}{r} + \dots. \qquad (4.12)$$

we also have the Taylor series the components of the energy-momentum tensor,

$$T_{\mu\nu}(t-|\vec{x}-\vec{x}\,'|,\vec{x}\,') \simeq T_{\mu\nu}(t-r,\vec{x}\,') + \hat{n}\cdot\vec{x}\,'\,\dot{T}_{\mu\nu}(t-r,\vec{x}\,') + \dots \qquad (4.13)$$ {Texp}

(*ii*) Slow dynamics: We assume that the dynamics of the source is slow, so that time derivatives can be neglected. This is an expansion in $v/c \ll 1$ and it is called Post-Newtonian expansion.

For non-gravitational interactions, this is an independent approximation, on top of the weak grav field. But if the source is held together by gravity, then this is a necessary approximation in order to be consistent with the weak-field approximation. In the Newtonian approximation, energy conservation gives $\mu v^2 = G\mu m/r$ and thus $v^2/c^2 = r_S/(2r)$. Weak gravitational fields hence implies small velocities.

This allows us to neglect higher-order terms in (4.13).

As a consequence of (*ii*), we can neglect the $O(1/r)$ term in (4.13) since it involve time variations. With these two approximations, which are by the way typical also in electromagnetic theory, we arrive at

$$h_{\mu\nu}(x) = \frac{\kappa^2}{4\pi r}\int d^3x'\, T_{\mu\nu}(t-r,\vec{x}\,'). \tag{4.14}$$

Indeed we can confirm this result from the leading order approximation we just derived, which gives[6]

$$h_{00} = \frac{\kappa^2}{4\pi r}\int d^3x'\, T_{00}(t-r,\vec{x}') = \frac{4G}{c^4}\frac{M}{r}. \tag{4.16}$$

This reproduces the Newtonian result (and notice M constant as a consequence of the conservation law).

Concerning the radiation part, we write in TT gauge

$$h^{\rm TT}_{ab} = \frac{\kappa^2}{4\pi r}P^{\rm TT}\int d^3x'\, T_{ab}(t-r,\vec{x}\,'), \tag{4.17}$$

where the propagation direction in the TT projector is $\hat{r}$.[7] This formula can be further manipulated to bring out the multipoles of the source. Also a static source cannot radiate so better to make this explicit bringing in time derivatives. Conservation laws implies that the integrand vanishes if the source is static. In fact, from the conservation law one has the identity (4.7). and we recall that $T^{00} = \rho$ is the energy density of matter. This is the second momentum of the energy distribution. This quantity differs from the quadrupole moment only by a trace term, see (**??**). But the trace term is irrelevant because of the projector, therefore we can freely replace the right-hand side with $\ddot{Q}^{ab}/2$. We conclude that

$$h^{\rm TT}_{ab}(t,\vec{x}) = \frac{2G}{c^4}\frac{1}{r}\ddot{Q}^{\rm TT}_{ab}(t-r). \tag{4.18}$$ {Qformula}

[6]This result may look unfamiliar, since the Schwarzschild metric in static coordinates differs from it by a factor of 2. But recall that our analysis assumes De Donder gauge – otherwise (4.10) is immediately incompatible with conservation of $T_{\mu\nu}$! –, aka harmonic gauge. The Schwarzschild metric in harmonic gauge reads

$$ds^2 = -\frac{\rho - r_{\rm s}/2}{\rho + r_{\rm s}/2}dt^2 + \frac{\rho + r_{\rm s}/2}{\rho - r_{\rm s}/2}d\rho^2 + (\rho + r_{\rm s}/2)^2 d\Omega, \qquad \rho = r - r_{\rm s}/2,$$

and then

$$h_{00} = \frac{2M}{\rho}, \qquad h_{rr} = \frac{2M}{\rho}, \qquad h_{AB} = 2M\rho h^{\rm S^2}_{AB}, \qquad h = -h_{00} + h_{aa} = \frac{4M}{\rho}. \tag{4.15}$$

So the factor 4 instead of 2 is because we are in traceless gauge, which is not the usual coordinates in which we write the metric.

[7]This is line with the far-away approximation we are taking: the structure of the source is neglected, and the direction of propagation of the wave identified with the direction between the source and the observer.

This is the celebrated *quadrupole formula*, derived by Einstein in 1918: The dominant radiation in the slow-motion approximation arises from the acceleration of the quadrupole moment. From the derivation, we see that it is the first term in a multipolar expansion of the field. Similarly to what we did with (4.7), the higher order multipolar corrections can also be rewritten in terms of the energy and momentum of the source. For instance, the next term coming from (4.13) depends on $\dot{T}^{ab}$, and this can be rewritten in terms of $\dddot{T}^{00}$ and $\ddot{T}^{0a}$.

The fact that the lowest contribution is quadrupole means that there is no monopole nor dipole. Absence of monopole is familiar already from Maxwell theory. In particular, if a source has a varying energy distribution but preserves spherical symmetry, it does not radiate GWs.

Absence of dipole comes on the other hand from the conservation of the energy momentum tensor. In e-m, the total charge is conserved, but not the charge dipole:

$$\vec{D} = \sum_i q_i \vec{x}_i, \qquad \dot{\vec{D}} = \sum_i q_i \vec{v}_i, \qquad \ddot{\vec{D}} = \sum_i q_i \vec{a}_i. \tag{4.19}$$

In gravity (even Newtonian), the mass dipole is conserved:

$$\vec{D} = \sum_i m_i \vec{x}_i, \qquad \dot{\vec{D}} = \sum_i \vec{p}_i, \qquad \ddot{\vec{D}} = \sum_i \dot{\vec{p}}_i = 0 \tag{4.20}$$

and similarly the magnetic dipole from ang. mom. conservation.

Absence of monopole and dipole radiation should be expected on general grounds. At the linearized level, it follows from $\dot{M} = \dot{\vec{P}} = 0$. These are no longer valid through interactions and back reaction which lead to a gravitational system losing energy. However they are replaced by other conservation equations that guarantee absence of monopole and dipole radiation to all orders.

Let's make some estimates. Imagine that the matter distribution M in a volume of radius R, with typical time-scale T. Then $Q \sim MR^2$, and $\ddot{Q} \sim MR^2/T^2 = M(v/c)^2$. Hence an order-of-magnitude estimate gives

$$h \sim \frac{1}{r}\frac{GM}{c^2}\frac{v^2}{c^2} = 5 \times 10^{-19} \left(\frac{M}{10M_\odot}\right)\left(\frac{1\text{Mpc}}{r}\right)\frac{v^2}{c^2}. \tag{4.21}$$

at 100 Mps for 10 solar mass BHs at orbital distance of 10 rS and relativistic velocities we get 10^{-21}.

Angular distribution: Using (2.33) with $\hat{p} = \hat{z}$,

$$Q^{\text{TT}}_{ab} = P^{\text{TT}}(Q) = P^{\text{TT}}(M) = \begin{pmatrix} \frac{1}{2}(M_{11} - M_{22}) & I_{12} & 0 \\ M_{12} & -\frac{1}{2}(M_{11} - M_{22}) & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{4.22}$$

namely for a wave travelling in the z direction,

$$h_+(t,r) = \frac{G}{c^4}\frac{1}{r}(\ddot{M}_{11} - \ddot{M}_{22})|_{t-r}, \qquad h_\times = \frac{2G}{c^4}\frac{1}{r}\ddot{M}_{12}(t-r) \tag{4.23}$$

say the north pole. Then if travelling in an arbitrary direction at an angle (θ, ϕ) from the north pole, we can reach this point with a rotation $R(\theta,\phi) := R_{\hat{z}}(\phi)R_{\hat{x}}(\theta)$, under which $M' = R(\theta,\phi)MR^T(\theta,\phi)$, leading to

$$h_+ = \frac{G}{c^4}\frac{1}{r}\Big(\ddot{M}_{11}(\cos^2\phi - \sin^2\phi\cos^2\theta) + \ddot{M}_{22}(\sin^2\phi - \cos^2\phi\cos^2\theta) - \ddot{M}_{33}\sin^2\theta$$
$$-\ddot{M}_{12}\sin 2\phi(1+\cos^2\theta) + \ddot{M}_{13}\sin\phi\cos 2\theta + \ddot{M}_{23}\cos\phi\sin 2\theta\Big) \tag{4.24}$$

$$h_\times = \frac{2G}{c^4}\frac{1}{r}\left(\ddot{M}_{12}\cos 2\phi\cos\theta - \ddot{M}_{13}\cos\phi\sin\theta + \ddot{M}_{23}\sin\phi\sin\theta + \frac{\ddot{M}_{11}-\ddot{M}_{22}}{2}\sin 2\phi\cos\theta\right) \tag{4.25}$$

Summary of approximations:

- long-distances, Far away, namely 1/r and related to multipole
- smalle velocities, PN slow motion of source
- We are still only at first order in G! PM expansion.

Adding higher-order terms, and calling $T_{00} = \rho$,

$$h_{00} = \frac{\kappa^2}{4\pi r}\int d^3x' \left[\rho + \hat{n}\cdot\vec{x}'\,\dot\rho + \frac{1}{r}\hat{n}\cdot x'\,\rho + \dots\right]|_{t-r} = \frac{4G}{c^4}\left(\frac{M}{r} - \frac{1}{r}\hat{n}\cdot\frac{\vec{P}}{c} + \frac{\hat{n}\cdot\vec{M}}{r^2} + \dots\right)|_{t-r} \tag{4.26}$$

where we used $\dot{T}_{00} = \partial_a T^a_0$ followed by an integration by parts. We see the multipolar and PN expansions arising at once.

Similarly,

$$h^{\rm TT}_{ab}(t,\vec{x}) = \frac{4G}{c^4}\frac{1}{r}P^{\rm TT}[S_{ab} + \frac{1}{c}n^c\dot{S}_{ab,c} + \frac{1}{2c^2}n^c n^d\ddot{S}_{ab,cd}](t-r). \tag{4.27}$$

and the NLO triple time der of mass octupole and double time der of current quadrupole:

$$\ddot{Q}_{ab} + \frac{1}{6}\hat{n}_c\dddot{M}^{abc} + \frac{2}{3}(\ddot{P}^{(ab)c} - \ddot{P}^{cab}). \tag{4.28}$$

5 Energy of GWs

As we know from a basic course in general relativity, there is no universal notion of energy, just like there is no universal notion of time. By diffeomorphism invariance, there cannot be any scalar or tensorial local quantity that fully describes the gravitational energy.[8] In particular, the Hamiltonian one finds from the Legendre transform of the Lagrangian is a sum of constraints and thus identically zero when evaluated on solutions. Any attempt to work around these facts and define quasi-local observables representing the gravitational energy unavoidably run into trouble with ambiguities and dependence on purely kinematical structures. The only well-defined resolution to this problem is to work with global notions of energy. Such global notions are useful to describe isolated systems, namely

[8]Such quantity will have to be zero in a local free-falling frame where the effects of gravity are absent, and if it were a tensor, it would then be zero in any frame. A tensorial quantity capturing *some* aspects of gravitational energy can be constructed using the Bel-Robinson tensor, but it is fourth-order in derivatives, therefore does not have the right physical dimensions, and will capture only higher-order terms of the gravitational energy. {footenergy}

spacetimes that are fully dynamical in a certain region, but become well approximated by flat spacetime at large distances from this region. In this case, one can introduce a physically meaningful notion of boundary to the spacetime, and exploit the fact that the Hamiltonian picks up a boundary contribution which is non-vanishing on solutions. The resulting *surface charges* can be used to characterise the total energy of the system, as well as other quantities such as angular momentum associated with the isometries of the flat metric at the boundary.[9]

These classical difficulties with the definition of gravitational energy arise already at the linearized level, as we are about to see. Let us look at the gravitational wave perturbation simply as a spin-2 field moving on Minkowski. Thanks to the Poincaré invariance of the background, we can apply Noether's theorem and derive a conserved energy-momentum tensor for $h_{\mu\nu}$. An explicit calculation starting from the linearized Lagrangian and using the TT gauge giveS

$$t^{\rm N}_{\mu\nu} = \frac{1}{2\kappa^2}\left(\partial^\mu h^{\alpha\beta}\partial_\nu h_{\alpha\beta} - \frac{1}{2}\delta^\mu_\nu \partial_\lambda h_{\rho\sigma}\partial^\lambda h^{\rho\sigma}\right), \tag{5.1}$$ {tNoether}

where the label N stands for Noether. See (A.27) in the Appendix for details. This tensor is conserved, namely $\partial_\mu t^{{\rm N}\mu\nu} = 0$, but has no physical meaning, because it is *not* gauge-invariant: It changes under a linearized diffeomorphism (2.9), and consequently assigns non-zero value to pure gauge modes. Furthermore, it is a tensor only with respect to the global Lorentz transformations, and not with respect to arbitrary diffeomorphisms. Therefore it can have arbitrary values in an arbitrary coordinate system. For instance, we can make it to vanish at any point using Riemann normal coordinates, since in these coordinates the first derivatives of the metric vanish at that point. We are thus seeing explicitly the difficulties sketched at the beginning of the section. This lack of gauge-invariance is a direct consequence of the equivalence principle.[10]

A partial resolution to this problem is to consider spatial averages. We consider a region L whose size is much larger than the typical wavelength λ of the perturbation, but much smaller than the typical wavelength λ_B of the background (which is infinite for a flat background). This definition as such is coordinate dependent, but there exist a more precise way to make it covariant and coordinate-independent. Then we can safely assume that the perturbations vanishes at the boundary of the region. Therefore expressions under the averaging sign can be freely integrated by parts in space and, upon going on-shell, also in time derivatives since a wave propagates on the light-cone. For instance,

$$\langle \partial_\mu h_{\alpha\beta}\partial^\mu h^{\alpha\beta}\rangle = -\langle h_{\alpha\beta}\Box h^{\alpha\beta}\rangle = 0 \tag{5.2}$$

in vacuum. Under this procedure, we define

$$t_{\mu\nu} := \langle t^{\rm N}_{\mu\nu}\rangle = \frac{1}{\kappa^2}\langle \frac{1}{2}\partial^\mu h^{\alpha\beta}\partial_\nu h_{\alpha\beta}\rangle = \frac{1}{\kappa^2}\langle \frac{1}{2}\partial^\mu h^{\alpha\beta}_{\rm TT}\partial_\nu h^{\rm TT}_{\alpha\beta}\rangle. \tag{5.3}$$

The resulting tensor is gauge-invariant!

[9]At spatial infinity, these are the ten Poincaré charges, namely energy and momentum, angular momentum and center-of-mass location. At null infinity, where the gravitational waves go, there is an infinite amount of additional charges, called super-translation charges, and associated with memory effects. The symmetry group is called BMS group, and contains an infinite number of different Poincaré subgroups, one for each memory configuration.

[10]It is instructive to put this problem in perspective with what happens in the electromagnetic case. If one computes the canonical energy-momentum tensor of Maxwell's theory using the Noether formula, one finds a meaningless gauge-dependent expression. However, the Noether construction only defines the tensor up to total divergences, and it is possible to find one such that the resulting is gauge inv. Furthermore, it coincides with he metric one. So the construction can be completes. In gravity we have a similar problem, but the construction cannot be completed. Only total energy well defined, as discussed earlier.

This is in line with Einstein's initial derivation. It is valid, but has important limitations: first, it relies heavily on the special background chosen, and had we worked with a non-isometric one, then there would be no Noether charge possible. Second, it is not clear how to extend this construction to higher orders in perturbation theory.

These shortcomings can be resolved if we look at the actual back-reaction on the metric caused by the waves. In fact the actual "effective" source that determines the second-order metric perturbation is not (5.1), but rather the second order expansion of the Einstein tensor, $G^{(2)}_{\mu\nu}(h)$. To see this, let us look at the higher orders of the expansion of Einstein's equations:

$$\underbrace{G^{(0)}_{\mu\nu}(\eta)}_{0} + \underbrace{G^{(1)}_{\mu\nu}(h)}_{1} + \underbrace{G^{(2)}_{\mu\nu}(h) + G^{(1)}_{\mu\nu}(h^{(2)})}_{2} + \ldots = 0. \tag{5.4}$$ {G012}

The zeroth order term imposes that the background is a solution. The first order term has given us the linearized gravitational wave solution. Using this solution in the first second order term, we see that back-reaction on the metric must be included, in the form of a second-order correction $h^{(2)}_{\mu\nu}$ to be added to (2.2), otherwise the Einstein's equations are violated.[11] Keeping the second order terms alone, we can rearrange the equation they have to satisfy as

$$G^{(1)}_{\mu\nu}(h^{(2)}) = \frac{\kappa^2}{2} t^{\mathrm{G}}_{\mu\nu}, \qquad t^{\mathrm{G}}_{\mu\nu} := -\frac{2}{\kappa^2} G^{(2)}_{\mu\nu}(h), \tag{5.5}$$

where the label G is to remind us that this tensor is built out of the Einstein tensor. It is conserved in flat spacetime, but an explicit calculation – not reported here – shows that it is different from (5.1). The difference is a term $\partial^\rho\partial^\sigma U_{\mu\rho\nu\sigma}$ where U is an arbitrary local quadratic function of $h_{\mu\nu}$ with the same symmetries of the Riemann tensor. This structure guarantees that both tensors have vanishing divergence.

This second candidate definition of energy-momentum tensor of gravity actually depends on second derivatives of the metric, so it cannot be made to vanish at any give point. However it is still not gauge-invariant: hence its value depends on arbitrary choices of coordinates. Therefore one has to invoke again the averaging procedure. Upon doing so, one finds out that

$$\langle t^{\mathrm{G}}_{\mu\nu} \rangle = t_{\mu\nu}, \tag{5.6}$$

so the two procedures give the same answer! So t^{G} offers a generalization of Einstein's construction that is valid for an arbitrary background, and furthermore can be now systematically extended to any order in perturbation theory. The extension is however not straightforward, since one has to recompute the higher order expansions of $G_{\mu\nu}$ and evaluate them on the perturbed solutions in order to identify it. Another shortcoming is that the interpretation of a piece of the field equations as energy-momentum tensor is ambiguous: it depends on the way the equations are written, and the variables used. For example, in the Landau-Lifshitz reformulation of the Einstein's equations one uses a density-weighted inverse metric as fundamental variable, and the field equations are arranged in a different way. Then an expansion like the one used in (5.4) leads to a third candidate for the energy-momentum tensor, which again differs from the previous two by a term $\partial^\rho\partial^\sigma U_{\mu\nu\rho\sigma}$. This third candidate, known as Landau-Lifshitz pseudo-tensor, has also the same sort of problems, like gauge-dependence and vanishing at a point in Riemann normal coordinates, and coincides at leading order under the averaging procedure:

$$\langle t^{\mathrm{LL}}_{\mu\nu} \rangle = t_{\mu\nu} + O(h^3). \tag{5.7}$$

[11] A valid alternative is to change the background instead, which is for instance the approach taken in Maggiore, see p.47.

Even if coordinate-dependent, this approach has the merit of being set up in a way that makes it very natural to develop a systematic perturbative expansion, since the pseudo-tensor is defined already in an exact form, and does not need to be calculated order by order as in the previous example. For this reason, this reformulation is widely used by the community working in the post-Newtonian expansion, and it is briefly reviewed in Appendix C.

Then for the emitted power we have

$$\frac{dE}{dt} = \int_\Sigma \dot{t}_{00} d^3x = \int_\Sigma \partial_a t^a{}_0 d^3x = \oint_{\partial\Sigma} t^a{}_0 u_a dS = \oint_{\partial\Sigma} t_{0r} dS, \tag{5.8}$$

using (4.1), Stokes theorem and in the last step we choose as boundary a 2-sphere of radius r, hence the outgoing unit normal is simply $u_a = \partial_a r$. Continuing and using the explicit form of t,

$$\frac{dE}{dt} = \oint_{\partial\Sigma} t_{0r} dS = \frac{1}{2\kappa^2} \oint_{\partial\Sigma} \langle \dot{h}^{\rm TT}_{ab} \partial_r h^{\rm TT}_{ab} \rangle dS = -\frac{G}{8\pi c^5} \frac{1}{r^2} \oint_{\partial\Sigma} (\dddot{Q}^{\rm TT}_{ab})^2 dS, \tag{5.9}$$

{Energyloss}

where in the last step we used (4.18) and its consequence that $\partial_r h^{\rm TT}(ct-r) = -(1/c)\partial_t h^{\rm TT}(ct - r) + o(r^{-1})$. To evaluate the integral, we observe that the only angular dependence occurs in the TT projector (2.55). Using the following formula,

$$\oint_{S^2} P^{{\rm TT}\,cd}_{ab}\, d^2\Omega = \frac{2\pi}{15}(11\delta^c_a\delta^d_b - 4\delta_{ab}\delta^{cd} + \delta^d_a\delta^c_b), \tag{5.10}$$

we find

$$\frac{dE}{dt} = -\frac{G}{5c^5}(\dddot{Q}_{ab})^2(t-r). \tag{5.11}$$

{Energyloss1

(Notice no more TT part). This is the second famous quadrupole formula of Einstein, here recovered in a more modern perspective (with the averages) which is amenable to avoid the criticism of the historical debate, and to set up a systematic perturbative expansion.

Add numerical estimates

Comments. This formula gives the instantaneous power radiated at a distance r from the source and a time t, as a function of the energy drained from the system at the retarded time $t - r$. Notice that the notion of retarded time used relies heavily on the assumption of a flat background. When higher orders are included, the relation becomes more complicated. But also, another tricky effect comes in: the waves self-interact, unlike e-m waves. This creates a delay in part of the signal, which starts travelling inside the light-cone, similar to light slowing down in a medium due to interactions with the medium. Then the total GW consists of the WF plus a tail that arrives later. Then hard to have an expression like the quadrupole formula in general, see Maggiore 5.3.5

To get some numerical estimates from the formula we just derived, (2.64) with $h_\times = 0$,

$$t^{\rm Eff}_{tt} = \frac{\omega^2 h_+}{2\kappa^2} = 1.5 \frac{mW}{m^2} \left(\frac{h_+}{10^{-22}}\right)^2 \left(\frac{\nu}{1{\rm kHz}}\right)^2 \tag{5.12}$$

6 Emission of GWs by a binary system

As an example of production of gravitational waves, we will consider a binary system.

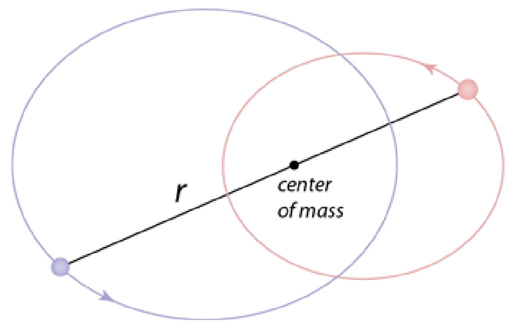


Figure 4: Fig1 *Keplerian orbits*

6.1 Preamble: integrability of the Newtonian 2-body problem

Consider two stars of mass m_1 and m_2 in orbit around each other. We define the center of mass location and the relative distance via

$$\vec{r}_c := m_1\vec{r}_1 + m_2\vec{r}_2, \qquad \vec{r} := \vec{r}_1 - \vec{r}_2. \tag{6.1}$$

Inverting,

$$\vec{r}_1 = \vec{r}_c + \frac{m_2}{m}\vec{r}, \qquad \vec{r}_2 = \vec{r}_c - \frac{m_1}{m}\vec{r}. \tag{6.2}$$

We choose for convenience the origin of the coordinate system such that $\vec{r}_c = 0$, and choose the center-of-mass frame in which $\vec{v}_c = 0$. In this frame, the total momentum is conserved in the absence of external sources. The total energy of the system is then

$$E = \sum_i \frac{1}{2} m_i \vec{v}_i^{\,2} - \frac{G m_1 m_2}{|\vec{r}_1 - \vec{r}_2|} = \frac{1}{2}\mu\vec{v}^{\,2} - \frac{Gm\mu}{r}, \tag{6.3}$$

which we see is equivalent to the energy of a single body of mass μ under an effective acceleration $\ddot{\vec{r}} = -Gm/r^2\hat{r}$.

We now specialize to circular orbits. For circular orbits,

$$\vec{r} = (d\cos\omega t, d\sin\omega t, 0), \qquad \omega^2 d = \frac{Gm}{d^2}, \qquad d = \left(\frac{Gm}{\omega^2}\right)^{1/3} \tag{6.4}$$ {om2d}

Now let's compute the mass moments.

$$\rho(t, \vec{x}) = \sum_{i=1}^{2} m_i \delta^{(3)}(\vec{x} - \vec{r}_i) \tag{6.5}$$

hence

$$M^{ab} = \sum_i m_i r_i^a r_i^b = \mu r^a r^b, \qquad \mu = \frac{m_1 m_2}{m_1 + m_2} \tag{6.6}$$

From this we can compute

$$\dot{M}^{ab} = 2\sum_i m_i r_i^{(a} v_i^{b)} \tag{6.7}$$

and we notice that under a global translation $\vec{r}_i \mapsto \vec{r}_i + \vec{a}$, we have $\dot{M} \mapsto \dot{M} + 2P^{(a}a^{b)}$. Therefore choosing the center-of-mass frame, conservation of total momentum in the absence of external forces also guarantees that the quadrupole momentum $\ddot{M}$ is independent of the choice of origin.

Then, (check signs...)

$$I^{ab} = \frac{1}{2}\mu d^2 \begin{pmatrix} 1-\cos 2\omega t & \sin 2\omega t & 0 \\ & 1+\cos 2\omega t & 0 \\ & & 0 \end{pmatrix} \tag{6.8}$$

and

$$\ddot{I}^{ab} = 2\mu d^2\omega^2 \begin{pmatrix} \cos 2\omega t & -\sin 2\omega t & 0 \\ & -\cos 2\omega t & 0 \\ & & 0 \end{pmatrix} \tag{6.9}$$

From the explicit form of the TT projector, we find that

$$Q^{\mathrm{TT}}_{ab} = P^{\mathrm{TT}}(Q) = P^{\mathrm{TT}}(M) = \begin{pmatrix} \frac{1}{2}(M_{11}-M_{22}) & M_{12} & 0 \\ M_{12} & -\frac{1}{2}(M_{11}-M_{22}) & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{6.10}$$

Thanks to the fact that the orbit is planar, $Q_{13} = Q_{23} = Q_{33} = 0$.

Therefore from (4.18) and (**??**), we read

$$h_+(t,\vec{x}) = \frac{4G\mu d^2\omega^2}{c^4}\frac{1}{r}\cos 2\omega t, \qquad h_+(t,\vec{x}) = \frac{4G\mu d^2\omega^2}{c^4}\frac{1}{r}\sin 2\omega t. \tag{6.11}$$ {hbinary}

Using (6.4), we can rewrite the magnitude of the amplitude as

$$h = \frac{4G^2 m}{c^4 d}\frac{1}{r}. \tag{6.12}$$

ADD estimates

Some observations: the frequency of the wave emitted is *twice* the frequency of the source. This is due to the perfect monochromaticity of the source. If we had a source with a superposition of two frequencies, e.g. ω and 2ω then the GW would be emitted with also lower frequencies, e.g. ω, 2ω, 3ω and 4ω. So smaller frequencies than the double. In general GWs are emitted at wavelengths *larger or equal* than the source's wavelength, the opposite of what happens in e-mag. More like sound waves, cannot be used to reconstruct image directly.

Now for the radiated power:

$$\dddot{M}^{ab} = 4\mu d^2\omega^3 \begin{pmatrix} -\sin 2\omega t & -\cos 2\omega t & 0 \\ & \sin 2\omega t & 0 \\ & & 0 \end{pmatrix} \tag{6.13}$$

and thus

$$(\dddot{Q}_{ab})^2 = (\dddot{M}_{ab})^2 - \frac{1}{3}(\dddot{M}_{aa})^2 = (\dddot{M}_{ab})^2 = 16\mu^2 d^4\omega^6(2\sin^2 2\omega t + 2\cos^2 2\omega t) = 32\mu^2 d^4\omega^6. \tag{6.14}$$

The from (5.9) we find

$$P = -\frac{G}{5c^5}(\dddot{Q}_{ab})^2(t-r) = -\frac{32G\mu^2 d^4\omega^6}{5c^5} \tag{6.15}$$ {Elossbinary

$$P = -\frac{32}{5}\frac{G\mu^2}{c^5}d^4\omega^6 = -\frac{32}{5}\frac{G^4m^3\mu^2}{c^5d^5} \tag{6.16}$$ {ElossC}

and in one period, $T = 2\pi\omega$,

$$E_T^{\mathrm{quad}} = \frac{64\pi}{5}\frac{G\mu^2}{d}\left(\frac{v}{c}\right)^5 \tag{6.17}$$

TODO: plug in numbers for the earth.

6.2 Adding the inclination of the source

In general, the orbital plane will be tilted with respect to our line of sight. We can describe this tilt with the help of a rotation connecting the orbital frame (in which we have chosen the z axis to be perpendicular to the orbital motion) to the observer's frame (in which we have chosen the z axis to be the line of sight). **add drawing** Let us denote by ι the angle of inclination, and by φ the angle between the chosen x axis of the orbital frame, and the one of the observer's frame.[12]

$$\hat{n} = R\hat{z}, \qquad R_{\hat{z}}(\varphi)R_{\hat{y}}(\iota). \tag{6.18}$$

Then starting from the orbital frame, the projected, rotated quadrupole is

$$P^{\mathrm{TT}}\left(R^T\ddot{Q}R\right) = \begin{pmatrix} Q_{11}^{\mathrm{TT-R}} & Q_{12}^{\mathrm{TT-R}} & 0 \\ Q_{12}^{\mathrm{TT-R}} & -Q_{11}^{\mathrm{TT-R}} & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{6.19}$$

with

$$Q_{11}^{\mathrm{TT-R}} = \frac{1}{2}(Q_{11}(\cos^2\varphi - \sin^2\varphi\cos^2\iota) + Q_{22}(\sin^2\varphi - \cos^2\varphi\cos^2\iota) - Q_{12}\sin 2\varphi\cos^2\iota), \tag{6.20}$$

$$Q_{12}^{\mathrm{TT-R}} = \frac{1}{2}(Q_{11} - Q_{22})\sin 2\varphi\cos\iota + Q_{12}\cos 2\varphi\cos\iota. \tag{6.21}$$

These extra angles introduce a non-trivial bending of the basic sinusoidal curves

$$h_+(t,\vec{x}) = \frac{4G\mu d^2\omega^2}{c^4}\frac{1}{r}\frac{1+\cos^2\iota}{2}\cos 2\omega t, \qquad h_+(t,\vec{x}) = \frac{4G\mu d^2\omega^2}{c^4}\frac{1}{r}\cos\iota\sin 2\omega t. \tag{6.22}$$ {hbinary}

Remarks.

- If the system is edge-on, namely $\iota = \pi/2$, then $h_\times = 0$: we have linear polarization of the waves
- If the system is face-on, namely $\iota = 0$, then $h_\times(t) = h_+(t + \pi/2)$ (check): circular polarization
- Intermediate configurations give elliptic polarizations

Hence, by measuring the polarization of the waves, we can derive ι.

[12] If the x axis are determined by some predetermined conventions one may need two independent $\hat{z}$ rotations, see e.g. longitude of pericenter and of ascending nodes used in celestial mechanics.

6.3 The chirp amplitude

Where does the energy emitted from GW comes from? in this approximation, of test bodies slowly orbiting, can only come from the gravitational energy of the system.[13] We have seen that in the quadrupolar approximation, the energy loss by gravitational waves for the simplest case of circular waves, scales like the inverse fifth power of the distance between the stars. Therefore as the system loses energy and inspirals, the energy loss increases. Therefore the two stars inspiral towards one another until they coalesce.

The potential energy of the orbit is

$$E = -\frac{Gm\mu}{2d}, \tag{6.23}$$

hence using (6.16),

$$\dot{E} = \frac{Gm\mu}{2d^2}\dot{d} = -\frac{32}{5}\frac{G^4m^3\mu^2}{c^5d^5}. \tag{6.24}$$

From this formula we can deduce the orbital distance loss, and consequently the orbital frequency increase, via

$$\dot{d} = -\frac{64}{5}\frac{G^3m^2\mu}{c^5d^3} \tag{6.25}$$

and

$$\frac{\dot{\omega}}{\omega} = -\frac{3}{2}\frac{\dot{d}}{d} = -\frac{96}{5}\frac{G^{5/3}m^{2/3}\mu}{c^5}\omega^{8/3}, \tag{6.26}$$

where we used (6.4) again. The quantity

$$M_c^{5/3} := \mu m^{2/3} \tag{6.27}$$ {Mc}

will be useful below. In terms of this quantity,

$$\dot{\omega} = -\frac{96}{5}\left(\frac{GM_c}{c^3}\right)^{5/3}\omega^{11/3}. \tag{6.28}$$

Integrating we find

$$-\frac{3}{8}\omega^{-8/3}\Big|_{\omega_c}^{\omega} = \frac{96}{5}\left(\frac{GM_c}{c^3}\right)^{5/3}(t-t_c). \tag{6.29}$$

The frequency will stop increasing once we reach coalescence (actually even before this approximation will break down, but let's forget about that). Neglecting that much higher contribution to the LHS, we can rewrite it as

$$-\frac{3}{8}\omega^{-8/3} = \frac{96}{5}\left(\frac{GM_c}{c^3}\right)^{5/3}(t-t_c), \tag{6.30}$$

hence

$$\omega = \left(\frac{5}{256}\frac{1}{t_c-t}\right)^{3/8}\left(\frac{GM_c}{c^3}\right)^{-5/8} \tag{6.31}$$

(Recall that the monochromatic emission will be at 2ω)

[13] In fact corrections depending on the internal structure and the fact that energy can be released also from there only enter at a much higher approximation level

To get some estimates, in terms of the frequency of the wave. Consider two stars of mass $1.4M_\oplus$. The chirp mass is then $1.21M_\oplus$, and

$$\nu_{\rm GW} = \frac{\omega}{\pi} = 134{\rm Hz}\left(\frac{1.21M_\oplus}{M_c}\right)^{5/8}\left(\frac{1s}{t_c - t}\right)^{3/8}. \tag{6.32}$$

In the range $10 - 100 - 1000$Hz we get the radiation emitted from 17 minutes to coalescence, the last two seconds, and the last few milliseconds. For the kHz frequence, the Kepler's radius is only 30km!

Number of cycles in a detector's bandwidth:

$$\int_{t_{\omega_{\rm min}}}^{t_{\omega_{\rm max}}} \nu dt \tag{6.33}$$

Then from

$$h = \frac{4m\mu}{d}\frac{1}{r} \propto \omega^{2/3} \propto (t_c - t)^{-1/4} \tag{6.34}$$

so the amplitude increases as we approach the coalescence. See the picture.

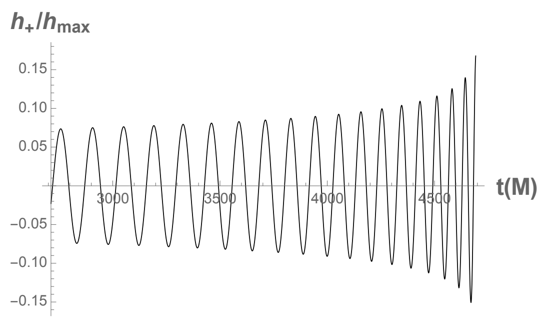


Figure 5: Figchirp *Example of a chirp amplitude.*

As the GWs are emitted, the amplitude and the frequency both increase. For this reason it is called a chirping, and the quantity (6.27) is the parameter that control the chirp, this is why it is called chirp mass.

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Appendix

A Perturbative expansion around arbitrary backgrounds

{AppLin}

We report in this Appendix explicit formulas useful to describe the perturbative expansion. We begin with general formulas around an arbitrary background, and then specialize to Minkowski. The idea is to to write

$$g_{\mu\nu} = \bar g_{\mu\nu} + h_{\mu\nu}, \tag{A.1}$$

and expand all relevant quantities in powers of h. From this we compute

$$g^{\mu\nu} = \bar{g}^{\mu\nu} - h_{\mu\nu} + h^{\mu\rho}h_\rho{}^\nu + O(h^3), \tag{A.2}$$

$$\sqrt{-g} = \sqrt{-\bar{g}}\left[1 + \frac{1}{2}h + \frac{1}{2}\left(\frac{1}{4}h^2 - \frac{1}{2}h^2_{\mu\nu}\right)\right] + O(h^3), \tag{A.3}$$

$$\sqrt{-g}g^{\mu\nu} = \sqrt{-\bar{g}}(\bar{g}^{\mu\nu} + \frac{1}{2}\bar{g}^{\mu\nu}h - h^{\mu\nu}) + O(h^2). \tag{A.4}$$ {gg}

From the last formula we see that

$$\partial_\mu(\sqrt{-g}g^{\mu\nu}) = -\sqrt{-g}g^{\mu\rho}\Gamma^\nu_{\mu\rho} = \partial_\mu(\sqrt{-\bar{g}}\bar{g}^{\mu\nu})(1+h) - \sqrt{-\bar{g}}(\bar{\nabla}_\mu h^{\mu\nu} - \frac{1}{2}\bar{\nabla}^\nu h) + O(h^2), \tag{A.5}$$ {pgg}

where $\bar{\nabla}$ is the background covariant derivative. This equation allows us to understand the relation between the De Donder gauge and harmonic coordinates. Harmonic coordinates are defined by

$$\Box x^\mu = 0 \quad \Leftrightarrow \quad g^{\nu\rho}\Gamma^\mu_{\nu\rho} = 0, \tag{A.6}$$

and using (A.5) we see that the harmonic condition is preserved if the perturbation satisfies

$$\bar{\nabla}_\mu h^{\mu\nu} - \frac{1}{2}\bar{\nabla}^\nu h = 0. \tag{A.7}$$

Using the expansion formulas in the definition of the Riemann tensor, one finds

$$\Gamma^\mu_{\nu\rho} = \bar{\Gamma}^\mu_{\nu\rho} + \Gamma^{(1)\mu}{}_{\nu\rho} + \Gamma^{(2)\mu}{}_{\nu\rho} + O(h^3), \tag{A.8}$$

$$\Gamma^{(1)\mu}{}_{\nu\rho} = \frac{1}{2}\bar{g}^{\mu\sigma}(2\bar{\nabla}_{(\nu}h_{\rho)\sigma} - \bar{\nabla}_\sigma h_{\nu\rho}), \tag{A.9}$$

$$\Gamma^{(2)\mu}{}_{\nu\rho} = -\frac{1}{2}h^{\mu\sigma}(2\bar{\nabla}_{(\nu}h_{\rho)\sigma} - \bar{\nabla}_\sigma h_{\nu\rho}), \tag{A.10}$$

$$R_{\mu\nu\rho\sigma} = \bar{R}_{\mu\nu\rho\sigma} + R^{(1)}_{\mu\nu\rho\sigma} + R^{(2)}_{\mu\nu\rho\sigma} + O(h^3), \tag{A.11}$$

$$R^{(1)}_{\mu\nu\rho\sigma} = \tag{A.12}$$

$$R_{\mu\nu} = \bar{R}_{\mu\nu} + R^{(1)}_{\mu\nu} + R^{(2)}_{\mu\nu} + O(h^3), \tag{A.13}$$

$$R^{(1)}_{\mu\nu} = -\frac{1}{2}\bar{\Box}h_{\mu\nu} + \bar{\nabla}_{(\mu}\bar{\nabla}_\rho h^\rho{}_{\nu)} - \frac{1}{2}\bar{\nabla}_\mu\bar{\nabla}_\nu h + \bar{R}_{\rho(\mu}h_{\nu)}{}^\rho - \bar{R}_{\mu\rho\nu\sigma}h^{\rho\sigma} \tag{A.14}$$

$$\begin{aligned} R^{(2)}_{\mu\nu} = {} & \frac{1}{2}h^{\rho\sigma}\bar{\nabla}_\nu\bar{\nabla}_\mu h_{\rho\sigma} - h^{\rho\sigma}\bar{\nabla}_\rho\bar{\nabla}_{(\mu}h_{\nu)\sigma} + \frac{1}{2}h^{\rho\sigma}\bar{\nabla}_\rho\bar{\nabla}_\rho h_{\mu\nu} - (\bar{\nabla}_\rho h^\rho{}_\sigma - \frac{1}{2}\bar{\nabla}_\sigma h)(\bar{\nabla}_{(\mu}h^\sigma{}_{\nu)} - \frac{1}{2}\bar{\nabla}^\sigma h_{\mu\nu}) \\ & + \frac{1}{4}\bar{\nabla}_\mu h_{\rho\sigma}\bar{\nabla}_\nu h^{\rho\sigma} + \bar{\nabla}_\rho h_{\sigma\mu}\bar{\nabla}^{[\rho}h^{\sigma]}{}_\nu + O(h^3), \end{aligned} \tag{A.15}$$

$$R = \bar{R} + R^{(1)} + R^{(2)} + O(h^3), \tag{A.16}$$

$$R^{(1)} = \bar{\nabla}_\mu\bar{\nabla}_\nu h^{\mu\nu} - \bar{\Box}h - \bar{R}_{\mu\nu}h^{\mu\nu}, \tag{A.17}$$

$$R^{(2)} = g^{\mu\nu}R^{(2)}_{\mu\nu} - h^{\mu\nu}R^{(1)}_{\mu\nu} + h^{\mu\rho}h^\nu_\rho\bar{R}_{\mu\nu} \tag{A.18}$$

$$\begin{aligned} &= h^{\rho\sigma}\bar{\Box}h_{\rho\sigma} + \frac{3}{4}\bar{\nabla}_\rho h_{\mu\nu}\bar{\nabla}^\rho h^{\mu\nu} - \frac{1}{2}\bar{\nabla}_\mu h_{\nu\rho}\bar{\nabla}^\nu h^{\mu\rho} - 2h^{\mu\nu}\bar{\nabla}_\mu\bar{\nabla}_\rho h^\rho{}_\nu + h^{\mu\nu}\bar{\nabla}_\mu\bar{\nabla}_\nu h \\ &\quad - \left(\bar{\nabla}_\mu h^\mu{}_\nu - \frac{1}{2}\bar{\nabla}_\nu h\right)^2 + \bar{R}_{\mu\nu\rho\sigma}h^{\mu\rho}h^{\nu\sigma} \end{aligned} \tag{A.19}$$

$$= -\frac{1}{4}(\bar{\nabla}_\mu h_{\nu\rho})^2 + \frac{1}{2}(\bar{\nabla}_\mu h^\mu_\nu)^2 - \frac{1}{4}(\bar{\nabla}_\mu h)^2 + \frac{1}{2}\bar{R}_{\mu\nu\rho\sigma}h^{\mu\rho}h^{\nu\sigma} + \frac{1}{2}\bar{R}_{\mu\nu}h^{\mu\rho}h^\nu_\rho + b.t. \tag{A.20}$$

where in the last equality we used

$$\bar{\nabla}_\mu h_{\nu\rho}\bar{\nabla}^\nu h^{\mu\rho} = b.t. + (\bar{\nabla}_\mu h^\mu_\nu)^2 + \bar{R}_{\mu\nu\rho\sigma}h^{\mu\rho}h^{\nu\sigma} - \bar{R}_{\mu\nu}h^{\mu\rho}h^\nu_\rho. \tag{A.21}$$

To find the linearized Einstein's equations, it is sufficient to look at the $O(h)$ terms in the Ricci expansions. For the Lagrangian on the other hand, one needs the quadratic order, hence

$$L = \sqrt{-g}(R-2\Lambda) = \sqrt{-\bar{g}}\bar{L} + \sqrt{-\bar{g}}L^{(1)} + \sqrt{-\bar{g}}L^{(2)} + O(h^3), \tag{A.22}$$ {EHlin}

$$\bar{L} = \bar{R} - 2\Lambda, \tag{A.23}$$

$$L^{(1)} = \bar{\nabla}_\mu(\bar{\nabla}_\nu h^{\mu\nu} - \bar{\nabla}^\mu h) - h^{\mu\nu}(\bar{G}_{\mu\nu} + \Lambda\bar{g}_{\mu\nu}) \tag{A.24}$$

$$\begin{aligned} L^{(2)} &= \frac{1}{2}\left(\frac{1}{4}h^2 - \frac{1}{2}h^2_{\mu\nu}\right)(\bar{R}-2\Lambda) + \frac{1}{2}hR^{(1)} + R^{(2)} \\ &= -\frac{1}{4}(\bar{\nabla}_\rho h_{\mu\nu})^2 + \frac{1}{2}\bar{\nabla}_\rho h_{\mu\nu}\bar{\nabla}^\mu h^{\nu\rho} - \frac{1}{2}\bar{\nabla}_\mu h^{\mu\nu}\bar{\nabla}_\nu h + \frac{1}{4}(\bar{\nabla}_\mu h)^2 \\ &\quad + (h^{\mu\rho}h^\nu_\rho - \frac{1}{2}hh^{\mu\nu})\bar{R}_{\mu\nu} + \frac{1}{4}(\frac{1}{2}h^2 - h^2_{\mu\nu})\bar{L} + \bar{\nabla}_\mu v^\mu, \end{aligned}$$

where

$$v^\mu = h^{\nu\rho}\bar{\nabla}^\mu h_{\nu\rho} - h^{\mu\nu}(\bar{\nabla}_\rho h^\rho_\nu - \bar{\nabla}_\nu h) - h^{\nu\rho}\bar{\nabla}_\nu h^\mu_\rho + \frac{1}{2}hC^\mu \tag{A.25}$$

A.1 Minkowski background

We now specialize the above formulas to Minkowski background. When the background is Minkowski, we have $\bar{g}_{\mu\nu} = \eta_{\mu\nu}$ and $\bar{\nabla}_\mu = \partial_\mu$ in Cartesian coordinates. With an abuse of notation, we will use $\Box$ to mean the flat space d'Alambertian, even if sometimes the less ambiguous symbol ∂^2 is also used in the literature. The expansions of the Riemann and Ricci tensors simplify to (2.6) and (2.7) in the main text, and the linearized EEs give

$$G^{(1)}_{\mu\nu} = -\frac{1}{2}\Box h_{\mu\nu} + \partial_{(\mu}\partial_\rho h^\rho{}_{\nu)} - \frac{1}{2}\partial_\mu\partial_\nu h - \frac{1}{2}\eta_{\mu\nu}(\partial_\rho\partial_\sigma h^{\mu\nu} - \Box h) = \frac{\kappa^2}{2}T_{\mu\nu}. \tag{A.26}$$

As for the linearized action, we take (A.22) with $\Lambda = 0$, and using

$$R^{(2)} = -\frac{1}{4}(\partial_\rho h_{\mu\nu})^2 + \frac{1}{2}(\partial_\mu h_{\mu\nu})^2 - \frac{1}{4}(\partial_\mu h)^2, \tag{A.27}$$ {L2Mink}

$$\frac{1}{2}hR^{(1)} = -\frac{1}{2}\partial_\mu h^\mu{}_\nu\partial^\nu h + \frac{1}{2}\partial_\mu h^2, \tag{A.28}$$

we get

$$L^{(2)} = (\sqrt{-g}R)^{(2)} = -\frac{1}{4}h^2_{\mu\nu,\rho} + \frac{1}{2}h^\mu{}_{\nu,\mu}{}^2 - \frac{1}{2}h^\mu{}_{\nu,\mu}\partial^\nu h + \frac{1}{4}\partial_\mu h^2 = -\frac{1}{2}h^{\mu\nu}G^{(1)}_{\mu\nu}, \tag{A.29}$$

often called Pauli-Fierz Lagrangian in the literature. Adding the matter term (notice with opposite sign due to $\delta g^{\mu\nu} = -h^{\mu\nu}$),

$$L^{(2)}_{\text{tot}} = L^{(2)} + \frac{\kappa^2}{2}T_{\mu\nu}h^{\mu\nu}. \tag{A.30}$$

From the vacuum Lagrangian we can compute the Noether energy-momentum tensor, namely

$$t^\mu{}_\nu = -\frac{1}{\kappa^2}\left(\frac{\partial L}{\partial\partial_\mu h_{\alpha\beta}}\partial_\nu h_{\alpha\beta} - \delta^\mu_\nu L\right). \tag{A.31}$$

We have

$$\frac{\partial L^{(2)}}{\partial\partial_\mu h_{\alpha\beta}}\partial_\nu h_{\alpha\beta} = -\frac{1}{2}\partial^\mu h^{\alpha\beta}\partial_\nu h_{\alpha\beta} + \partial_\nu h^{\mu\alpha}(\partial^\rho h_{\rho\alpha} - \frac{1}{2}\partial_\alpha h) - \frac{1}{2}\partial_\nu h\partial_\alpha h^{\mu\alpha} + \frac{1}{2}\partial^\mu h\partial_\nu h. \tag{A.32}$$

This is quadratic in the field, and conserved. However it is not symmetric, and furthrmore not g.i. As discussed in the main text, this expression is not gauge-invariant, and indeed there cannot be any gauge-invariant two-derivative energy momentum tensor for gravity. In the De Donder gauge,

$$L^{(2)}_{\text{DeD}} = -\frac{1}{4}h^2_{\mu\nu,\rho} + \frac{1}{2}(\partial_\mu h)^2, \qquad \frac{\partial L^{(2)}_{\text{DeD}}}{\partial\partial_\mu h_{\alpha\beta}}\partial_\nu h_{\alpha\beta} = -\frac{1}{2}\partial^\mu h^{\alpha\beta}\partial_\nu h_{\alpha\beta} + \frac{1}{4}\partial^\mu h\partial_\nu h, \tag{A.33}$$

and

$$t^\mu{}_\nu = \frac{1}{\kappa^2}\left(\frac{1}{2}\partial^\mu h^{\alpha\beta}\partial_\nu h_{\alpha\beta} - \frac{1}{4}\partial^\mu h\partial_\nu h - \eta_{\mu\nu}(\frac{1}{4}h^2_{\mu\nu,\lambda} - \frac{1}{2}(\partial_\mu h)^2)\right), \tag{A.34}$$

which is (5.1) in the main text. In the TT gauge, which we recall can always be chosen in vacuum, this reduces further to

$$t^\mu{}_\nu = \frac{1}{\kappa^2}(\frac{1}{2}\partial^\mu h^{\alpha\beta}\partial_\nu h_{\alpha\beta} - \frac{1}{4}\delta^\mu_\nu h^2_{\mu\nu,\lambda}). \tag{A.35}$$

Finally, invoking the averaging procedure that allows us to integrate by parts and use the field equations in vacuum, the second term vanishes and we end up with

$$t^{\text{Eff}}_{\mu\nu} = \frac{1}{2\kappa^2}\langle\partial_\mu h_{\alpha\beta}\partial_\nu h^{\alpha\beta}\rangle. \tag{A.36}$$ {teffA}

It is then easy to see that this is gauge invariant, since replacing one h by $\partial\xi$ and integrating by parts we get zero by the divergenceless condition in vacuum. Hence, we can replace h with h^{TT}.

A.2 Gauge-invariance and $t_{\mu\nu}$

The linearized EH Lagrangian around Minkowski has *two* different symmetries. The first is under linearized diffeomorphisms, acting as

$$x^\mu \mapsto x^\mu - \xi^\mu(x), \qquad h_{\mu\nu} \mapsto h_{\mu\nu} + 2\partial_{(\mu}\xi_{\nu)}. \tag{A.37}$$ {local}

This is a gauge symmetry: Noether's theorem gives on-shell a vanishing charge up to boundary terms, which are the linearized version of the Komar charge.

The second is under global Poincaré transformations, split in translations acting as

$$x^\mu \mapsto x^\mu - \epsilon^\mu, \qquad h_{\mu\nu} \mapsto h_{\mu\nu} + \epsilon^\rho\partial_\rho h_{\mu\nu}, \tag{A.38}$$ {global}

and Lorentz transformations

$$x^\mu \mapsto x^\mu - \epsilon^\mu{}_\nu x^\nu, \qquad h_{\mu\nu} \mapsto h_{\mu\nu} + ... \tag{A.39}$$ {globalL}

These are standard symmetries of the action, in the sense that Noether charges are not zero, and give instead an energy-momentum tensor $t_{\mu\nu}$ quadratic in the fields. This tensor is invariant under (A.38), in agreement with Noether's theorem, but not under (A.37), as can be easily seen. This lack of invariance is not surprising, since $t_{\mu\nu}$ is by no means a charge for that symmetry.

Notice that the second is a special case of a diffeo, with constant parameter. However, the transformation induced on the field is *not* a special case of the first one: if we take constant ϵ^μ in the first one, we obtain an invariant h, just like for a global gauge transformation in electromagnetism.

To resolve this apparent puzzle, we must recall that both transformations arise from the unique diff-invariance of the full action. Under a diffeo the metric transforms as

$$g'_{\mu\nu}(x') = \frac{\partial x^\rho}{\partial x^\mu}\frac{\partial x^\sigma}{\partial x^\nu} g_{\rho\sigma}(x), \tag{A.40}$$

from which we read the infinitesimal generator

$$\delta_\xi g_{\mu\nu}(x) := g'_{\mu\nu}(x) - g_{\mu\nu}(x) = \pounds_\xi g_{\mu\nu}(x). \tag{A.41}$$ {gdiffeo}

Expanding both sides of (A.41) as background plus perturbation, $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$, we find

$$\delta_\xi h_{\mu\nu}(x) := h'_{\mu\nu}(x) - h_{\mu\nu}(x) = -\delta_\xi \bar{g}_{\mu\nu}(x) + \pounds_\xi \bar{g}_{\mu\nu}(x) + \pounds_\xi h_{\mu\nu}(x). \tag{A.42}$$ {totalVar}

We can treat this symmetry in two different ways in perturbation theory:

1. $\delta_\xi \bar{g}_{\mu\nu} = 0, \qquad \delta_\xi h_{\mu\nu} = \pounds_\xi \bar{g}_{\mu\nu} + \pounds_\xi h_{\mu\nu}$
2. $\delta_\xi \bar{g}_{\mu\nu} = \pounds_\xi \bar{g}_{\mu\nu}, \qquad \delta_\xi h_{\mu\nu} = \pounds_\xi h_{\mu\nu}$

Both transformations are symmetries of the Lagrangian at each order in the perturbative expansion. However, it is the first that is the one most interesting physically, because only then it makes sense to compare perturbations, since they are defined with respect to the same background. The implementation of the second transformation as symmetry is slightly subtler, because it mixes different orders of h. To see that, we expand the Lagrangian,

$$L(\bar{g} + h) = \bar{L} + L^{(1)\mu\nu} h_{\mu\nu} + \frac{1}{2} h_{\mu\nu} L^{(2)\mu\nu\rho\sigma} h_{\rho\sigma} + \dots \tag{A.43}$$

where $\bar{L} := L(\bar{g})$, and so on. Then using the first option for the transformation,

$$\begin{aligned} \delta_\xi L &= \bar{L}^{(1)\mu\nu} \delta_\xi h_{\mu\nu} + h_{\mu\nu} \bar{L}^{(2)\mu\nu\rho\sigma} \delta_\xi h_{\rho\sigma} + \dots \\ &= \bar{L}^{(1)\mu\nu} \pounds_\xi \bar{g}_{\mu\nu} + \bar{L}^{(1)\mu\nu} \pounds_\xi h_{\mu\nu} + h_{\mu\nu} \bar{L}^{(2)\mu\nu\rho\sigma} \pounds_\xi \bar{g}_{\rho\sigma} + h_{\mu\nu} \bar{L}^{(2)\mu\nu\rho\sigma} \pounds_\xi h_{\rho\sigma} + \dots \end{aligned} \tag{A.44}$$

The symmetry is exact order by order in perturbation theory since it is a symmetry of the full theory. At zeroth order this is obvious, since

$$\bar{L}^{(1)\mu\nu} \pounds_\xi \bar{g}_{\mu\nu} = ..\partial_\mu(\xi_\nu G^{((1))\mu\nu}) \tag{A.45}$$

thanks to the Bianchi identities, so it is a symmetry. Similarly for all higher order terms. Now let us look at the first-order term. It has two contributions, and in general they are both required to get a total derivative. However the first contribution vanishes if the background is on-shell. Only in this case, we recover that $\delta_\xi h_{\mu\nu} := \pounds_\xi \bar{g}_{\mu\nu}$ is a symmetry, and thus (A.37) when the background is flat. Notice that this was manifest in (2.9) since flat spacetime is a solution of the background field equations. The second-order term should be completed with an $L^{(3)}$ contribution. However, notice that

$$h_{\mu\nu} \bar{L}^{(2)\mu\nu\rho\sigma} \pounds_\xi h_{\rho\sigma} = \frac{1}{2} \bar{L}^{(2)\mu\nu\rho\sigma} \pounds_\xi (h_{\mu\nu} h_{\rho\sigma}) = \frac{1}{2} \xi^\alpha \partial_\alpha (\bar{L}^{(2)\mu\nu\rho\sigma} h_{\mu\nu} h_{\rho\sigma}) - \frac{1}{2} h_{\mu\nu} h_{\rho\sigma} \bar{L}^{(3)\mu\nu\rho\sigma\alpha\beta} \pounds_\xi \bar{g}_{\alpha\beta}. \tag{A.46}$$

Therefore $\delta_\xi h_{\mu\nu} = \pounds_\xi h_{\mu\nu}$ is also an off-shell symmetry of the truncated Lagrangian at quadratic order, but only in the special case in which the background has isometries. This is also consistent with what happens at the linear order, since for isometries $\pounds_\xi \bar{g}_{\mu\nu} = 0$ and then $\pounds_\xi h$ is also a symmetry off-shell at this order.

Taking $\bar{g} = \eta$, the isometries are the Poincaré transformations, namely $\xi^\mu = \epsilon^\mu{}_\nu x^\nu + \epsilon^\mu$, to which we find the global Noether charges. In particular for translations, we recover the $\delta_\epsilon h_{\mu\nu} = \epsilon^\rho \partial_\rho h_{\mu\nu}$.

So this is the precise sense in which they are two different symmetries, and their origin from the unique diff-invariance of the full action.

B Linearized canonical analysis

{AppCan}

We start from

$$L^{(2)} = (\sqrt{-g}R)^{(2)} = -\frac{1}{4}h^2_{\mu\nu,\rho} + \frac{1}{2}h^\mu{}_{\nu,\mu}{}^2 - \frac{1}{2}h^\mu{}_{\nu,\mu}\partial^\nu h + \frac{1}{4}\partial_\mu h^2 = -\frac{1}{2}h^{\mu\nu}G^{(1)}_{\mu\nu}, \tag{B.1}$$

After an integration by parts to remove spatial derivatives from h_{0a} , we find

$$\pi^{ab} := \frac{\partial L^{(2)}}{\partial \dot{h}_{ab}} = \frac{1}{2}\dot{h}^{ab} - \frac{1}{2}\delta^{ab}\dot{h}^c_c - \partial^{(a}h^{b)}{}_0 + \delta^{ab}\partial^c h_{0c}. \tag{B.2}$$

From this we compute

$$\pi = -\dot{h}^c_c + 2\partial^c h_{0c}, \qquad \dot{h}^{ab} = 2\pi^{ab} - \delta^{ab}\pi + 2\partial^{(a}h^{b)}{}_0, \tag{B.3}$$

and

$$S = \int dtd^3x \left(\pi^{ab}\dot{h}_{ab} - h_{00}\mathcal{H} - h_{0a}\mathcal{H}^a - H_0\right), \tag{B.4}$$

where

$$\mathcal{H} := \partial_a\partial_b h^{ab} - \partial^2 h^c_c, \qquad \mathcal{H}^a := -2\partial_b\pi^{ab}, \tag{B.5}$$

$$H_0 := \frac{1}{2}\pi^{ab}\pi_{ab} - \frac{1}{4}\pi^2 + V(h), \tag{B.6}$$

$$V(h) = \frac{1}{2}(\partial_a h_{bc})^2 - \partial_a h_{bc}\partial_b h_{ca} + \partial_a h^{ab}\partial_b h^c_c - \frac{1}{2}\partial_a h^b_b \partial^a h^c_c. \tag{B.7}$$

We see the appearence of the linearized constraints; 4 as in the full theory, and still first class, generating the linearized diffeomorphisms. The main novelty is the presence of a true Hamiltonian, due to the presence of a preferred time in the background metric.

From this analysis we can count the physical dofs and we have a physical hamiltonian. The problem of time has been resolved by the presence of a background Minkowski spacetime and its class of privileged inertial observers. On the other hand, the dual to it, the lack of localization of the energy, is still present, as we saw in the main text.

Would be nice to do it for arbitrary background, and keeping track also of the linear term. See explicitly what the constraints generate.

C Landau-Lifshitz approach

{AppLL}

It is a reformulation of the Einstein's equations based on a change of fundamental variables, motivated precisely by the search back in the day of an energy-momentum tensor for gravity. As a reformulation it is exact, and valid in any coordinate system; However, it is only convenient when studying the linearized expansion around flat metric in Cartesian coordinates. The main technical idea is to work with a density-weight one pseudo-tensor given by

$$\mathfrak{g}^{\alpha\beta} := \sqrt{-g}g^{\alpha\beta}. \tag{C.1}$$

From this, one constructs

$$H^{\alpha\mu\beta\nu} := \mathfrak{g}^{\alpha\beta}\mathfrak{g}^{\mu\nu} - \mathfrak{g}^{\alpha\nu}\mathfrak{g}^{\beta\mu} \tag{C.2}$$

which has the same symmetries as the Riemann tensor, and whose second derivatives can be related to the Einstein tensor as follows,

$$\partial_\mu\partial_\nu H^{\alpha\mu\beta\nu} = (-g)(2G^{\alpha\beta} + \kappa^2 t^{\alpha\beta}_{\rm LL}). \tag{C.3}$$

Here $t^{\alpha\beta}_{\rm LL}$ is a density-weight two pseudo-tensor, given explicitly by some horrible expression in terms of second derivatives of the metric. The use of densities instead of tensors is a sciagura from the point of view of general covariance. One must understand that right from the start, this is a brute force computational approach without any desire of using geometric quantities. In particular, one should avoid any temptation of giving any strong interpretation as a genuine energy-momentum. Being a pseudo-tensor for instance, it can be made to vanish at any given point of spacetime, by adopting Riemann normal coordinates in the neighbourhood. (Compare this with a genuine tensor like the Riemann tensor, if it can be made zero in a point in a given coordinate system, then it will be zero in any coordinate system)

Thanks to that identity, the EEs can be rewritten as

$$\partial_\mu\partial_\nu H^{\alpha\mu\beta\nu} = \kappa^2(-g)(T^{\alpha\beta} + t^{\alpha\beta}_{\rm LL}), \tag{C.4}$$

with "conservation law"

$$\partial_\mu\Big((-g)(T^{\alpha\beta} + t^{\alpha\beta}_{\rm LL})\Big) = 0 \tag{C.5}$$

valid only on-shell on the EEs and compatible with the identity $\partial_\alpha\partial_\mu\partial_\nu H^{\alpha\mu\beta\nu} = 0$. This is the value of this reformulation: it suggests a definition for the energy-momentum, which although as discussed is coordinate-dependent, can nonetheless provide a useful starting point for perturbation theory around Minkowski and around Cartesian coordinates.

$$P^\mu := \int (-g)(T^{\mu0} + t^{\mu0}_{\rm LL})d^3x \tag{C.6}$$

$$J^{\mu\nu} := \int (-g)2x^{[\mu}(T^{\nu]0} + t^{\nu]0}_{\rm LL})d^3x \tag{C.7}$$

Ten conserved quantities including the position of the center of mass.

Can be turned into surface integrals via

$$P^\mu := \frac{1}{\kappa^2}\int \partial_\nu H^{\mu\nu0\beta}dS_\beta \tag{C.8}$$

$$J^{\mu\nu} := \frac{1}{\kappa^2}\int (2x^{[\mu}\partial_\alpha H^{\nu]\alpha0\beta} + 2H^{[\mu0\nu]k})dS_\beta \tag{C.9}$$

Reduce to ADM values at infinity.