Geometrical Aspects of Supergravity

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Contents

From Supersymmetry to Supergravity

- Haag et all. : $\mathfrak{g} = \mathfrak{iso}(1,3) \oplus \mathfrak{g}_c \oplus \{Q_\alpha, \bar{Q}^{\dot{\alpha}}\}$.
- Supersymmetry transformations generated by spin-$\frac{1}{2}$ operators Q_α and $\bar{Q}^{\dot{\alpha}}$.
- Anticommuting parameters : ϵ and $\bar{\epsilon}$.

Composition of two Susy transformations :

$[\epsilon^\alpha Q_\alpha, \bar{\epsilon}_{\dot{\alpha}} \bar{Q}^{\dot{\alpha}}] = 2i\epsilon^\alpha \sigma^\mu{}_{\alpha\dot{\alpha}} \bar{\epsilon}^{\dot{\alpha}} P_\mu \Rightarrow$ space time translation.

- Promoting global Susy to local : $\epsilon \to \epsilon(x)$:
  - ⋆ Composition of two Susy transformations generates a Local translation.
  - ⋆ Local Supersymmetry $\equiv$ Supergravity

The Equivalence Principle

Equivalence Principle

The local equivalence of inertial and gravitational force

- In the instantaneous local referential frame $\mathcal{R}_0(p)$ where gravity is eliminated : we recover local Lorentz symmetry !
- Span a Flat tangent superspace at each point of the curved superspace

Dynamical Gauge Superfields

- Superconnection : $\Omega_{\tilde{M}M}{}^{N} \rightarrow$
  - ⋆ $\mathcal{L}_{\text{Sugra}}$ Invariance under local Lorentz transformations
  - ⋆ Covariant derivatives : $\mathcal{D}_{\tilde{M}} X^M = \partial_{\tilde{M}} X^M + \Omega_{\tilde{M}N}{}^{M} X^N$
- Vielbein : $E_{\tilde{M}}{}^{M} \rightarrow$
  - ⋆ $\mathcal{L}_{\text{Sugra}}$ Invariance under superdiffeormorphism
  - ⋆ $E_{\tilde{M}}{}^{M}$ Curved superspace $\rightarrow$ Flat tangent space

    $E_{M}{}^{\tilde{M}}$ Flat tangent space $\rightarrow$ Curved superspace

Example

$$\mathcal{D}_{\tilde{M}} X^M \rightarrow \mathcal{D}_N X^M = E_N{}^{\tilde{M}} \mathcal{D}_{\tilde{M}} X^M$$

The Covariant Derivatives, the generators of Supergravity

Generators of the Supergravity superalgebra

$$[\mathcal{D}_M, \mathcal{D}_N]_{|M||N|} = T_{MN}{}^{Q}\mathcal{D}_Q + \frac{1}{2}R_{MN\underline{\alpha}\underline{\beta}}J^{\underline{\beta}\underline{\alpha}}$$

- Torsion and Curvature tensor $\propto E_{\tilde{M}}{}^{M}$ and $\Omega_{\tilde{M}M}{}^{N}$
- Closes under Superdiffeomorphism $+$ Lorentz group transformation

The Bianchi Identities

- Bianchi Identities :

$$
\begin{aligned}
0 =&(-)^{|M_1||M_3|}\left[\mathcal{D}_{M_1},[\mathcal{D}_{M_2},\mathcal{D}_{M_3}]_{|M_2||M3|}\right]_{|M_1|(|M_2|+|M_3|)}\\
&+(-)^{|M_2||M_1|}\left[\mathcal{D}_{M_2},[\mathcal{D}_{M_3},\mathcal{D}_{M_1}]_{|M_3||M1|}\right]_{|M_2|(|M_3|+|M_1|)}\\
&+(-)^{|M_3||M_2|}\left[\mathcal{D}_{M_3},[\mathcal{D}_{M_1},\mathcal{D}_{M_2}]_{|M_1||M2|}\right]_{|M_3|(|M_1|+|M_2|)}
\end{aligned}
$$

- $M_i = \alpha, \dot{\alpha}, \mu$, i=1,2,3 $\rightarrow$ 40 Bianchi identities

The Supergravity Constrains

Violation of the Equivalence Principle

- $\exists\ \mathcal{R}$ such that $\Omega_{\tilde{M}M}{}^{N} = 0 \Rightarrow \emptyset$ Curvarture
- $[\mathcal{D}_M, \mathcal{D}_N]_{|M||N|} = T_{MN}{}^{Q}\mathcal{D}_Q \neq$Susy Algebra
- $T_{MN}{}^{Q}$ violates the Equivalence Principle !

Supergravity Constrains

$$T_{\alpha\dot{\alpha}}{}^{\mu} = T_{\dot{\alpha}\alpha}{}^{\mu} = -2i\sigma^{\mu}{}_{\alpha\dot{\alpha}},$$

$$T_{\underline{\alpha\beta}}{}^{\gamma} = 0, \qquad T_{\alpha\beta}{}^{\mu} = T_{\dot{\alpha}\dot{\beta}}{}^{\mu} = 0,$$

$$T_{\underline{\alpha}\mu}{}^{\nu} = T_{\mu\underline{\alpha}}{}^{\nu} = 0, \quad T_{\mu\nu}{}^{\rho} = 0.$$

Bianchi Identities + Supergravity constrains → Reduction of the Bianchi identities :

$$(1)\quad I_{\dot\alpha\dot\beta\mu\nu} : R_{\dot\alpha\dot\beta\mu\nu} = 2i\sigma_{\nu\gamma\dot\alpha} T_{\dot\beta\mu}{}^{\gamma} + 2i\sigma_{\nu\gamma\dot\beta} T_{\dot\alpha\mu}{}^{\gamma},$$

$$(2)\quad I_{\alpha\beta\dot\gamma\dot\delta} : R_{\alpha\beta\dot\gamma\dot\delta} = 2i\sigma^{\mu}{}_{\beta\dot\gamma} T_{\alpha\mu\dot\delta} + 2i\sigma^{\mu}{}_{\alpha\dot\gamma} T_{\beta\mu\dot\delta},$$

$$(3)\quad I_{\dot\alpha\dot\beta\dot\gamma\dot\delta} : R_{\dot\alpha\dot\beta\dot\gamma\dot\delta} + R_{\dot\beta\dot\gamma\dot\alpha\dot\delta} + R_{\dot\gamma\dot\alpha\dot\beta\dot\delta} = 0,$$

$$(4)\quad I_{\alpha\dot\beta\gamma\delta} : R_{\alpha\dot\beta\gamma\delta} + R_{\dot\beta\gamma\alpha\delta} = 2i\sigma^{\mu}{}_{\alpha\dot\beta} T_{\gamma\mu\delta} + 2i\sigma^{\mu}{}_{\gamma\dot\beta} T_{\alpha\mu\delta},$$

$$(5)\quad I_{\alpha\dot\beta\mu\nu} : R_{\alpha\dot\beta\mu\nu} = 2i\sigma_{\nu\gamma\dot\beta} T_{\alpha\mu}{}^{\gamma} - 2i\sigma_{\nu\alpha\dot\gamma} T_{\dot\beta\mu}{}^{\dot\gamma}$$

$$(6)\quad I_{\dot\alpha\dot\beta\mu\gamma} : \bar{\mathcal{D}}_{\dot\alpha} T_{\dot\beta\mu\gamma} + \bar{\mathcal{D}}_{\dot\beta} T_{\dot\alpha\mu\gamma} = 0,$$

$$(7)\quad I_{\alpha\mu\nu\rho} : R_{\alpha\mu\nu\rho} + R_{\nu\alpha\mu\rho} = 2i\sigma_{\rho\alpha\dot\beta} T_{\mu\nu}{}^{\dot\beta},$$

$$(8)\quad I_{\alpha\mu\beta\gamma} : R_{\alpha\mu\beta\gamma} + R_{\beta\mu\alpha\gamma} = \mathcal{D}_{\alpha} T_{\beta\mu\gamma} + \mathcal{D}_{\beta} T_{\alpha\mu\gamma},$$

$$(9)\quad I_{\mu\alpha\dot\beta\dot\gamma} : R_{\mu\alpha\dot\beta\dot\gamma} = -\mathcal{D}_{\alpha} T_{\dot\beta\mu\dot\gamma} - \bar{\mathcal{D}}_{\dot\beta} T_{\alpha\mu\dot\gamma} - 2i\sigma^{\nu}{}_{\alpha\dot\beta} T_{\nu\mu\dot\gamma},$$

$$(10)\quad I_{\mu\nu\dot\alpha\dot\beta} : R_{\mu\nu\dot\alpha\dot\beta} = -\mathcal{D}_{\mu} T_{\nu\dot\alpha\dot\beta} - \mathcal{D}_{\nu} T_{\dot\alpha\mu\dot\beta} - \bar{\mathcal{D}}_{\dot\alpha} T_{\mu\nu\dot\beta} + T_{\nu\dot\alpha}{}^{\underline{\gamma}} T_{\underline{\gamma}\mu\dot\beta} + T_{\dot\alpha\mu}{}^{\underline{\gamma}} T_{\underline{\gamma}\nu\dot\beta},$$

$$(11)\quad I_{\mu\nu\rho\sigma} : R_{\mu\nu\rho\sigma} + R_{\nu\rho\mu\sigma} + R_{\rho\mu\nu\sigma} = 0,$$

$$(12)\quad I_{\mu\nu\dot\alpha\beta} : -\mathcal{D}_{\mu} T_{\nu\dot\alpha\beta} - \mathcal{D}_{\nu} T_{\dot\alpha\mu\beta} - \bar{\mathcal{D}}_{\dot\alpha} T_{\mu\nu\beta} + T_{\nu\dot\alpha}{}^{\underline{\gamma}} T_{\underline{\gamma}\mu\beta} + T_{\dot\alpha\mu}{}^{\underline{\gamma}} T_{\underline{\gamma}\nu\beta} = 0,$$

$$(13)\quad I_{\mu\nu\rho\alpha} : -\mathcal{D}_{\mu} T_{\nu\rho\alpha} - \mathcal{D}_{\nu} T_{\rho\mu\alpha} - \mathcal{D}_{\rho} T_{\mu\nu\alpha} + T_{\mu\nu}{}^{\underline{\gamma}} T_{\underline{\gamma}\rho\alpha} + T_{\nu\rho}{}^{\underline{\gamma}} T_{\underline{\gamma}\mu\alpha} + T_{\rho\mu}{}^{\underline{\gamma}} T_{\underline{\gamma}\nu\alpha} = 0,$$

Resolution of the 13 Bianchi identities :

1) Vector - Bispinor Isomorphism : $v_\mu \to v_{\alpha\dot{\alpha}}$
2) Decomposition of the curvature and torsion tensor in $SL(2,\mathbb{C})$ irreducible representations

$\to$ The equations splits in different symmetry classes !

$\star$ Result : R_{MNPQ} and $T_{MNP} \propto$ 3 superfields only : $\mathcal{R}$, $G_{\alpha\dot{\alpha}}$, $W_{(\alpha\beta\gamma)}$.

Explicit Supergravity Algebra

$$\{\mathcal{D}_\alpha, \mathcal{D}_\beta\} = -\frac{1}{2} R_{\alpha\beta\gamma\delta} J^{\delta\gamma},$$

$$\{\bar{\mathcal{D}}_{\dot\alpha}, \bar{\mathcal{D}}_{\dot\beta}\} = -\frac{1}{2} R_{\dot\alpha\dot\beta\dot\gamma\dot\delta} J^{\dot\delta\dot\gamma},$$

$$\{\mathcal{D}_\alpha, \bar{\mathcal{D}}_{\dot\alpha}\} = T_{\alpha\dot\alpha}{}^{\mu} \mathcal{D}_\mu - \frac{1}{2} R_{\alpha\dot\alpha\beta\gamma} J^{\gamma\beta} - \frac{1}{2} R_{\alpha\dot\alpha\dot\beta\dot\gamma} J^{\dot\gamma\dot\beta},$$

$$[\mathcal{D}_\alpha, \mathcal{D}_\mu] = T_{\alpha\mu}{}^{\beta} \mathcal{D}_\beta - T_{\alpha\mu}{}^{\dot\beta} \bar{\mathcal{D}}_{\dot\beta} - \frac{1}{2} R_{\alpha\mu\beta\gamma} J^{\gamma\beta} - \frac{1}{2} R_{\alpha\mu\dot\beta\dot\gamma} J^{\dot\gamma\dot\beta},$$

$$[\bar{\mathcal{D}}_{\dot\alpha}, \mathcal{D}_\mu] = T_{\dot\alpha\mu}{}^{\beta} \mathcal{D}_\beta + T_{\dot\alpha\mu}{}^{\dot\beta} \bar{\mathcal{D}}_{\dot\beta} - \frac{1}{2} R_{\dot\alpha\mu\beta\gamma} J^{\gamma\beta} - \frac{1}{2} R_{\dot\alpha\mu\dot\beta\dot\gamma} J^{\dot\gamma\dot\beta},$$

$$[\mathcal{D}_\mu, \mathcal{D}_\nu] = T_{\mu\nu}{}^{\beta} \mathcal{D}_\beta - T_{\mu\nu}{}^{\dot\beta} - \frac{1}{2} R_{\mu\nu\beta\gamma} J^{\gamma\beta} - \frac{1}{2} R_{\mu\nu\dot\beta\dot\gamma} J^{\dot\gamma\dot\beta}.$$

The Physical degrees of freedom of Supergravity

- $E_{\tilde{M}}{}^{M}$ has $8 \times 8 \times 16 = 1024$ components

General Change of Coordinates

$$z^{\tilde{M}} \to z^{\tilde{M}} + \xi^{\tilde{M}}(z),$$

- $z^{\tilde{M}}$: Cordinates of the curved superspace
- Transformations parameters $\xi^{\tilde{M}}(z)$: superfields depending on the coordinates $z^{\tilde{M}}$.

⋆ Physical degrees of freedom :

Graviton : $e_{\tilde{\mu}}{}^{\mu}$

Gravitino : $(\psi_{\mu}{}^{\tilde{\alpha}}, \bar{\psi}_{\tilde{\mu}\dot{\alpha}})$

- $\mathcal{R}$, $G_{\alpha\dot{\alpha}}$ give us the two auxilliary field b_μ and M to close the supermultiplet off shell.

Annexe A : How do we get the Supergravity Constrains ?

⋆ In the referential frame where ${\Omega_{\tilde{M}M}}^N = 0 \Rightarrow \emptyset$ Curvarture because

$$
\begin{aligned}
{R_{MNS}}^Q =& (-)^{|\tilde{M}|(|\tilde{N}|+|N|)} {E_M}^{\tilde{M}} {E_N}^{\tilde{N}} [\partial_{\tilde{M}} {\Omega_{\tilde{N}S}}^Q - {\Omega_{\tilde{M}S}}^R {\Omega_{\tilde{N}R}}^Q \\
& - (-)^{|\tilde{M}||\tilde{N}|} (\partial_{\tilde{N}} {\Omega_{\tilde{M}S}}^Q - {\Omega_{\tilde{N}S}}^R {\Omega_{\tilde{M}R}}^Q)].
\end{aligned}
$$

⋆ The covariant derivatives reduce to $\mathcal{D}_{\tilde{M}} = \partial_{\tilde{M}}$.
Introducing the Lorentz indices, we obtain the Covariant derivatives of Susy :

$$
\begin{aligned}
{E_\mu}^{\tilde{M}} \partial_{\tilde{M}} &= D_\mu = \partial_\mu \\
{E_\alpha}^{\tilde{M}} \partial_{\tilde{M}} &= D_\alpha = \partial_\alpha + i \sigma^\mu_{\alpha\dot{\alpha}} \bar{\theta}^{\dot{\alpha}} \partial_\mu \\
{E_{\dot{\alpha}}}^{\tilde{M}} \partial_{\tilde{M}} &= \bar{D}_{\dot{\alpha}} = \bar{\partial}_{\dot{\alpha}} - i \theta^\alpha \sigma^\mu_{\alpha\dot{\alpha}} \partial_\mu ,
\end{aligned}
$$

Annexe A : How do we get the Supergravity Constrains ?

⋆ The superalgebra reduces to

$$\{D_\alpha, \bar{D}_{\dot{\alpha}}\} = T_{\alpha\dot{\alpha}}{}^{\beta} D_\beta + T_{\alpha\dot{\alpha}\dot{\beta}} \bar{D}^{\dot{\beta}} + T_{\alpha\dot{\alpha}}{}^{\mu} \partial_\mu$$
$$\{D_\alpha, D_\beta\} = T_{\alpha\beta}{}^{\gamma} D_\gamma + T_{\alpha\beta\dot{\beta}} \bar{D}^{\dot{\beta}} + T_{\alpha\beta}{}^{\mu} \partial_\mu$$
$$\{\bar{D}_{\dot{\alpha}}, \bar{D}_{\dot{\beta}}\} = T_{\dot{\alpha}\dot{\beta}}{}^{\beta} D_\beta + T_{\dot{\alpha}\dot{\beta}\dot{\gamma}} \bar{D}^{\dot{\gamma}} + T_{\dot{\alpha}\dot{\beta}}{}^{\mu} \partial_\mu$$

⋆ Algebra of the covariant derivatives in Supersymmetry :

$$\{D_\alpha, \bar{D}_{\dot{\alpha}}\} = -2i\sigma^{\mu}{}_{\alpha\dot{\alpha}} \partial_\mu$$
$$\{D_\alpha, D_\beta\} = \{\bar{D}_{\dot{\alpha}}, \bar{D}_{\dot{\beta}}\} = 0.$$

⇒ Constrain the Torsion tensor components !

$$T_{\underline{\alpha\beta}}{}^{\underline{\gamma}} = 0, \quad T_{\alpha\beta}{}^{\mu} = T_{\dot{\alpha}\dot{\beta}}{}^{\mu} = 0,$$
$$T_{\alpha\dot{\alpha}}{}^{\mu} = T_{\dot{\alpha}\alpha}{}^{\mu} = -2i\sigma^{\mu}{}_{\alpha\dot{\alpha}}.$$