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Total derivative in a Lagrangian versus improvement of a conserved Noether current

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Subject (concerns classical relativistic field theory in n-dimensional Minkowski space-time $M = \mathbb{R}^n$) :

Total derivative in a Lagrangian : $\mathcal{L}_1(\varphi, \partial_\mu \varphi) = \partial_\mu \Lambda^\mu(\varphi)$

(Locally) conserved Noether current (j^μ) : $\partial_\mu j^\mu \approx 0$

(Notation $\approx$ for an on-shell equality)

I. Reminder on conserved current densities (j^μ) :

- Equivalence relation :

$$j^\mu \sim j^\mu + \underbrace{\partial_\rho B^{\rho\mu}}_{\text{superpot. term}} + \underbrace{t^\mu}_{\approx 0}, \qquad \text{where } B^{\rho\mu} = -B^{\mu\rho}$$

- Note :

$$\partial_\mu(\partial_\rho B^{\rho\mu}) = 0 \qquad \text{(identically conserved)}$$

- Thus :

For two equivalent currents $(j_1^\mu), (j_2^\mu)$: $\qquad \partial_\mu j_1^\mu \approx \partial_\mu j_2^\mu$

- Hence :

(j_1^μ) is on-shell conserved $\iff$ (j_2^μ) is on-shell conserved

$$\text{Addition of } \underbrace{\partial_\rho B^{\rho\mu}}_{\text{superpot. term}} + \underbrace{t^\mu}_{\approx 0} : \quad \text{improvement of } j^\mu$$

Illustr. 1 : EMT of free electromagnetic field $(n = 4)$

Lagrangian : $$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} \,, \quad \text{with } F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

Canonical EMT : $$T^{\mu\nu}_{\text{can}} = -F^{\mu\rho} \partial^\nu A_\rho + \frac{1}{4} \eta^{\mu\nu} F^{\rho\sigma} F_{\rho\sigma}$$

- Rewrite the first term :

$$-F^{\mu\rho} \underbrace{\partial^\nu A_\rho}_{= F^\nu{}_\rho + \partial_\rho A^\nu} = F^{\mu\rho} F_\rho{}^\nu - F^{\mu\rho} \partial_\rho A^\nu$$

- Rewrite the last term by using the Leibniz rule :

$$-F^{\mu\rho} \partial_\rho A^\nu = \underbrace{\partial_\rho (-F^{\mu\rho} A^\nu)}_{\text{superpot. term}} + \underbrace{(\partial_\rho F^{\mu\rho})}_{\approx 0} A^\nu$$

Thus :

$$\boxed{T^{\mu\nu}_{\text{can}} \sim F^{\mu\rho}F_{\rho}{}^{\nu} + \frac{1}{4}\eta^{\mu\nu}F^{\rho\sigma}F_{\rho\sigma}}$$ [Minkowski 1908]

Improved EMT : gauge invariant, symmetric, traceless

Illustration 2 : Scale invariance for a free massless real scalar field in n dimensions

Action :
$$S[\phi] \equiv \int_M d^n x \, \mathcal{L} \equiv \frac{1}{2} \int_M d^n x \, (\partial^\mu \phi)(\partial_\mu \phi)$$

Translation invariance $\rightsquigarrow$ $T^{\mu\nu}_{\text{can}} = \partial^\mu \phi \, \partial^\nu \phi - \eta^{\mu\nu} \mathcal{L}$

Invariance of action under scale transformations ($\rho \in \mathbb{R}$)

$$x' = \mathrm{e}^\rho x \, , \quad \phi'(x') = \mathrm{e}^{-\rho \, d} \phi(x) \, , \quad \text{with} \quad d \equiv \frac{n-2}{2}$$

$\rightsquigarrow$ canonical dilatation current :
$$j^\mu_{\text{can}} = x_\nu T^{\mu\nu}_{\text{can}} + d \, \phi \partial_\mu \phi$$

Trace of canonical EMT :

$$T^{\mu}_{\mathrm{can}\,\mu} = \frac{2-n}{2}\,(\partial^{\mu}\phi)(\partial_{\mu}\phi) \;\not\approx\; 0 \qquad \text{unless} \quad n = 2$$

Improved dilatation current j^μ_{conf}
Improved EMT $T^{\mu\nu}_{\text{conf}}$ (= traceless on-shell) :

$$\boxed{j^\mu_{\text{conf}} = x_\nu T^{\mu\nu}_{\text{conf}}} \quad \text{hence} \quad \boxed{0 \approx \partial_\mu j^\mu_{\text{conf}} = T^{\mu}_{\text{conf}\,\mu}}$$

C. Callan, S. Coleman, R. Jackiw [1970] : With $\xi_n \equiv \frac{1}{4}\frac{n-2}{n-1}$

$$j^\mu_{\text{conf}} \equiv j^\mu_{\text{can}} + \xi_n\,\partial_\rho\left[(x^\mu\partial^\rho - x^\rho\partial^\mu)\phi^2\right]$$

$$T^{\mu\nu}_{\text{conf}} \equiv T^{\mu\nu}_{\text{can}} - \xi_n\,(\partial^\mu\partial^\nu - \eta^{\mu\nu}\Box)\phi^2$$

II. Adding a total derivative to the Lagrangian :

Integration by parts :

$$\frac{1}{2}\int_M d^n x \, (\partial_\mu \phi)(\partial^\mu \phi) = -\frac{1}{2}\int_M d^n x \, \phi \, \Box \phi + \frac{1}{2}\int_M d^n x \, \partial_\mu \underbrace{(\phi \, \partial^\mu \phi)}_{\propto \, \partial^\mu \phi^2}$$

$\rightsquigarrow$ Consider :

$$\boxed{\mathcal{L}_1 \equiv \partial_\mu k^\mu}\,, \qquad \text{with} \quad \boxed{k^\mu \equiv -\xi \, \partial^\mu \phi^2} \quad (\xi \in \mathbb{R})$$

Associated dilatation current :

$$\boxed{j_1^\mu = \partial_\rho B^{\rho\mu}} \qquad \text{with} \quad \boxed{B^{\rho\mu} \equiv x^\rho k^\mu - x^\mu k^\rho}$$

Associated EMT :

$$\boxed{T_1^{\mu\nu} = -\partial_\rho \chi^{\rho\mu\nu}} \qquad \text{with} \quad \boxed{\chi^{\rho\mu\nu} \equiv k^\rho \eta^{\mu\nu} - k^\mu \eta^{\rho\nu}}$$

One recovers the expressions of CCJ from $\mathcal{L} + \mathcal{L}_1$ with the choice

$$\xi = \xi_n \equiv \frac{1}{4}\frac{n-2}{n-1}$$

III. Another example : [F. Delduc]

For certain (deformed) two-dimensional sigma models the addition of a certain total derivative to the Lagrangian allows us to have an improved current which satisfies the zero curvature condition (and thereby allows to establish the integrability of the models).

IV. General relation between a total derivative in the Lagrangian and improvements of conserved currents

V. Other examples ? ? ?