Geometry of large biconditioned random trees

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joint work with Igor KORTCHEMSKI

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Outlook

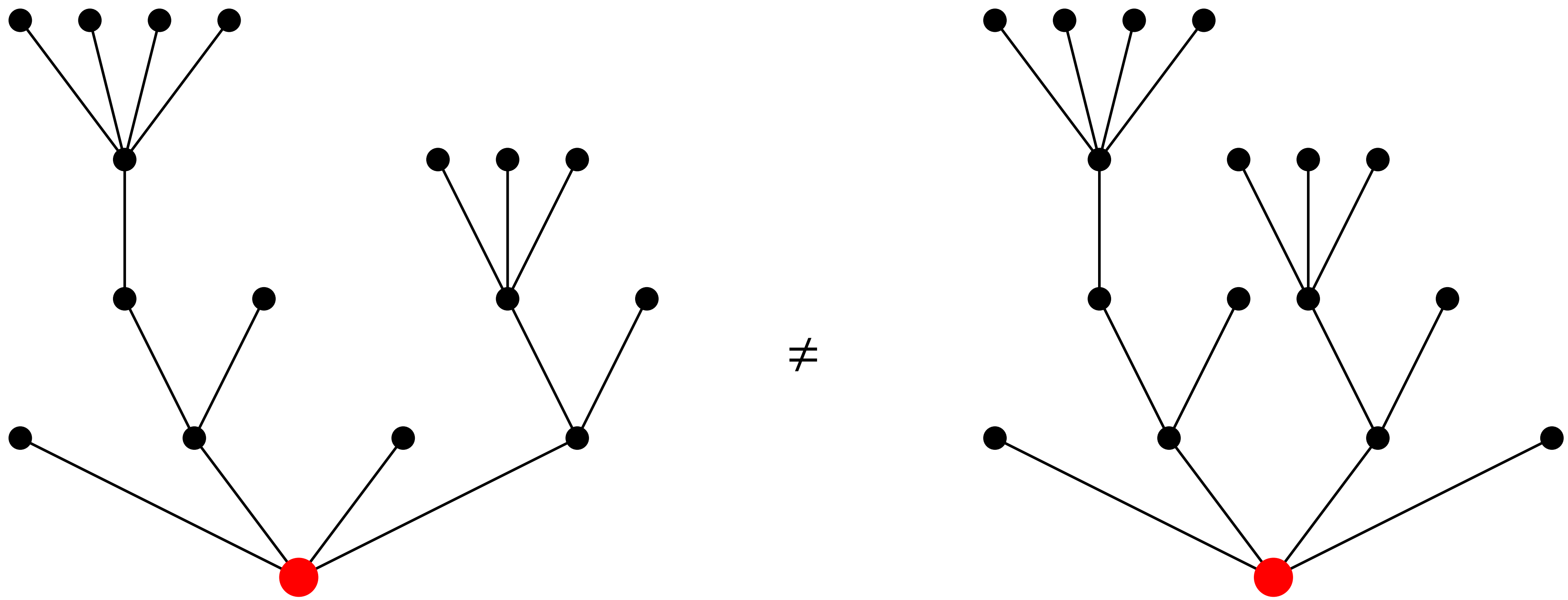
About the results:

- Motivation from random maps
- Answers in this framework
- Similar questions on trees left open

About the talk:

1. Model and questions on trees
2. From trees to excursions paths and then bridges
3. From bridges to nondecreasing paths
4. From nondecreasing paths to local limit estimates
5. Wrap up, further results, open questions
6. Brief discussion on maps?

Rooted plane trees



Genealogical tree:

- plane = siblings are ordered from left to right;
- rooted = ancestor and first child.

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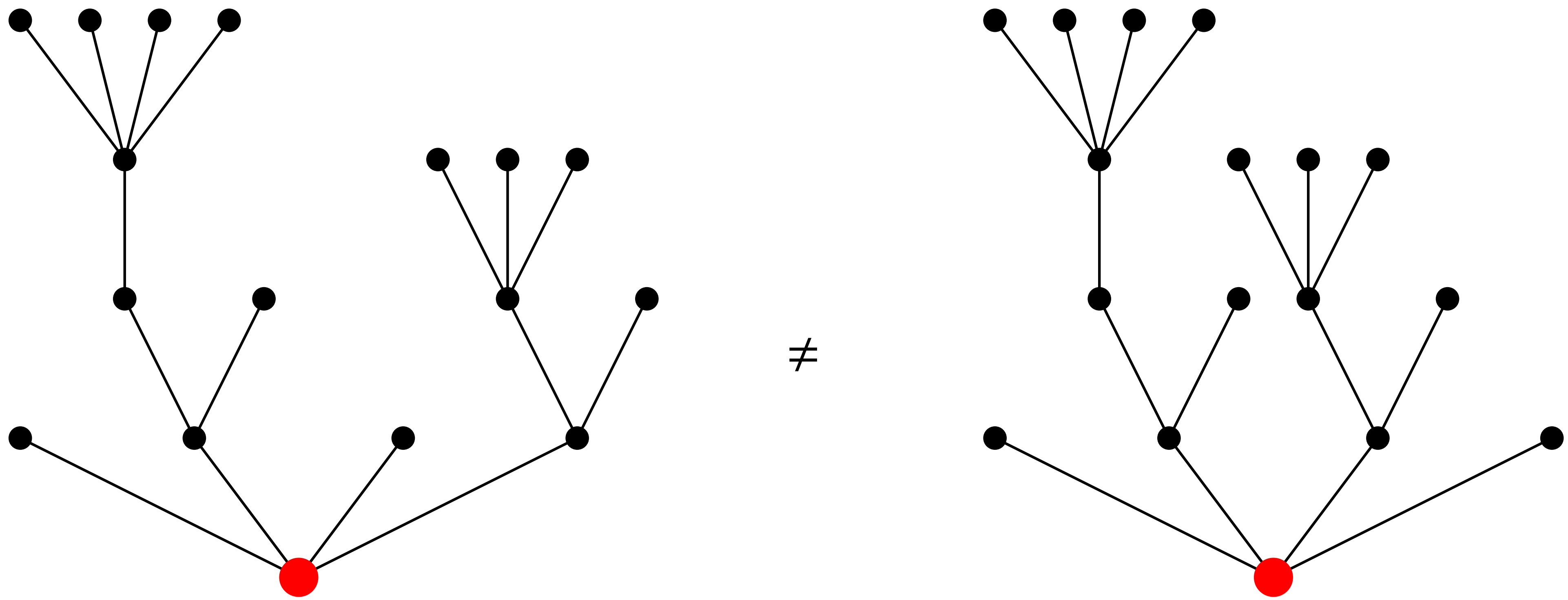

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Question: What does a random tree with n vertices look like when $n \to \infty$?

Random trees

Aldous '93: T_n uniform random tree with n vertices

$$\frac{1}{\sqrt{2n}} T_n \xrightarrow[n\to\infty]{(d)} \mathscr{T},$$

where the limit $\mathscr{T}$ is called the **Brownian tree**.

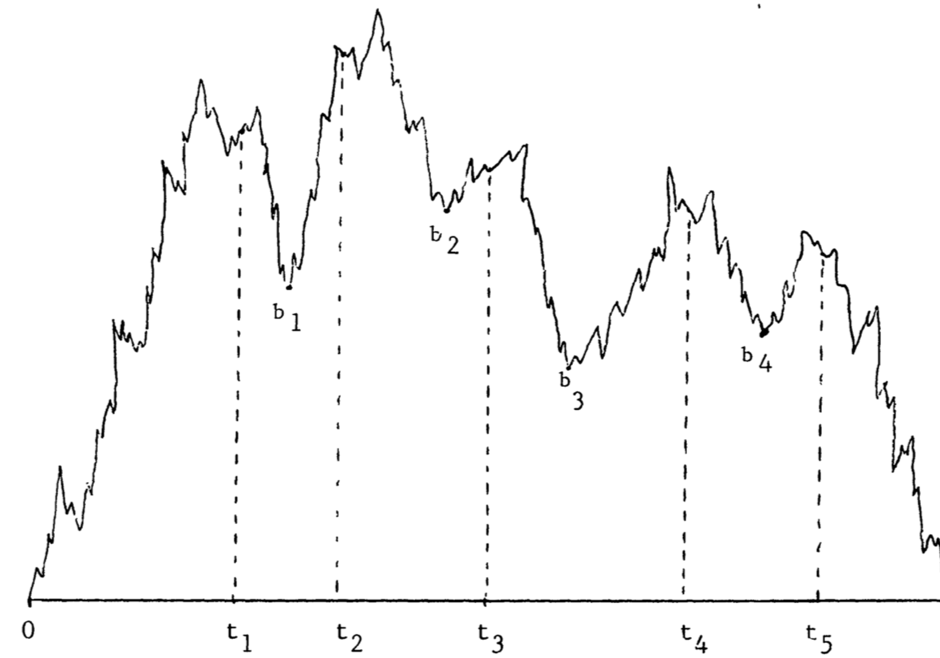

Fig. 2.

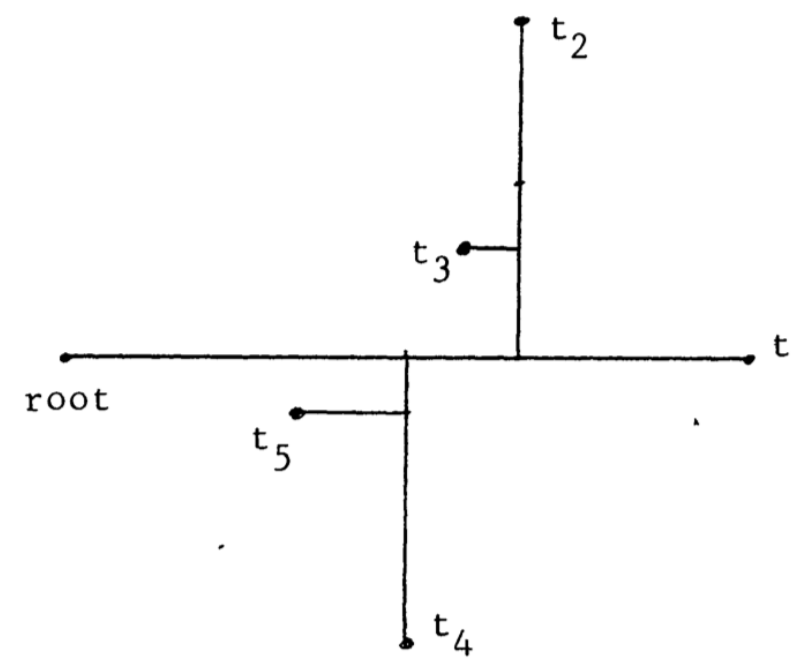

Fig. 3.

© David Aldous

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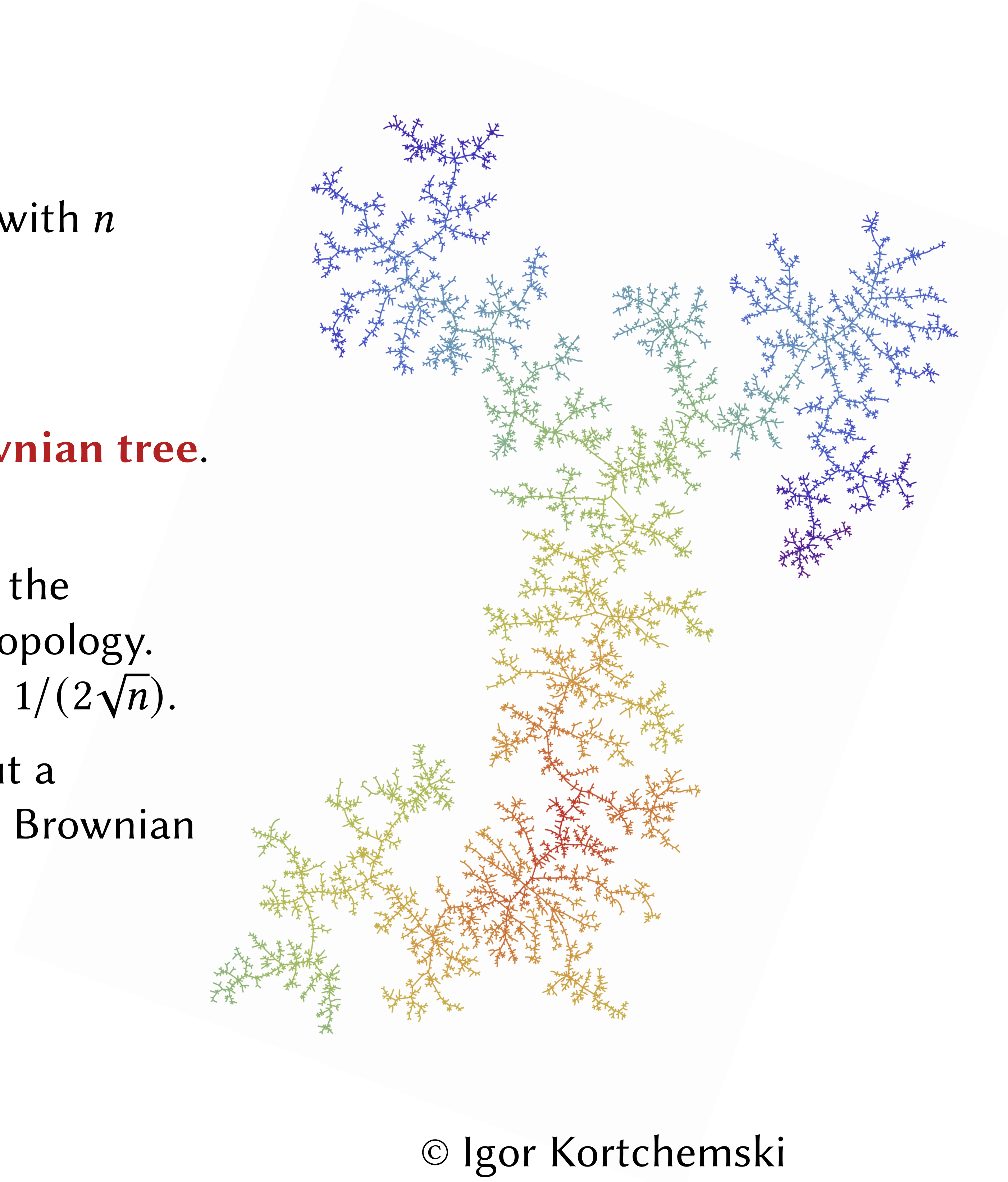
In modern language, the topology is the **Gromov–Hausdorff–Prokhorov** topology. Intuitively, each edge is given length $1/(2\sqrt{n})$.

T_∞ is not a discrete tree anymore, but a continuum one, and is related to the Brownian excursion.



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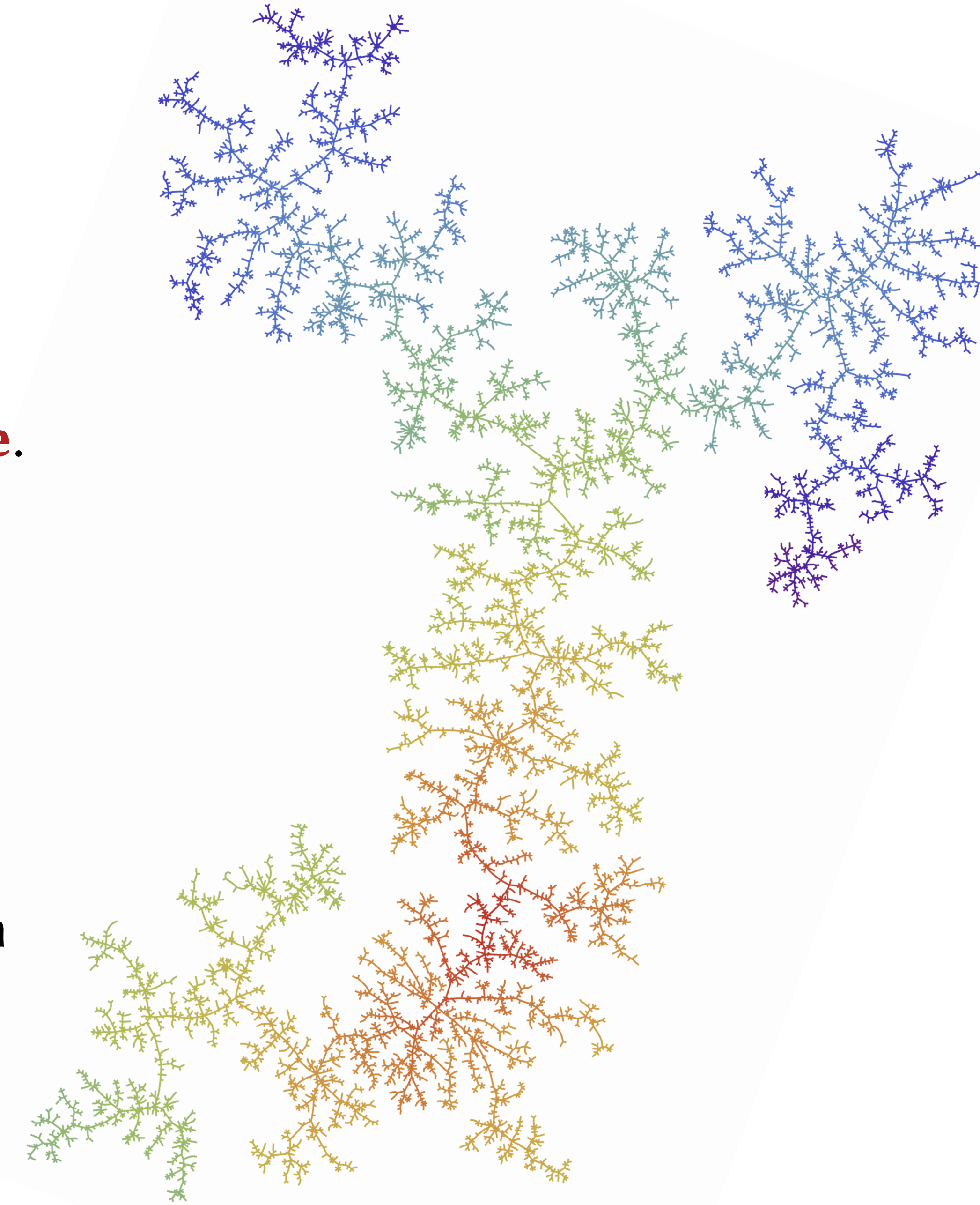
Universality. Aldous in fact considers size-conditioned **Bienaymé–Galton–Watson trees**, a (not so) particular case of **simply generated trees**.



Simply generated trees

Fix $\mathbf{q} = (q_k)_{k \geqslant 0} \in [0, \infty)^{\mathbf{Z}_+}$ and sample a tree t_n with n vertices with probability:

$$\mathbf{P}_n^{\mathbf{q}}(t_n) = \frac{1}{Z_n} \prod_{u \in t_n} q_{k_u},$$

where

- $u \in t_n$ is short for u is a vertex of t_n
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Remark: we must have $Z_n \neq 0$, which means that n must be compatible with the support of $\mathbf{q}$.

E.g. if $q_k \neq 0$ iff $k \in \{0, 2\}$, only binary trees, with odd size, are allowed.

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Examples:

- $q_k = 1$ for every $k \geqslant 1$, then $\mathbf{P}_n^{\mathbf{q}}$ is the uniform distribution on trees with n vertices.
- $q_k = 1$ if $k \in A$ and $q_k = 0$ otherwise, with $0 \in A$, then $\mathbf{P}_n^{\mathbf{q}}$ is the uniform distribution on trees with n vertices with offspring numbers in A.
- If $\mathbf{q}$ is a probability measure with mean 1, then $\mathbf{P}_n^{\mathbf{q}}$ is the law of a critical **Bienaymé–Galton–Watson** tree, i.e. each individual reproduces independently according to $\mathbf{q}$, and conditioned to have n vertices in total.

Limits of large simply generated trees

A straightforward calculation shows: if $\mathbf{p}$ and $\mathbf{q}$ are related by

$$p_k = ab^k q_k \quad \text{for every} \quad k \geqslant 0,$$

for some $a, b > 0$, then $\mathbf{P}_n^{\mathbf{q}} = \mathbf{P}_n^{\mathbf{p}}$.

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Given $\mathbf{q}$, if $G(s) = \sum_{k \geqslant 0} s^k q_k$ has radius of convergence $\rho > 0$, then for every $b \in (0, \rho)$, the sequence $p_k = G(b)^{-1} b^k q_k$ is a probability with mean $bG'(b)/G(b)$, which is increasing in b.

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Conclusion: if there exists $b \in (0, \rho]$ such that $bG'(b)/G(b) = 1$, then $\mathbf{P}_n^{\mathbf{q}}$ is the law of a critical Bienaymé–Galton–Watson tree conditioned to have n vertices.

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Theorem (Aldous '93) Suppose $\mathbf{p}$ has mean 1 and variance $\sigma^2 \in (0, \infty)$ and sample T_n from $\mathbf{P}_n^{\mathbf{p}}$, then

$$\frac{\sigma}{2\sqrt{n}} T_n \xrightarrow[n\to\infty]{(d)} \mathcal{T},$$

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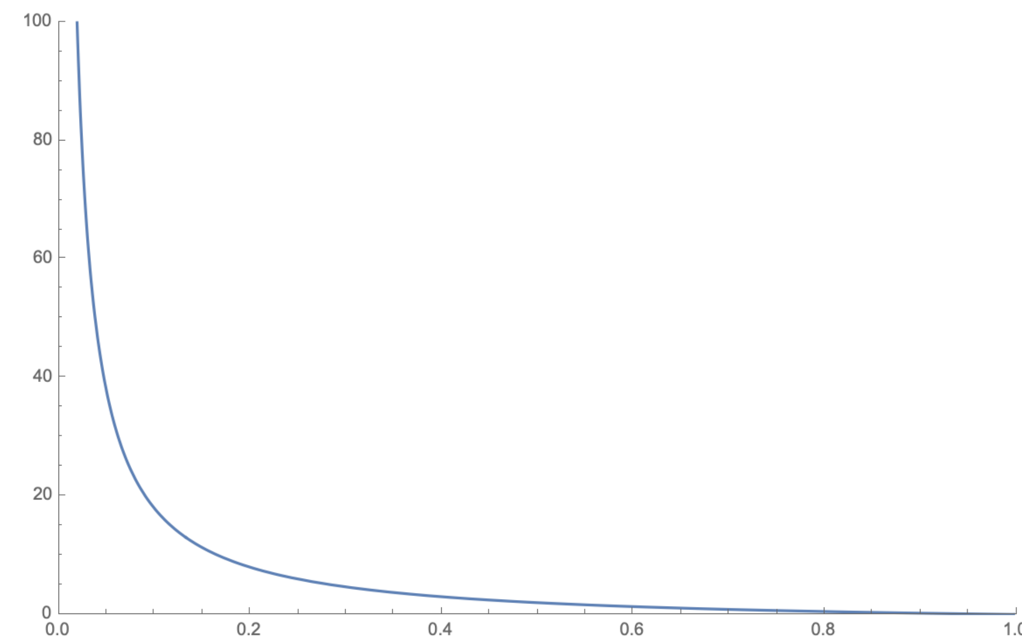
$$\frac{1}{\sqrt{ns(k_n/n)}} T_n \xrightarrow[n\to\infty]{(d)} \mathcal{T},$$

where $s(x) = 2(1-x)/x$ for every $x \in (0,1)$.



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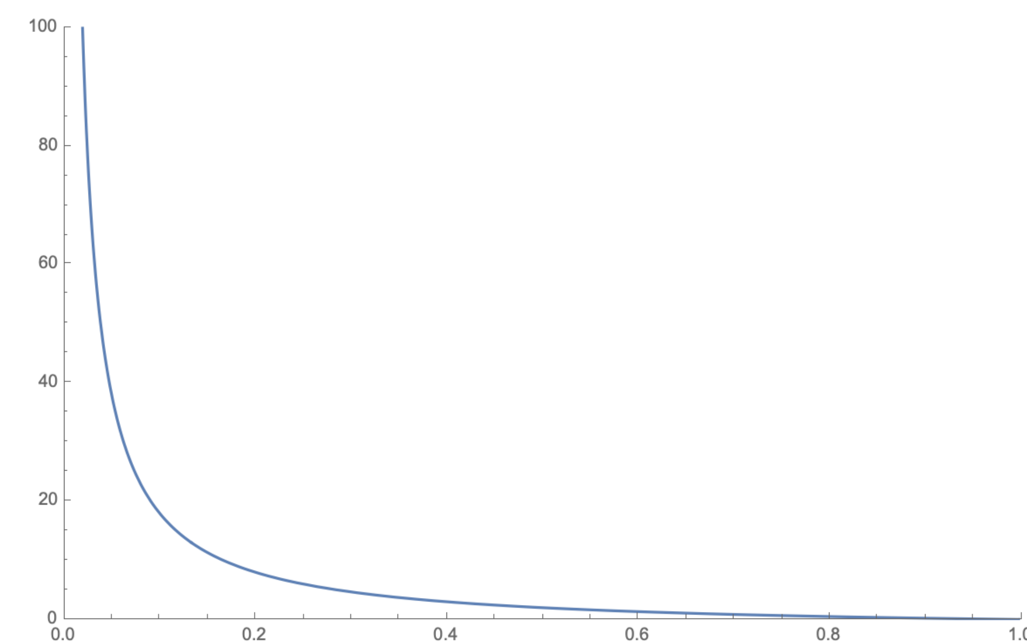

Question: What about more general biconditioned simply generated trees?

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We do not aim to control the **contour** or **height** process of the trees, but only their **Łukasiewicz path** $W_j = \sum_{i \leqslant j} w_i$.

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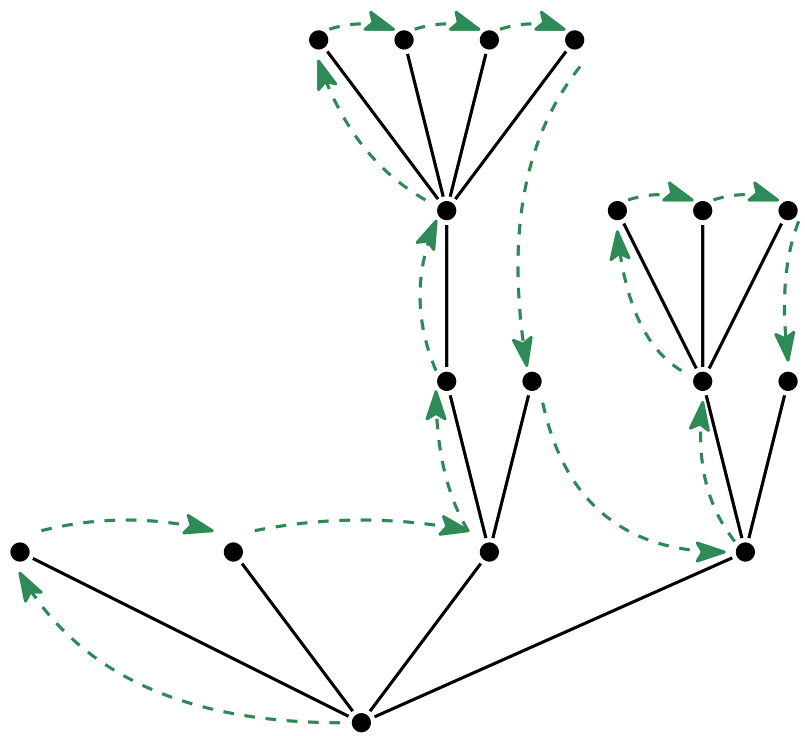

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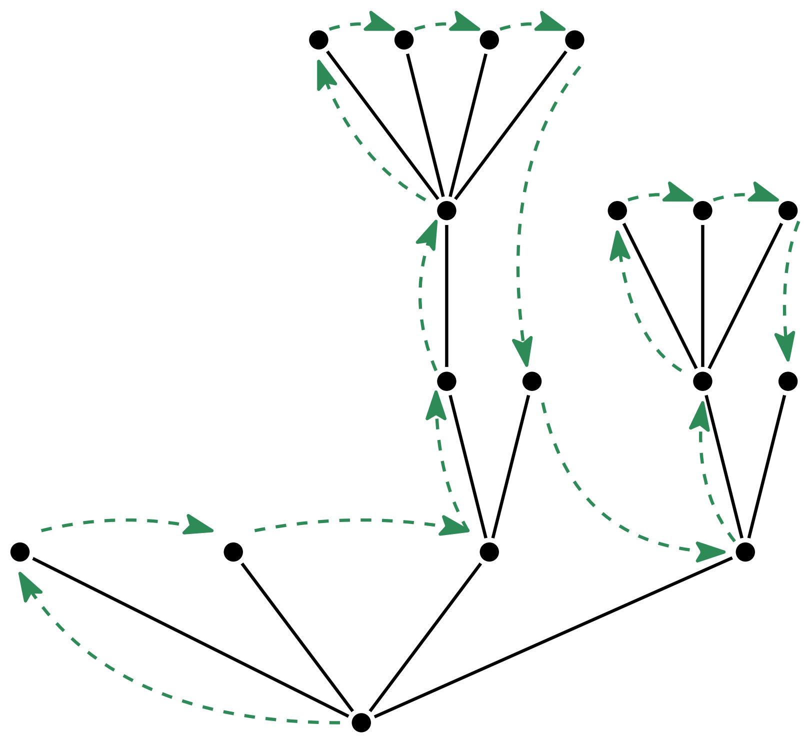

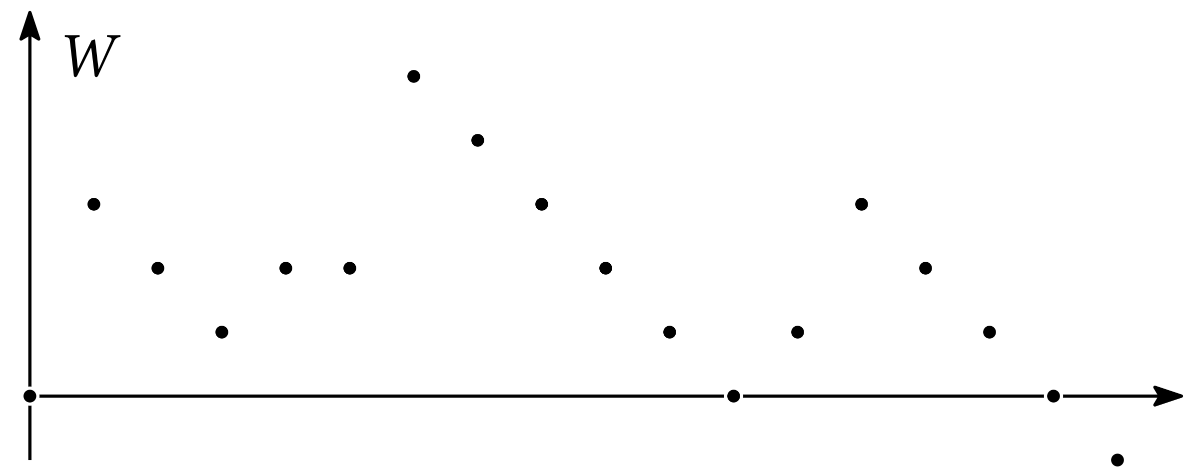

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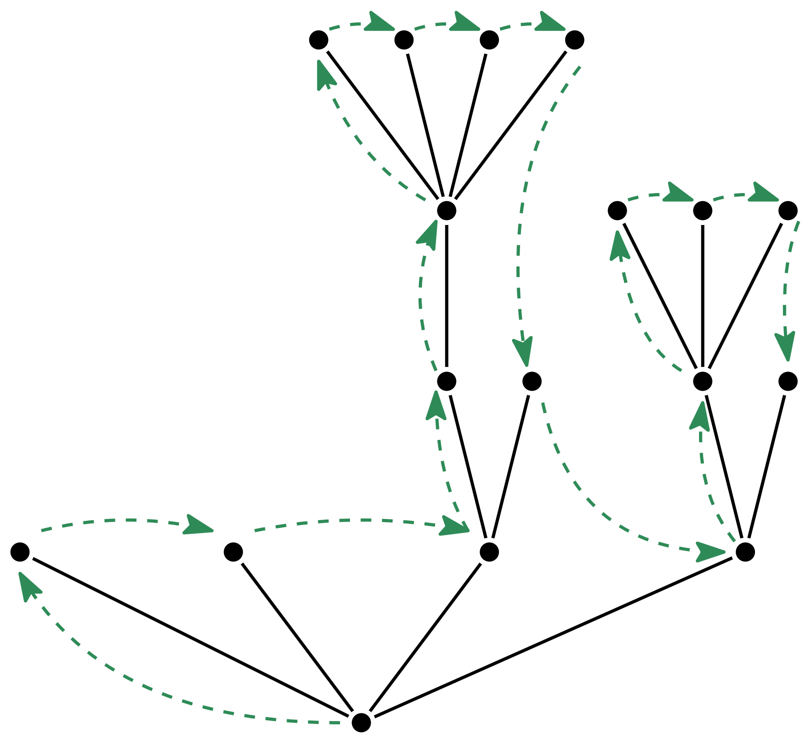

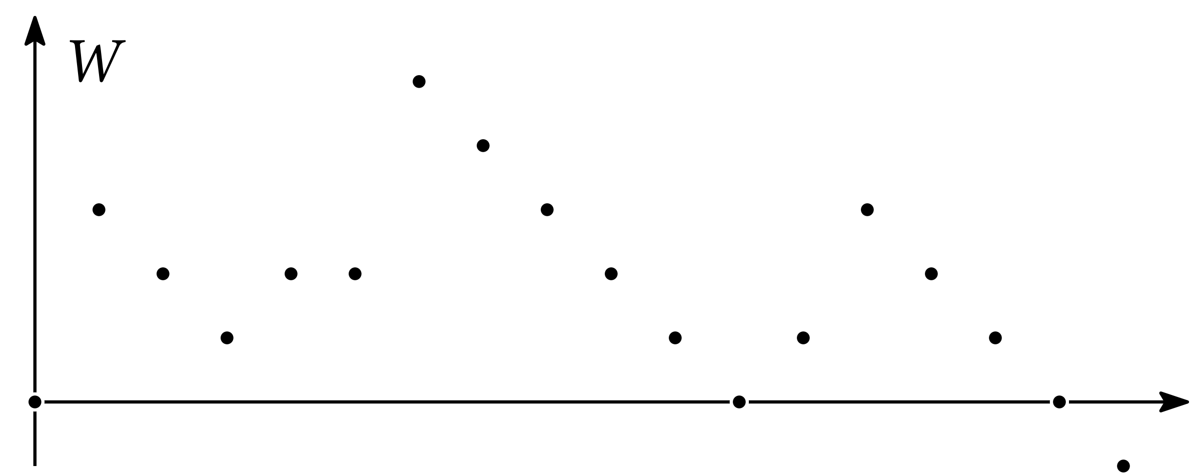

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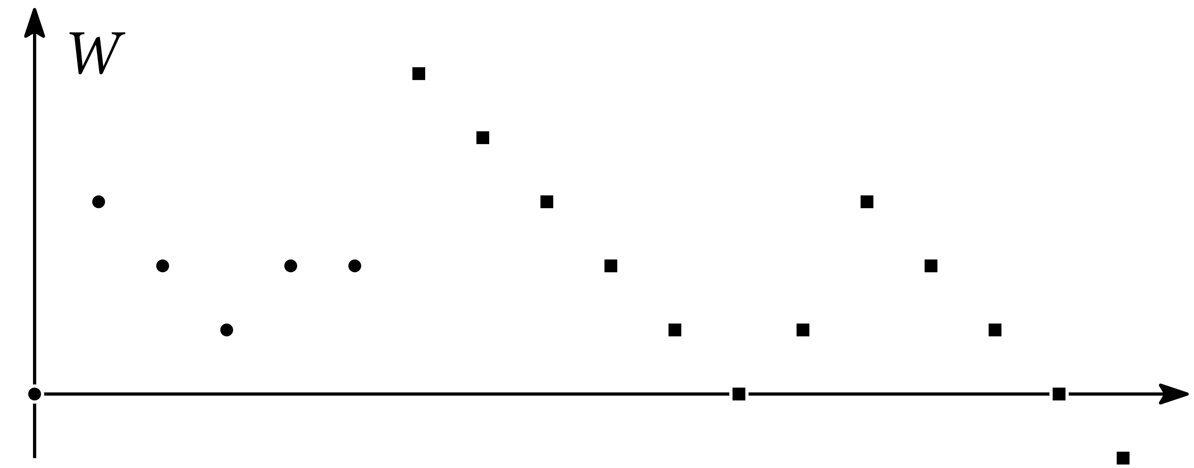
We want $a_n^{-1/2} W_{\lfloor nt \rfloor} \to B^{\mathrm{ex}}$ a Brownian excursion under $\mathbf{P}^{\mathbf{q}}(\,\cdot \mid n \text{ vertices } \& \ k_n \text{ leaves})$.

The conjugation trick

Recall: we want $a_n^{-1/2} W_{\lfloor nt \rfloor} \longrightarrow B^{\mathrm{ex}}$, a Brownian excursion under $\mathbf{P}^{\mathbf{q}}(\cdot \mid n \text{ vertices } \& \ k_n \text{ leaves})$.



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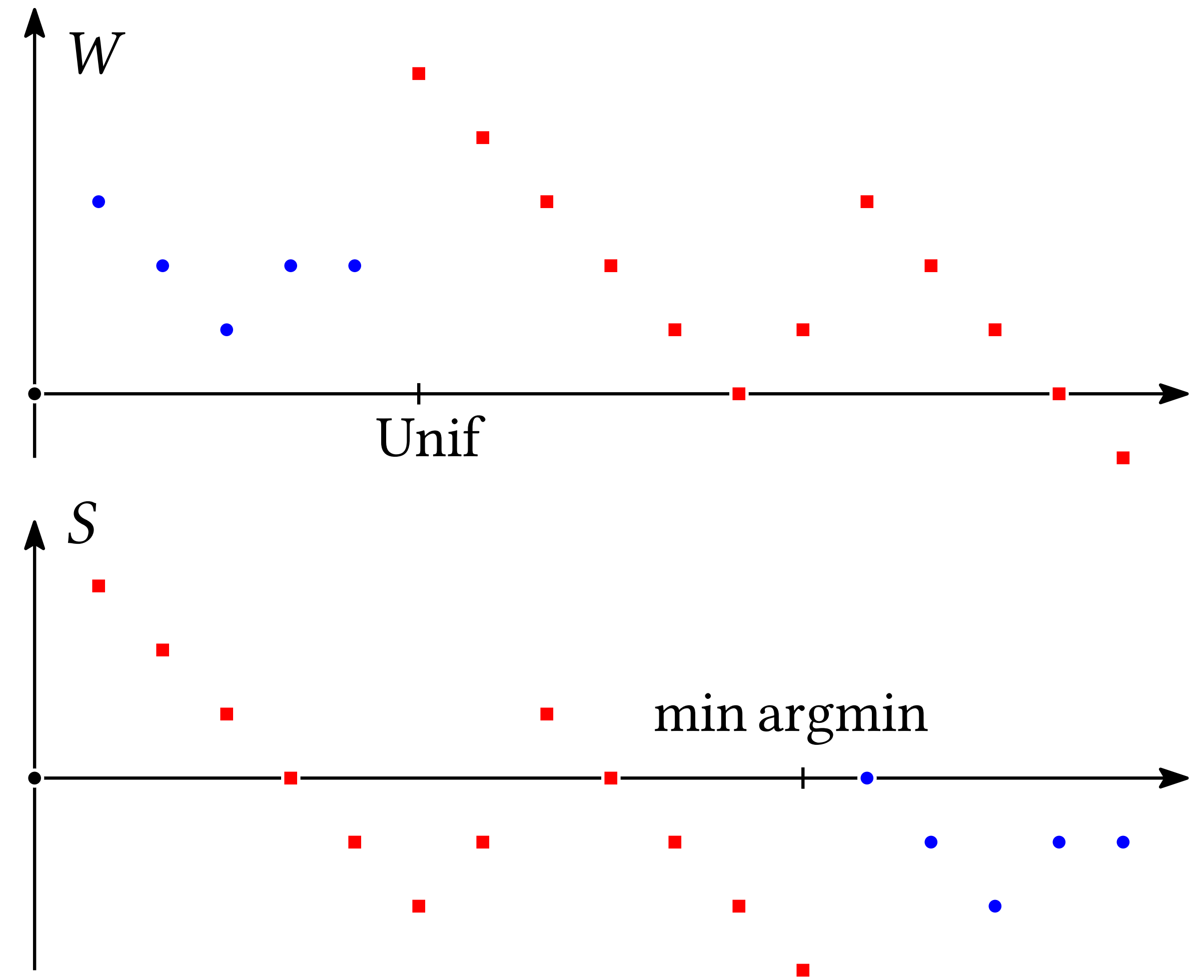

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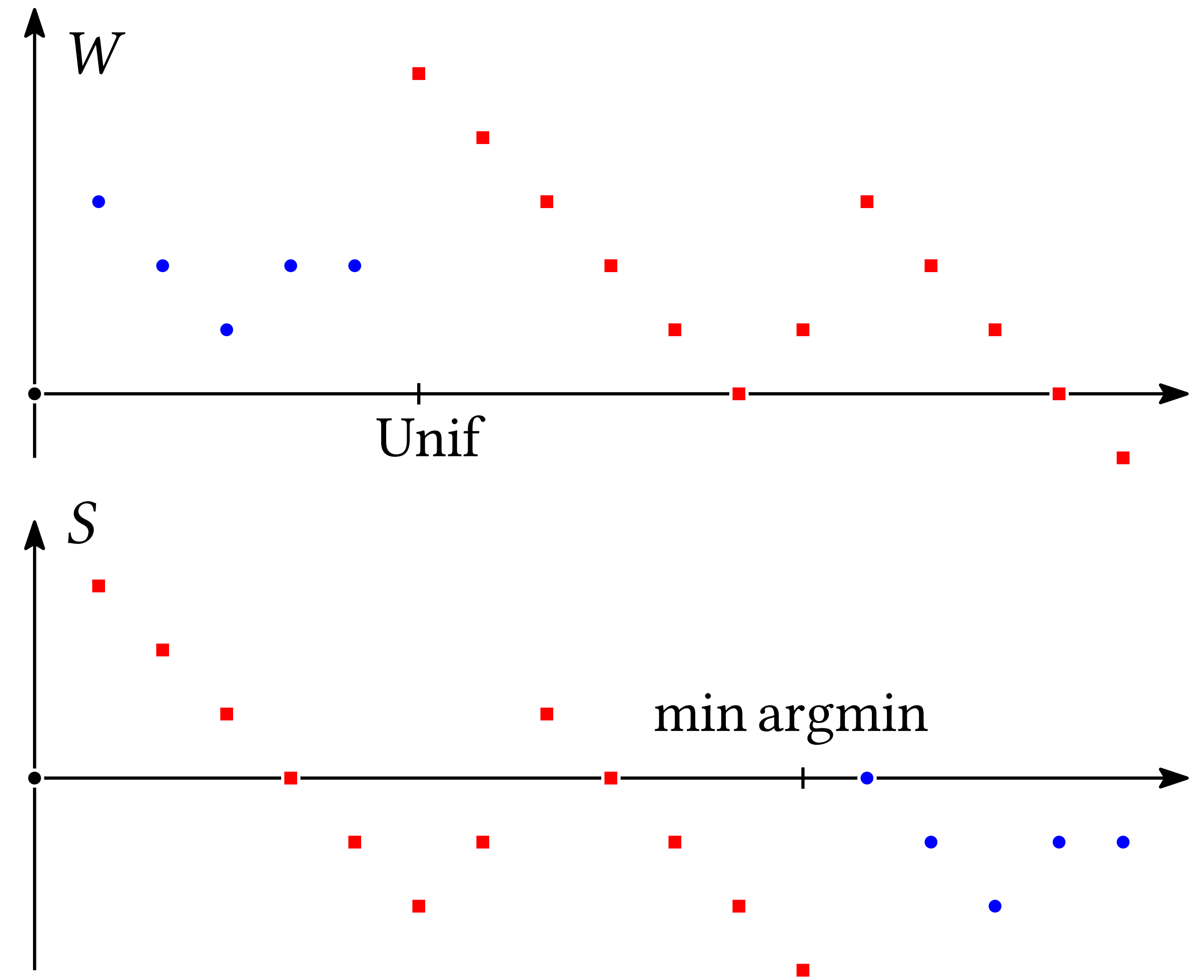

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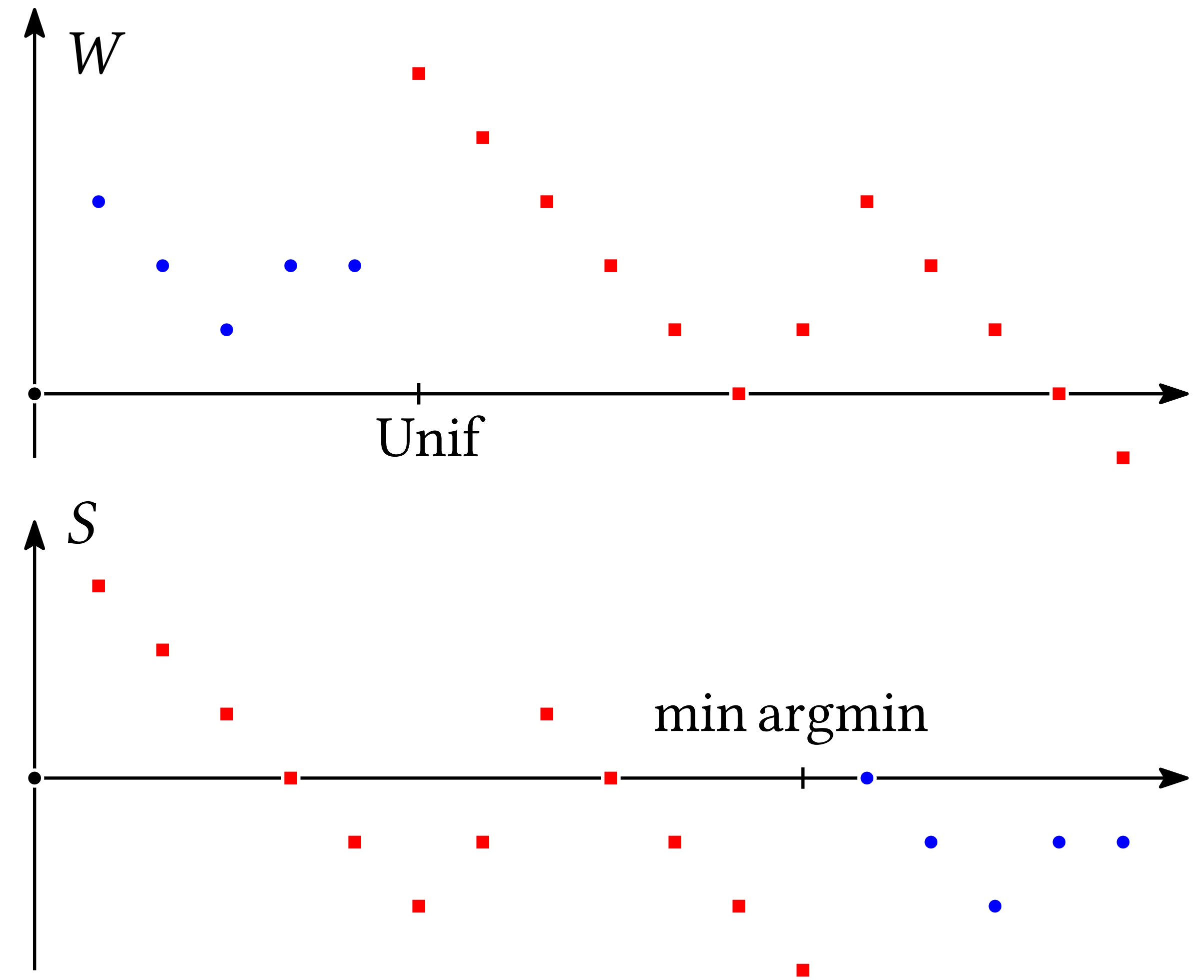
S is a random path whose increments are sampled with probability



$$\frac{1}{Z_n} \prod_{i=1}^{n} q_{s_i+1}$$

in the set $\left\{ (s_i)_{1 \leqslant i \leqslant n} \in \mathbf{Z}_{\geqslant -1}^{n} : \sum_{i=1}^{n} s_i = -1 \text{ and } \#\{i \leqslant n : s_i = -1\} = k_n \right\}.$

Simply generated bridges

Key observation: The position of the k_n negative increments of S is a uniform random choice. Therefore if we set $L_j = \{i \leqslant j : s_i = -1\}$, then it can be constructed from an urn.

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If we sample with replacement, then $L_j \sim \mathrm{Bin}(j, k_n/n)$ and then

$$\left(\frac{L_{\lfloor nt \rfloor} - k_n t}{\sqrt{k_n(n-k_n)/n}} \right)_{0 \leqslant t \leqslant 1} \xrightarrow[n\to\infty]{(d)} B.$$

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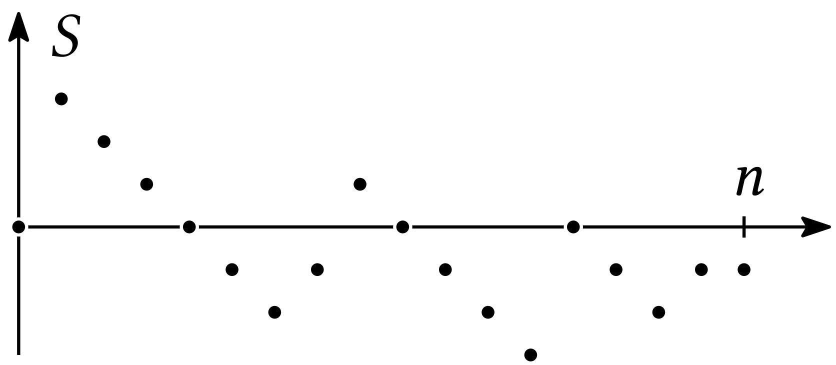
See e.g. the lecture notes from St-Flour by Aldous '85.

Simply generated bridges

Split the negative and nonnegative increments:



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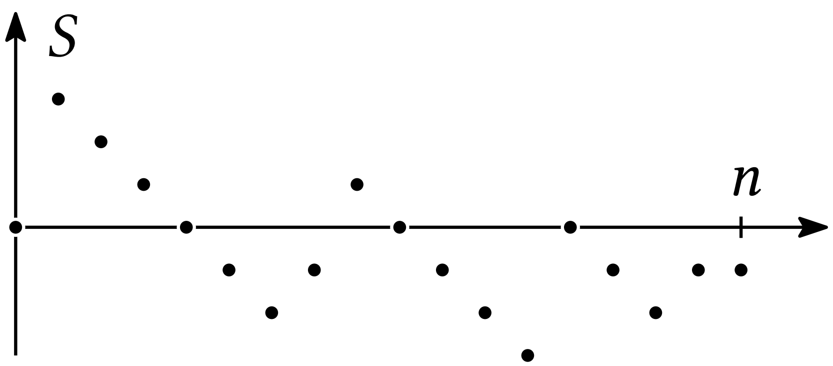

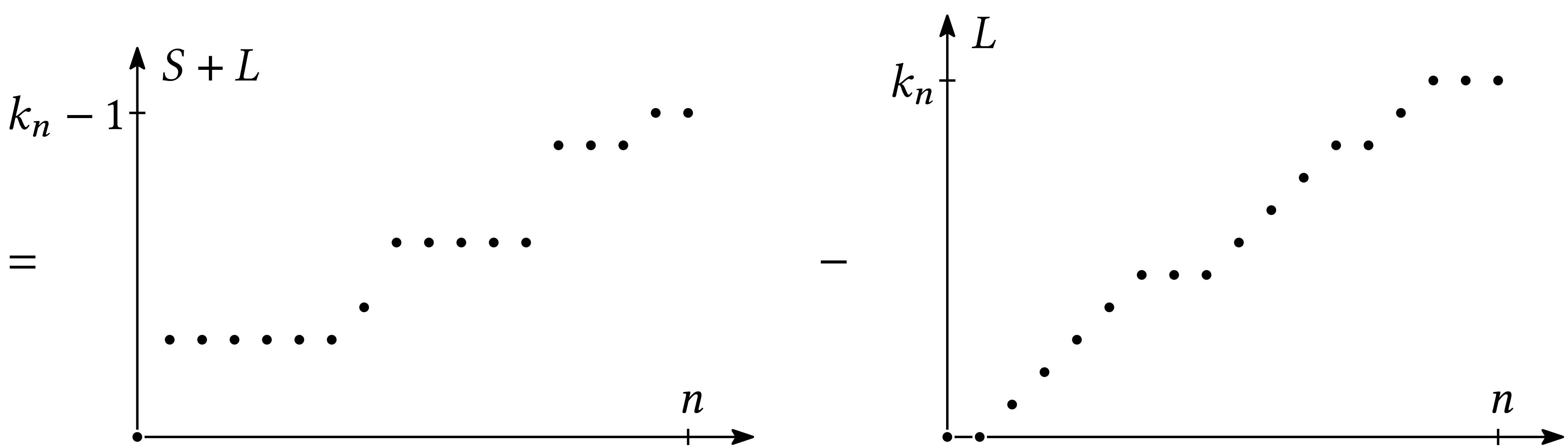

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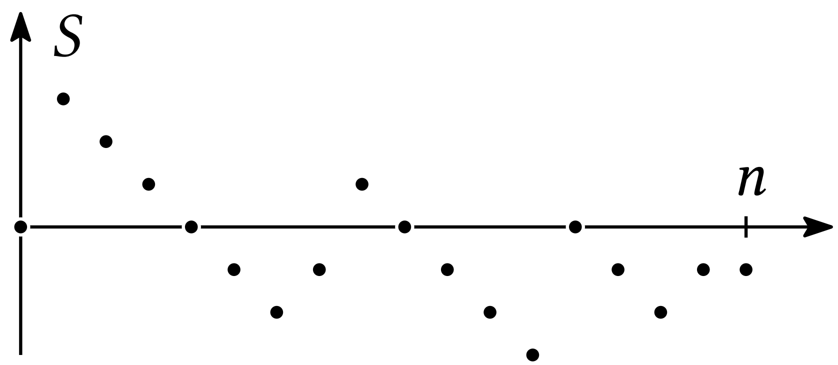

Recall that

$$\left(\frac{L_{\lfloor nt \rfloor} - k_n t}{\sqrt{k_n(n-k_n)/n}}\right)_{0\leqslant t\leqslant 1} \xrightarrow[n\to\infty]{(d)} B^{\mathrm{br}}.$$

It remains to study $S + L$, not independent from L.

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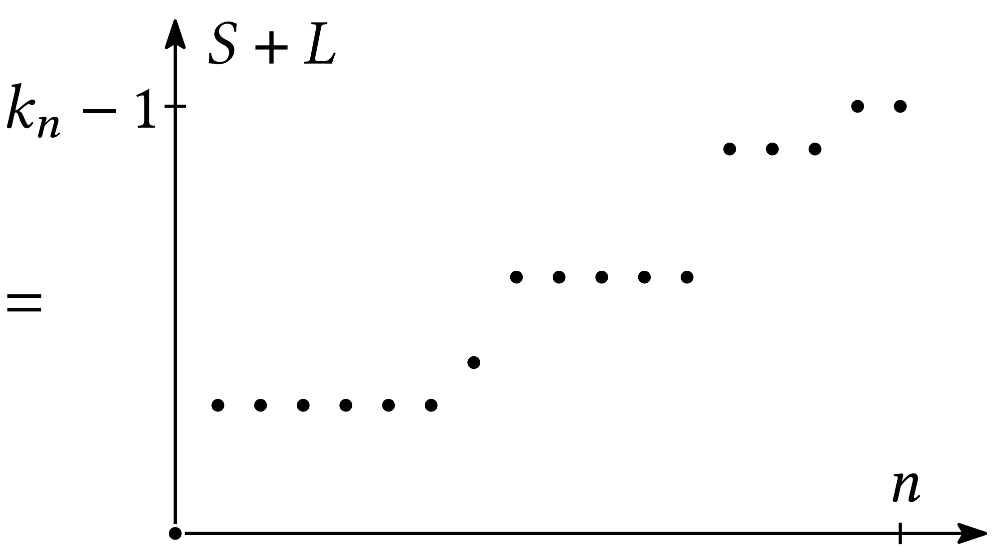

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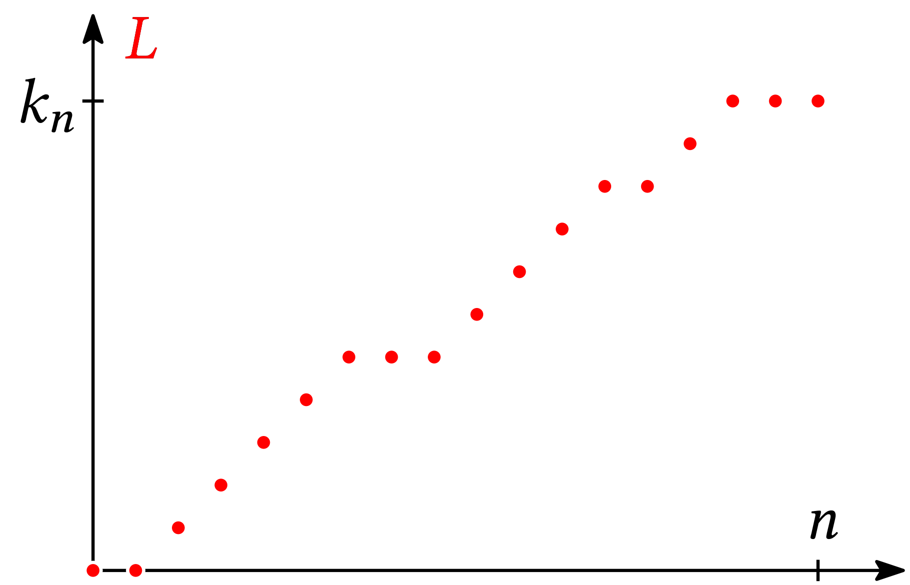

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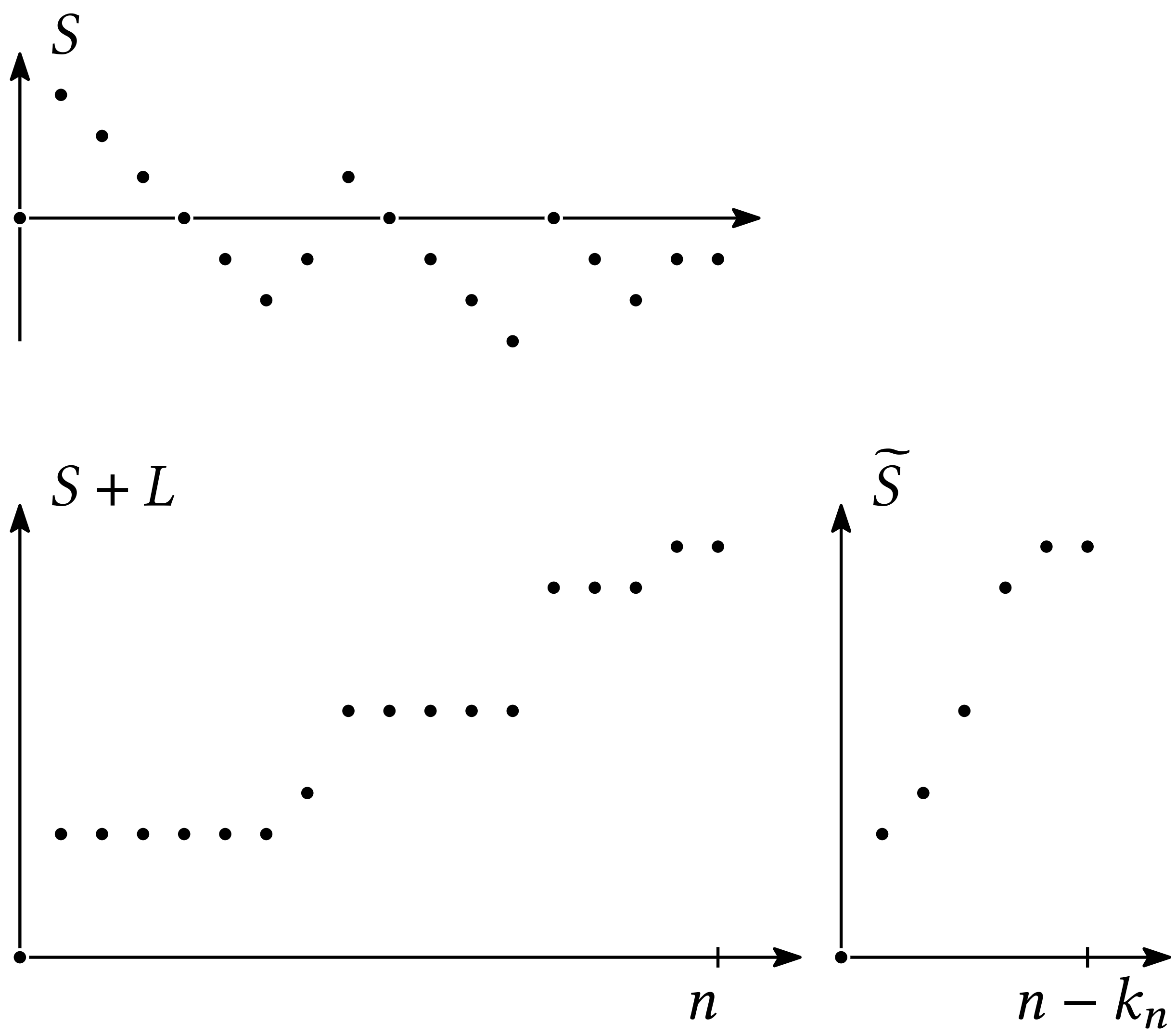
Remove the negative increments from S to get $(\widetilde{S}_i)_{0 \leqslant i \leqslant n-k_n}$, now independent from L.



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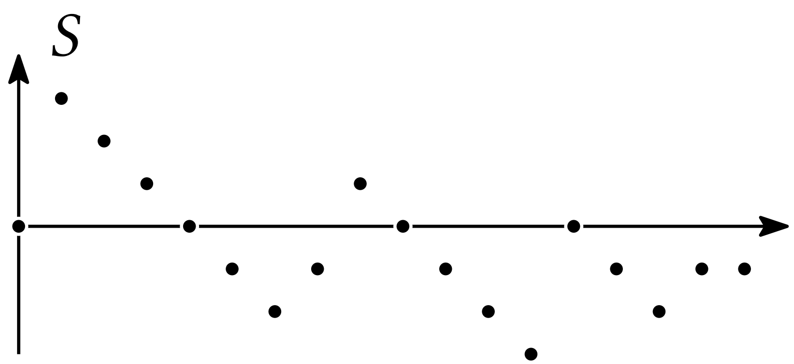

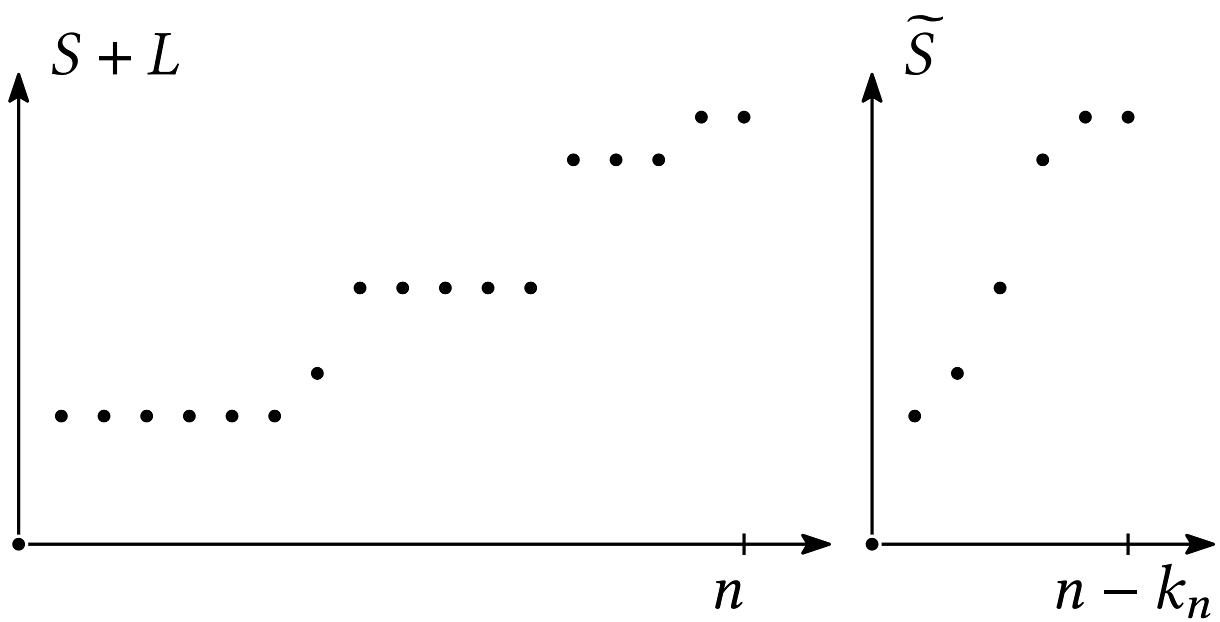

The increments of $\widetilde{S}$ belong to

$$\left\{(\widetilde{s}_i)_{1\leqslant i\leqslant n-k_n} \in \mathbf{Z}_{\geqslant 0}^{n-k_n} : \sum_{i=1}^{n-k_n} \widetilde{s}_i = k_n - 1\right\},$$

and are sampled with probability

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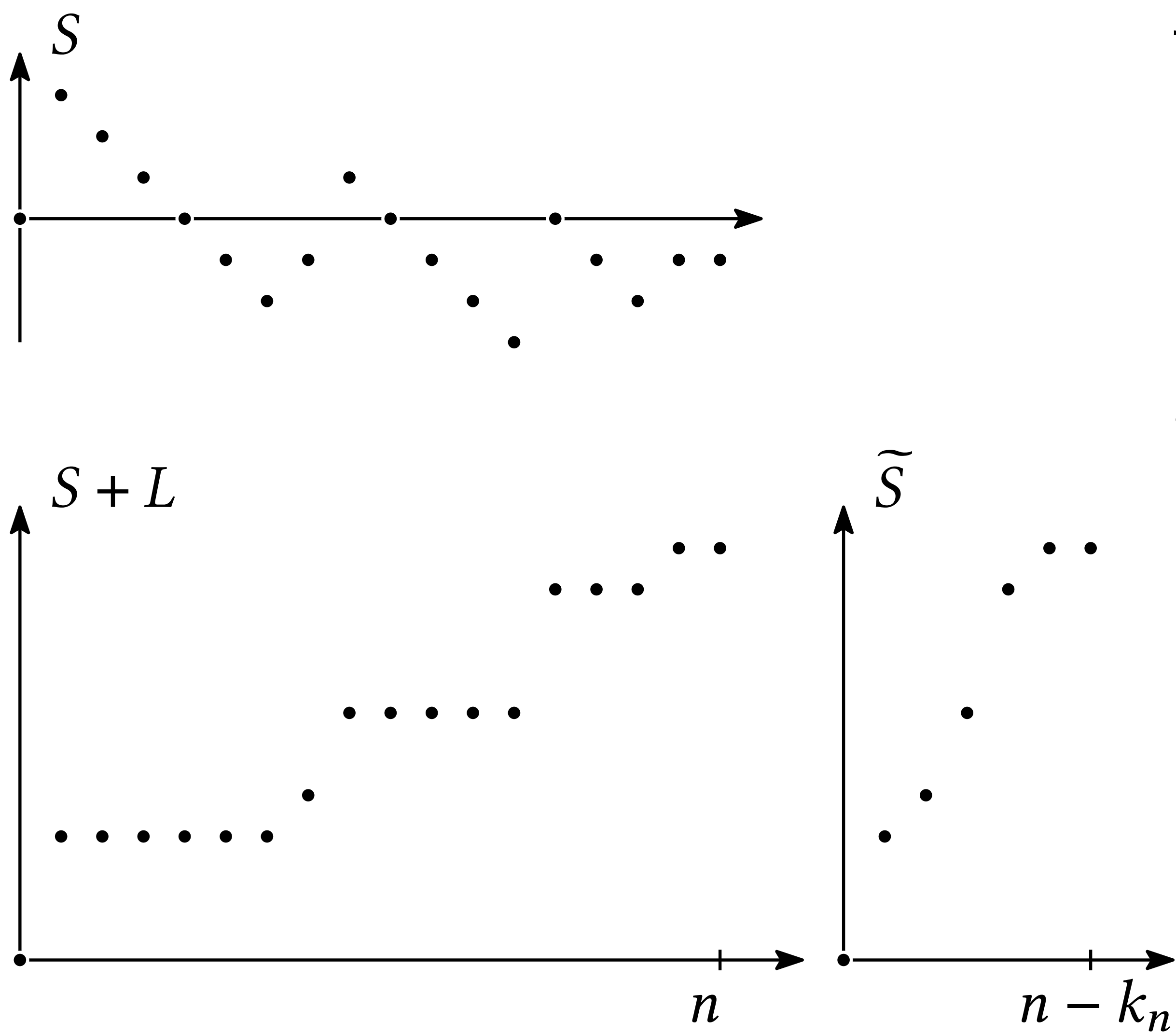

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Nota: $\frac{nt - L_{\lfloor nt \rfloor}}{n-k_n} \to t$ in proba. so

$$(S+L)_{\lfloor nt \rfloor} = \widetilde{S}_{\lfloor nt - L_{\lfloor nt \rfloor} \rfloor} \approx \widetilde{S}_{\lfloor (n-k_n) t \rfloor}$$

Scaling limits of simply generated bridges

It all boils down to proving a convergence of the form

$$\left(\frac{S_{\lfloor nt \rfloor} - x_n t}{\sqrt{a_n}}\right)_{0 \leqslant t \leqslant 1} \xrightarrow[n\to\infty]{(d)} B^{\mathrm{br}},$$

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Let us suppose that there exists a probability measure $\mathbf{p}$ of the form $p_k = ab^k q_k$. Then S is a $\mathbf{p}$-random walk conditioned on $S_n = x_n$.

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It all boils down to proving a convergence of the form

Change of notation!

$$\left(\frac{S_{\lfloor nt \rfloor} - x_n t}{\sqrt{a_n}}\right)_{0 \leqslant t \leqslant 1} \xrightarrow[n\to\infty]{(d)} B^{\mathrm{br}},$$

where S is an nondecreasing bridge from 0 to x_n in n steps, with weight sequence $\mathbf{q}$.

Let us suppose that there exists a probability measure $\mathbf{p}$ of the form $p_k = ab^k q_k$. Then S is a $\mathbf{p}$-random walk conditioned on $S_n = x_n$.

It suffices to restrict to the interval $[0, 1-\varepsilon]$ for $\varepsilon > 0$ fixed. One can then argue by **absolute continuity**: by the Markov property,

$$\mathbf{E}\left[F\left(\left(\frac{S_{\lfloor nt \rfloor} - x_n t}{\sqrt{a_n}}\right)_{0 \leqslant t \leqslant 1-\varepsilon}\right) \,\middle|\, S_n = x_n\right]$$
$$= \mathbf{E}\left[F\left(\left(\frac{S_{\lfloor nt \rfloor} - x_n t}{\sqrt{a_n}}\right)_{0 \leqslant t \leqslant 1-\varepsilon}\right) \cdot \frac{\mathbf{P}(S'_{n-\lfloor n(1-\varepsilon)\rfloor} = x_n - S_{\lfloor n(1-\varepsilon)\rfloor})}{\mathbf{P}(S_n = x_n)}\right],$$

where S and S' are two independent random walks with step distribution $\mathbf{p}$.

Simply generated bridges & local limit estimates

Easy case: when $\mathbf{p}$ has mean μ and finite variance σ^2 and $x_n - \mu n = o(\sqrt{n})$.

Then the Local Limit Theorem states that with $g_t(x) = (2\pi t)^{-1/2} \exp(-x^2/(2t))$,

$$\sup_{k \in \mathbf{Z}} \left| \sqrt{n\sigma^2}\, \mathbf{P}(S_n = \lfloor \mu n \rfloor + k) - g_1\left(\frac{k}{\sqrt{n\sigma^2}}\right) \right| \underset{n\to\infty}{\longrightarrow} 0.$$

Also

$$\left(\frac{S_{\lfloor nt \rfloor} - \mu n t}{\sqrt{n\sigma^2}} \right)_{0 \leqslant t \leqslant 1} \xrightarrow[n\to\infty]{(d)} B,$$

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Then with the previous decomposition,

$$\mathbf{E}\left[F\left(\left(\frac{S_{\lfloor nt \rfloor} - x_n t}{\sqrt{n\sigma^2}}\right)_{0 \leqslant t \leqslant 1-\varepsilon}\right) \middle| S_n = x_n\right] \underset{n\to\infty}{\longrightarrow} \mathbf{E}\left[F\left((B_t)_{0 \leqslant t \leqslant 1-\varepsilon}\right) \cdot \frac{g_\varepsilon(-B_{1-\varepsilon})}{g_1(0)}\right],$$

and the right-hand side equals $\mathbf{E}\left[F((B_t^{\mathrm{br}})_{0 \leqslant t \leqslant 1-\varepsilon})\right]$.

Simply generated bridges & local limit estimates

More generally, given x_n, one looks for a probability $\mathbf{p}^n$ of the form $p_k^n = a_n b_n^k q_k$ and with mean close to x_n/n, for which we can prove for some $a_n \to \infty$,

$$\sup_{k \in \mathbf{Z}} \left| \sqrt{a_n}\, \mathbf{P}(S_n^n = x_n + k) - g_1\left(\frac{k}{\sqrt{a_n}}\right) \right| \underset{n \to \infty}{\longrightarrow} 0.$$

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Theorem (Kortchemski & ☺ '21+). This estimates holds in each of the following cases:

1. $\lim_n x_n/n \in (i_{\mathbf{q}}, \rho G'(\rho)/G(\rho))$ where $i_{\mathbf{q}} = \min\{i : q_i > 0\}$ and $G(s) = \sum_k s^k q_k$ with radius of convergence ρ. Here
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2. $\lim_n x_n/n = 0$, $q_0, q_1 > 0$. Here $a_n = x_n$.
3. $\lim_n x_n/n = \infty$, G is Δ-analytic, and there exist $c, \alpha > 0$ such that $G(\rho - z) \sim c z^{-\alpha}$ as $z \to 0$ with $\mathrm{Re}(z) > 0$. Here $a_n = x_n^2/(\alpha n)$.

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The last case was motivated by uniform random bipartite maps which are related to $q_k = \binom{2k+1}{k+1}$, which satisfies all the assumptions and $i_{\mathbf{q}} = 0$ and $\rho G'(\rho)/G(\rho) = \infty$.

Other behaviours

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Then an unconditioned random walk S satisfies

$$\left(n^{-1/\alpha}\left(S_{\lfloor nt \rfloor} - \mu nt\right)\right)_{t \geqslant 0} \xrightarrow[n\to\infty]{(d)} X^\alpha,$$

where X^α is an **α-stable Lévy process** with no negative jump.

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3. The path $1_{U \leqslant t} - t$ where $U \sim \text{Unif}(0,1)$ when $\lambda = \infty$.

Final remarks & open questions

Question: What about the height process? By a more general work (in preparation, hopefully Kortchemski & ☺ '21b) we have the convergence of the marginals, but tightness is missing in general (only available for the moment in a finite variance regime).

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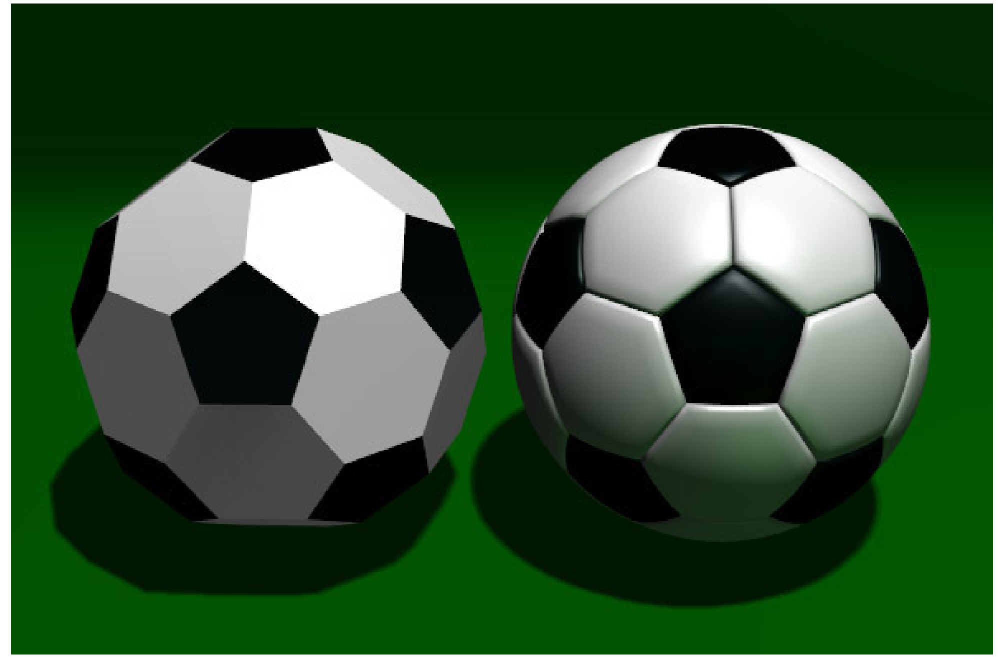
Thank you!

Planar maps

A (planar) map can be seen as the topological gluing of polygons, by identifying pairs of edges, to form a sphere.



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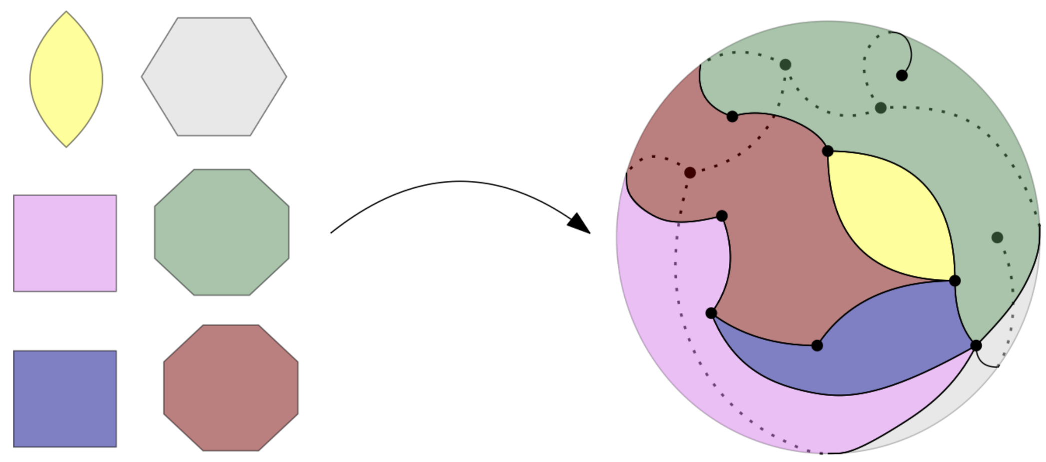

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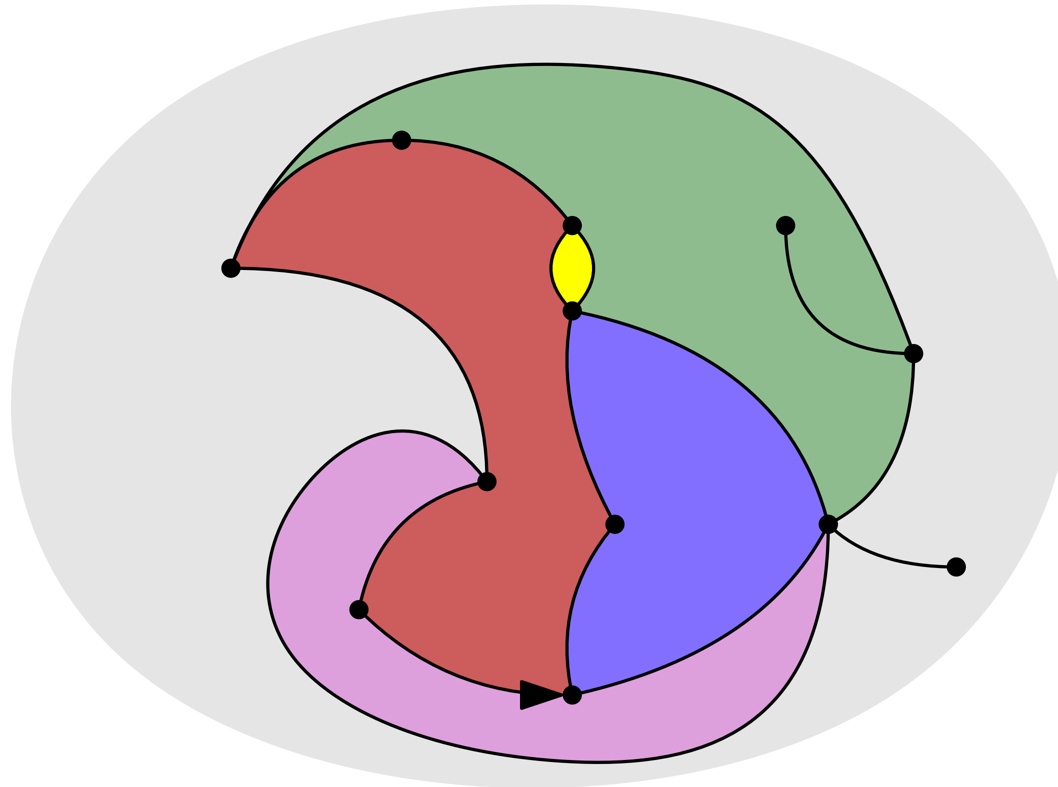

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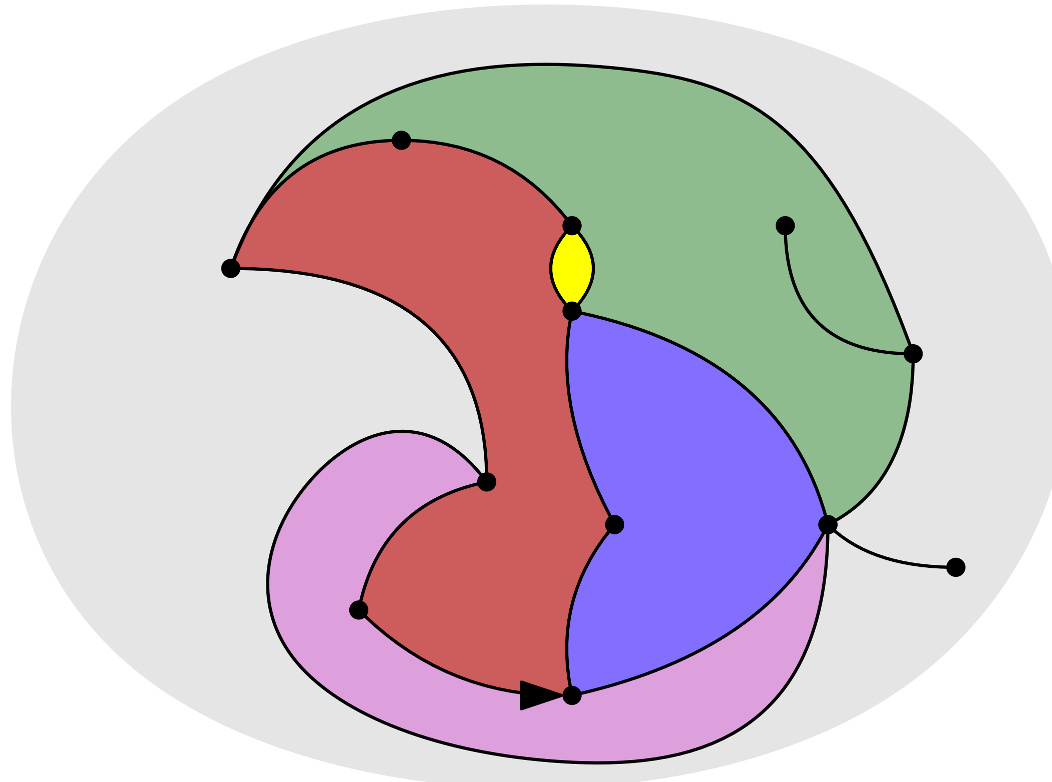

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Interest in planar maps:

- combinatorics: enumeration formulae, bijections;
- theoretical physics: matrix integral, quantum gravity;
- probability: behaviour of large random maps
  - model of discrete surfaces, scaling limit towards continuum surfaces?
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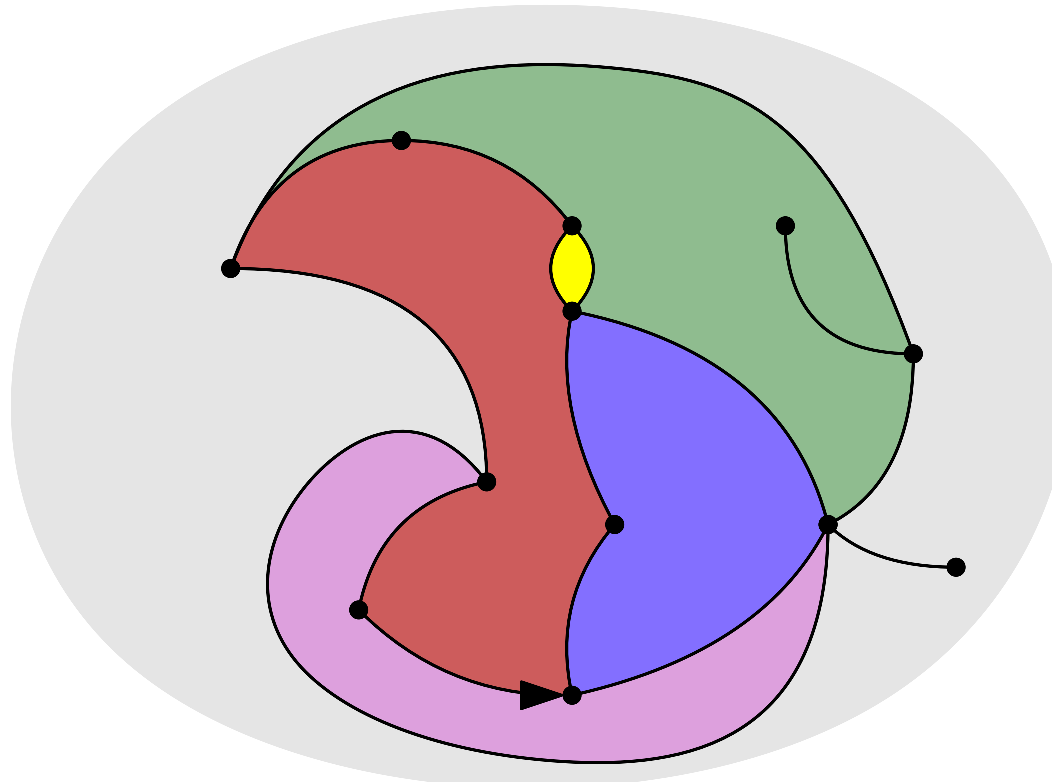

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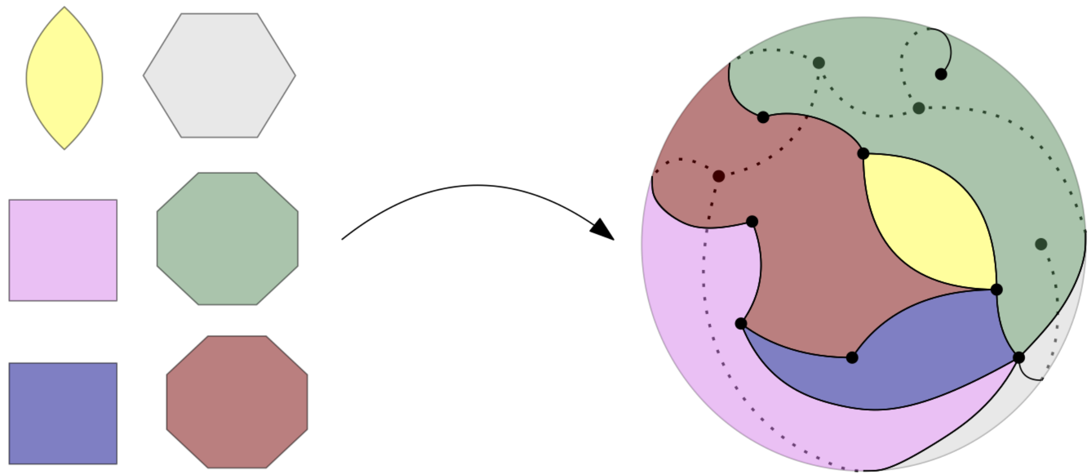
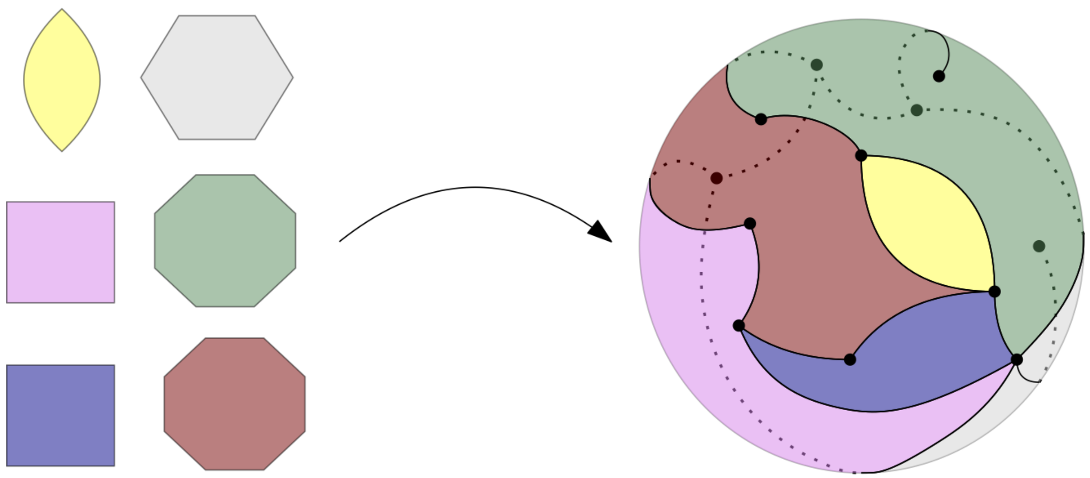
Technical restriction: We only consider **bipartite** maps.

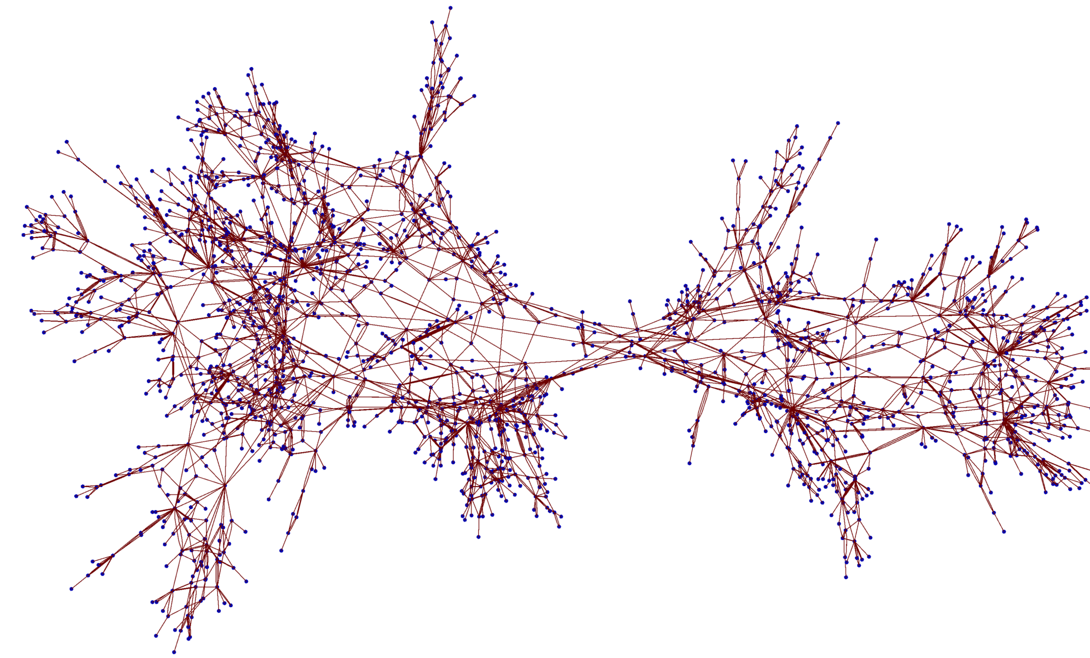
Convergence of maps



What topology do we put on maps?

Convergence of maps





As for trees, we extract the theoretical graph, and forget about the embedding, and give to each edge a length with tends to 0 with the size of the map.

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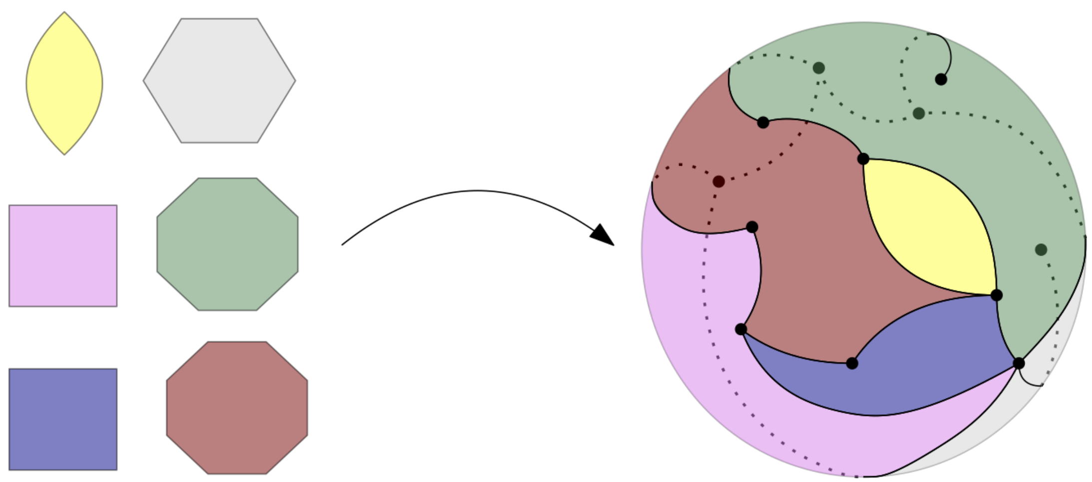

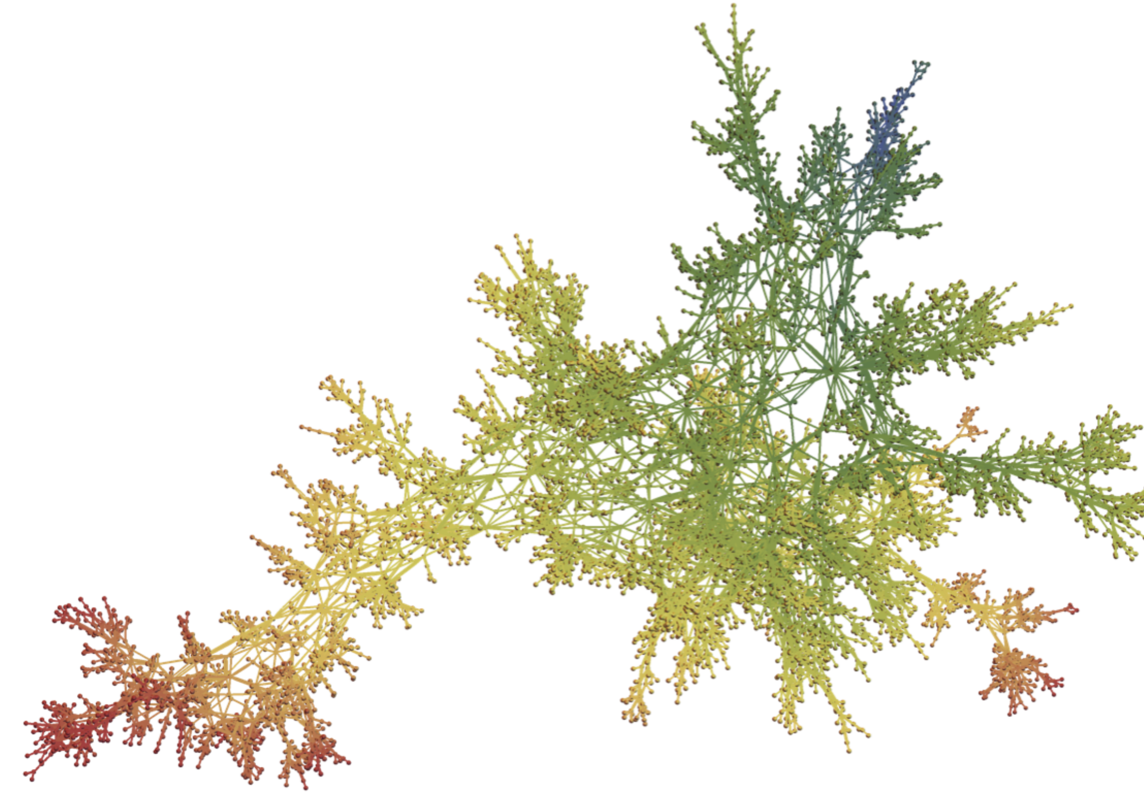

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Theorem (Le Gall '13 and Miermont '13)
If Q_n is a quadrangulation with n faces sampled uniformly at randon, then

$$\left(\frac{9}{8n}\right)^{1/4} Q_n \xrightarrow[n\to\infty]{(d)} \mathcal{S},$$

where $\mathcal{S}$ is the **Brownian sphere**.

$\mathcal{S}$ has the topology of the sphere (Le Gall & Paulin '08, Miermont '08) and Hausdorff dimension 4 (Le Gall '07).

Extended since to many other models of random maps, but always using the known case of quadrangulations as an input.

Boltzmann random maps

General model: fix $\mathbf{q} = (q_k)_{k \geqslant 1} \in [0, \infty)^{\mathbb{N}}$ and sample a map m_n with **size** n with probability:

$$\mathbf{P}_n^{\mathbf{q}}(m_n) = \frac{1}{Z_n} \prod_{\text{face } f} q_{\deg(f)/2},$$

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Theorem (☺ '21+). If M_n sampled from $\mathbf{P}_n^{\mathbf{q}}$ satisfies with high probability $\max_f \deg(f)(\deg(f)-2) \ll \sum_f \deg(f)(\deg(f)-2)$, then

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Application. If $\mathbf{q}$ satisfies some criticality and finite variance assumption, then

$$\left(\frac{c}{n}\right)^{1/4} M_n \xrightarrow[n\to\infty]{(d)} \mathcal{S},$$

where c depends both on $\mathbf{q}$ and the notion of size: either vertices, edges, or faces.

Biconditioned random maps

What about $\mathbf{q}$-Boltzmann maps with n edges and k_n vertices, and so $n - k_n + 2$ faces by Euler's formula? We assume both $k_n, n - k_n \to \infty$.

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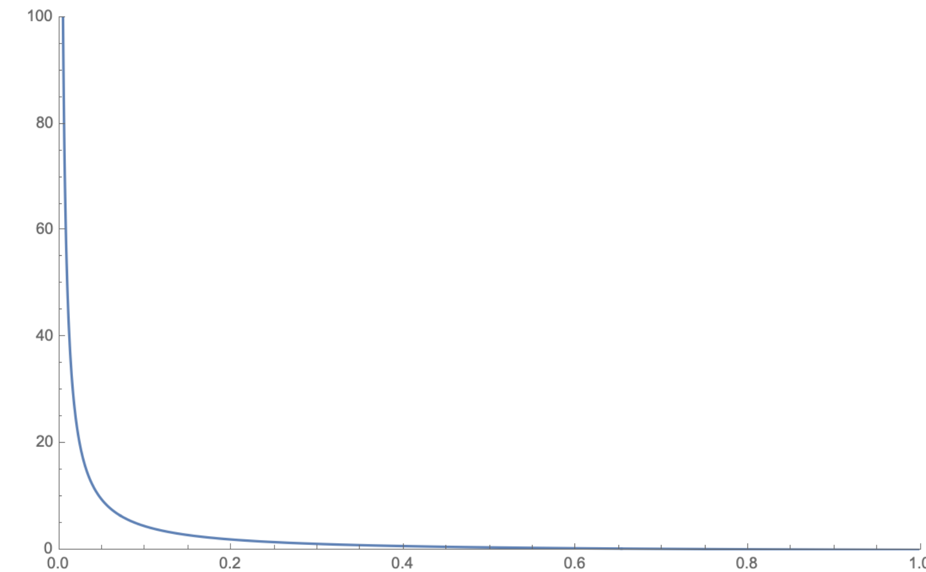
Theorem (Kortchemski & ☺ '21+). If M_n is a bipartite map with n edges and k_n vertices sampled uniformly at random, then

$$\left(s\left(\frac{k_n}{n}\right)\frac{9}{4n}\right)^{1/4} M_n \xrightarrow[n\to\infty]{(d)} \mathcal{S},$$

where $s(x) = (1-x)(3+x+\sqrt{(1-x)(9-x)})/(12x)$.



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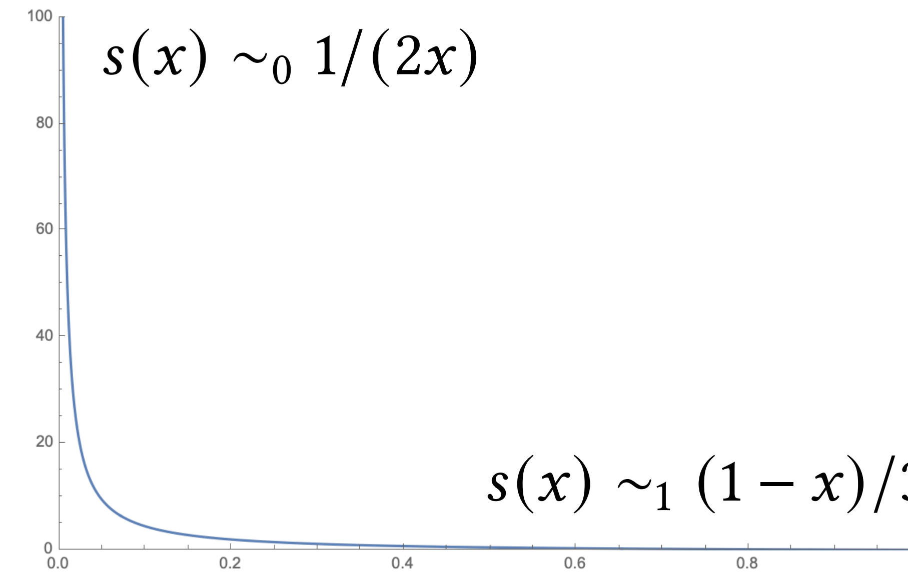

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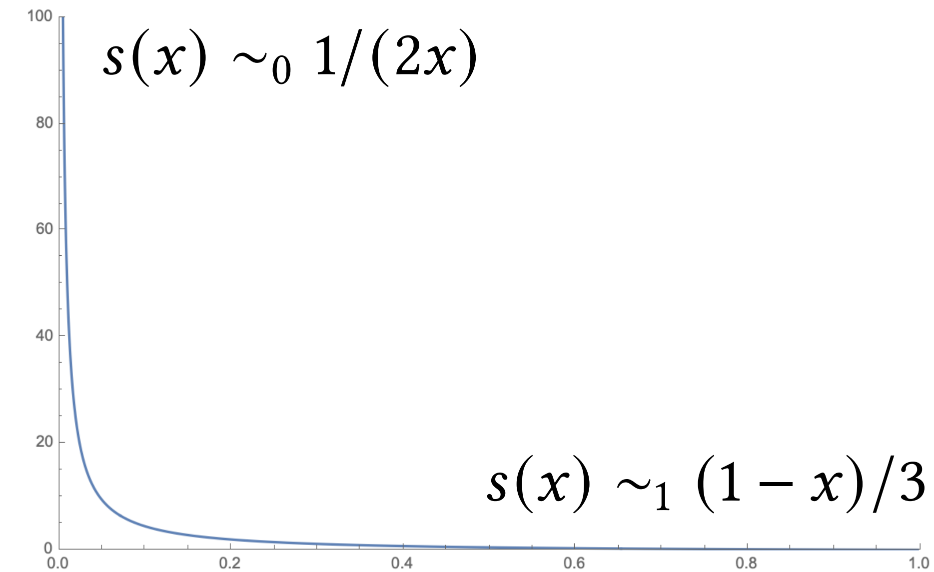

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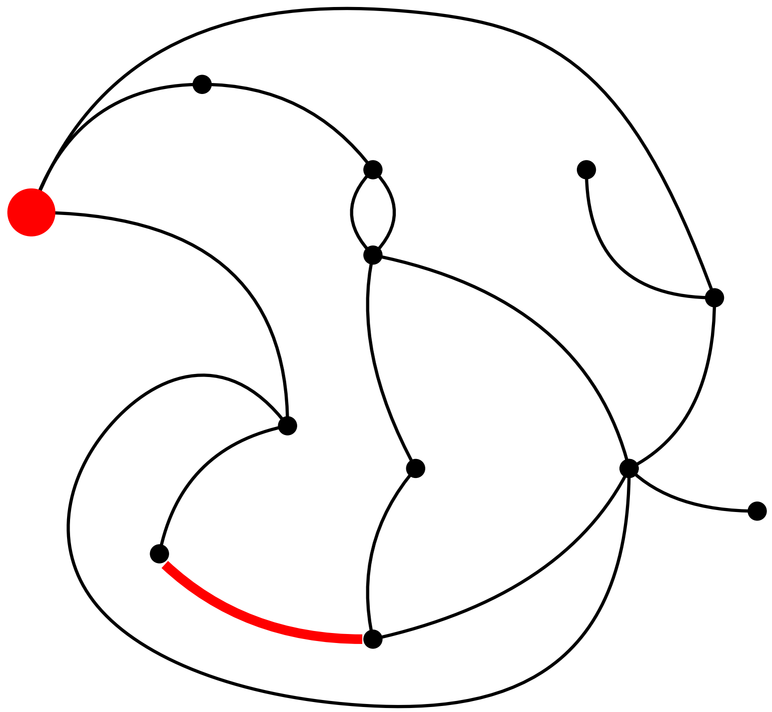
Actually nothing special about the uniform distribution, it is just a Boltzmann law with a sequence $\mathbf{q}$ with nice properties.

Back to trees



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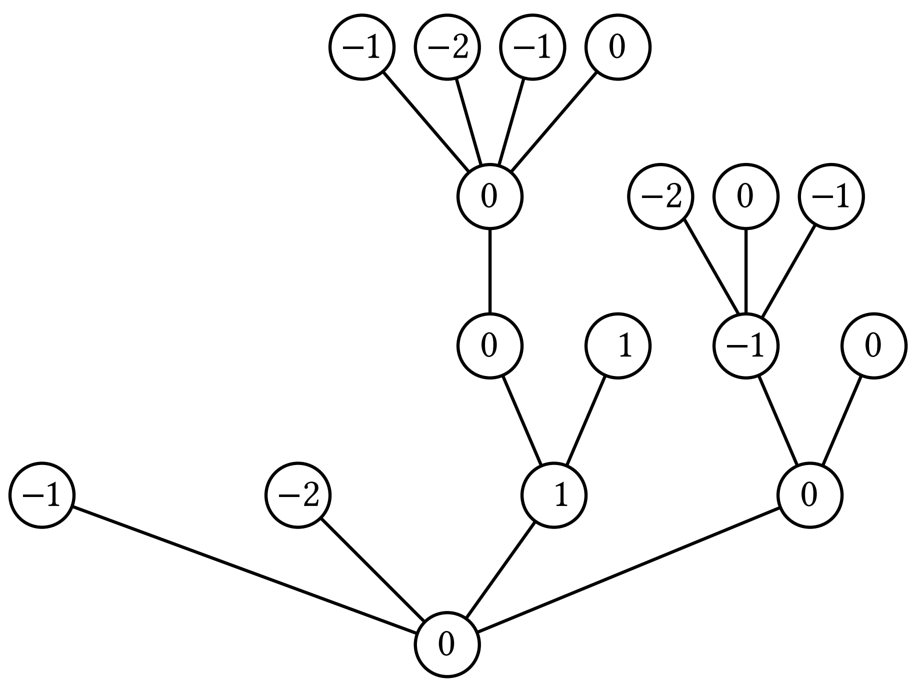
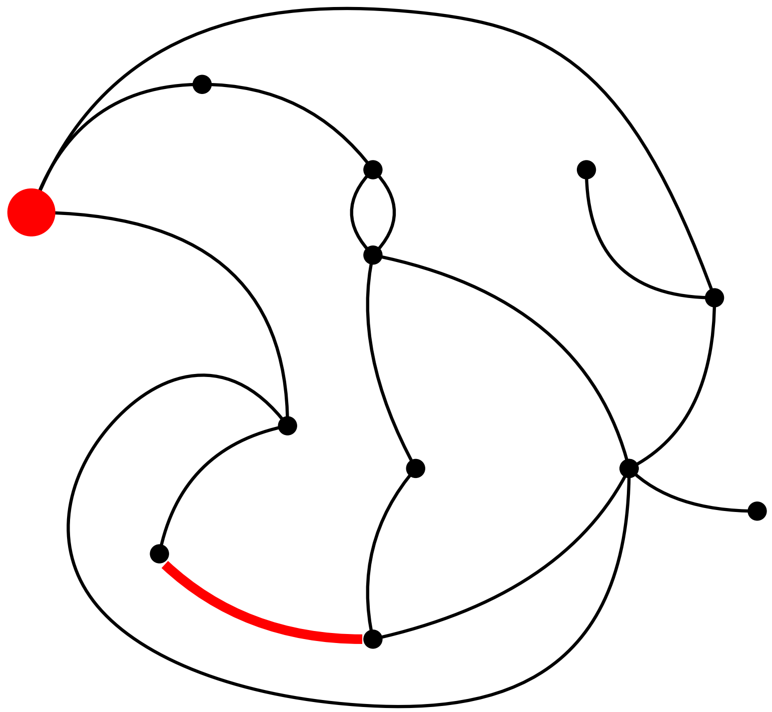
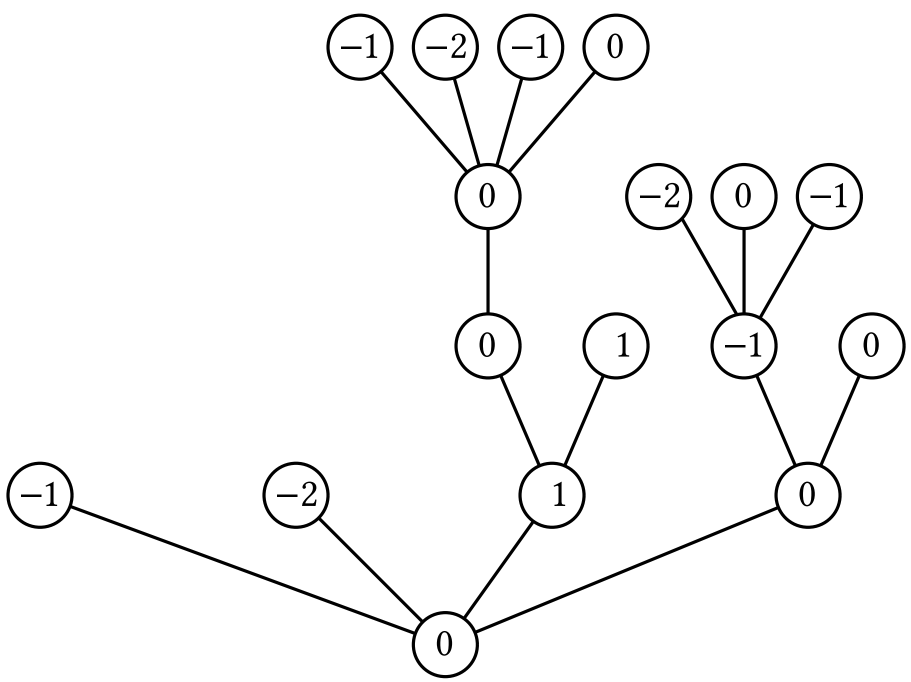


Back to trees





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Key properties of the bijection $M \leftrightarrow T$:

1. faces of $M \leftrightarrow$ internal vertices of T and the number of children is half the degree of the face;
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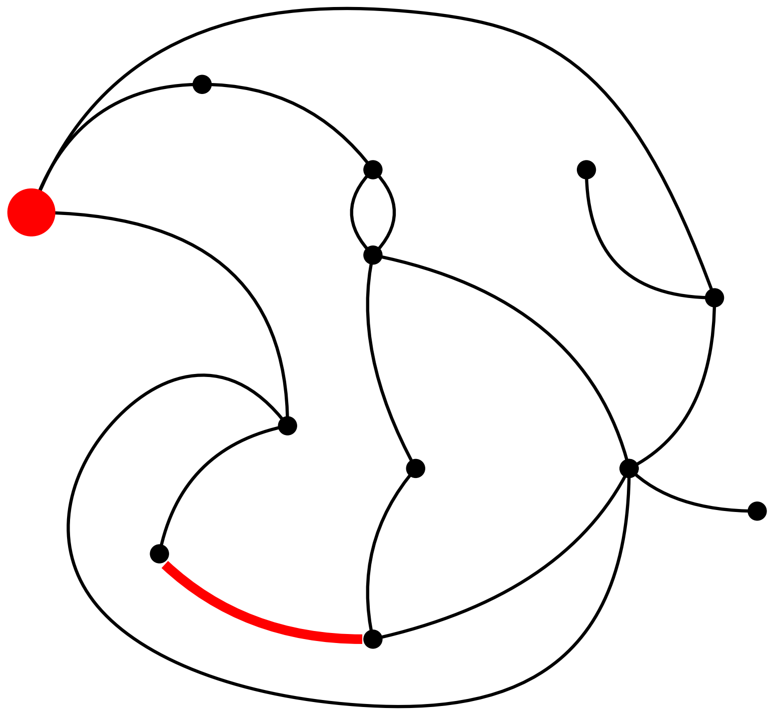
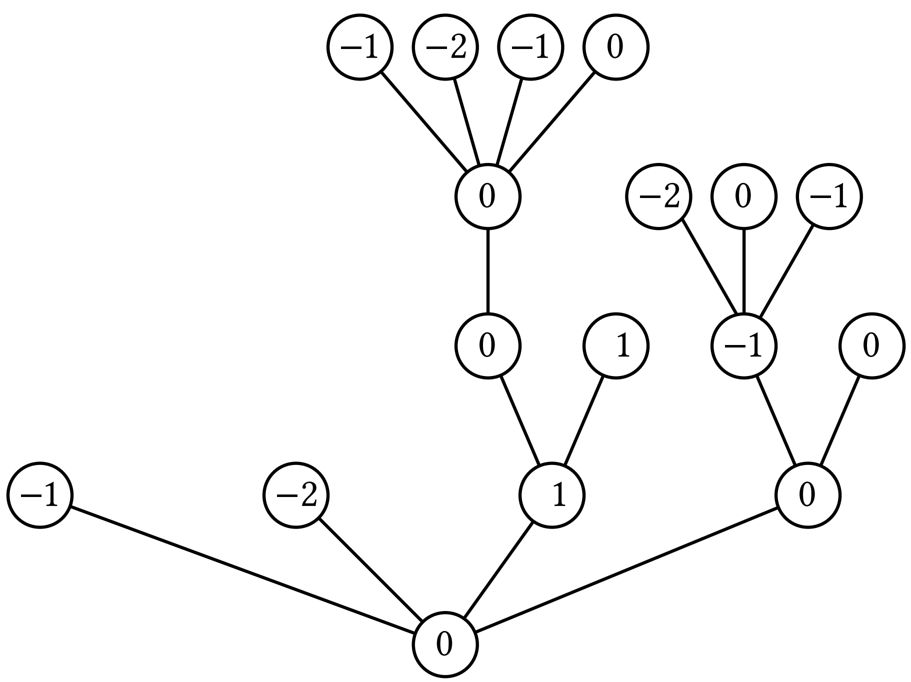
Back to trees





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Consequence: a $\mathbf{q}^M$-Boltzmann map with n edges and k_n vertices corresponds to a simply generated tree with $n+1$ vertices and $k_n - 1$ leaves, sampled from the weights

$$q_0^T = 1 \qquad \text{and} \qquad q_k^T = \binom{2k-1}{k-1} q_k^M \quad (k \geqslant 1).$$

Back to trees

Conclusion: In order to deduce that, for some deterministic sequence $a_n \to \infty$,

$$\left(\frac{9}{4a_n}\right)^{1/4} M_n \xrightarrow[n\to\infty]{(d)} \mathcal{S},$$

when M_n is a $\mathbf{q}^M$-Boltzmann bipartite map conditioned to have n edges and k_n vertices, it suffices to prove that, in a $\mathbf{q}^T$ simply generated tree with $n+1$ vertices and $k_n - 1$ leaves, where

$$q_0^T = 1 \qquad \text{and} \qquad q_k^T = \binom{2k-1}{k-1} q_k^M \quad (k \geqslant 1),$$

it holds that

$$\frac{\sum_u k_u(k_u-1)}{a_n} \xrightarrow[n\to\infty]{\mathbf{P}} 1 \quad \text{and} \quad \frac{\max_u k_u(k_u-1)}{a_n} \xrightarrow[n\to\infty]{\mathbf{P}} 0.$$

Final remarks & open questions

When $\lim_n x_n/n = \rho G'(\rho)/G(\rho) < \infty$, one needs to look closer and the behaviour depends on the speed of convergence. If $\mathbf{q}$ is a probability wih finite mean and in the domain of attraction of a stable law with index $\alpha \in (1, 2)$, the we can get a Brownian bridge, or a bridge of a stable process with a drift, or a one-big-jump principle.

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Thank you!